Irrational Square Roots:

\[ \sqrt{2} \quad \text{and} \quad \sqrt{n} \quad \text{when} \quad n \quad \text{is a Positive Integer and Not a Perfect Square} \]

[ It is recommended that you review Theorem (NIB) 3 in the handout "Theorems (NIB) 1, 2, and 3." ]

Theorem 4.6.1:  \( \sqrt{2} \) is irrational.

Proof: [ Proof by Contradiction ]

Suppose, by way of contradiction, that \( \sqrt{2} \) is rational.

Since \( \sqrt{2} \) is rational and positive, there exist positive integers \( m \) and \( n \), with \( n \neq 0 \), such that \( \sqrt{2} = \frac{m}{n} \), and we can assume that \( \frac{m}{n} \) is written in lowest terms, so that \( m \) and \( n \) have no common prime factor.

[ The author mistakenly says that \( m \) and \( n \) “have no common factor”, but 1 is always a common factor. ]

Since \( \sqrt{2} = \frac{m}{n} \), \( 2 = (\sqrt{2})^2 = \left( \frac{m}{n} \right)^2 = \frac{m^2}{n^2} \) by substitution .

Since \( 2 = \frac{m^2}{n^2} \), \( 2n^2 = m^2 \).

[ The contradiction that we will establish is that \( 2 | m \) and \( 2 | n \),

which contradicts the fact that \( m \) and \( n \) have no common prime factor. ]

Since \( m^2 = 2n^2 \) and \( n^2 \) is an integer, \( 2 | m^2 \), by definition of “divides”.

\( \therefore \) Since \( 2 | m^2 \) and 2 is prime, \( 2 | m \), by Theorem (NIB) 3 .

\( \therefore \) There exists an integer \( k \) such that \( m = 2k \), by definition of “divides”. Recall that \( 2n^2 = m^2 \).

\( \therefore 2n^2 = (2k)^2 = 2(2k^2) \), by substitution and the rules of algebra.

Dividing by 2, we conclude that \( n^2 = 2k^2 \), and \( k^2 \) is an integer .

\( \therefore 2 | n^2 \), by definition of “divides”.

\( \therefore \) Since \( 2 | n^2 \) and 2 is prime, \( 2 | n \), by Theorem (NIB) 3 .

\( \therefore 2 | m \) and \( 2 | n \), which contradicts the fact that \( m \) and \( n \) have no common prime factors.

Therefore, \( \sqrt{2} \) is irrational, by proof-by-contradiction

QED

[ You might consider how this proof can be adapted to prove that \( \sqrt{5} \) and \( \sqrt{7} \) are irrational. ]
To Prove: For all positive integers \( n \), if \( n \) is not a perfect square, then \( \sqrt{n} \) is irrational.

[ This is the statement to be proved in Problem #22 of Section 4.6. ]

Proof: [ by Contraposition ]

Let \( n \) be any positive integer.

Suppose that \( \sqrt{n} \) is rational. [ We need to show that \( n \) is a perfect square. ]

Since \( \sqrt{n} \) is rational and positive, there exist positive integers \( a \) and \( b \) with \( b \neq 0 \) such that \( \sqrt{n} = \frac{a}{b} \), and we can assume that \( \frac{a}{b} \) is written in lowest terms, so that \( a \) and \( b \) have no common prime factor.

Since \( \sqrt{n} = \frac{a}{b} \), \( n = \left( \sqrt{n} \right)^2 = \left( \frac{a}{b} \right)^2 = \frac{a^2}{b^2} \). Since \( n = \frac{a^2}{b^2} \), \( b^2 n = a^2 \).

[ We next prove that \( b = 1 \) using a proof-by-contradiction. ]

Suppose, by way of contradiction, that \( b \neq 1 \). (***)

\[ \therefore \text{ Since } b > 0 \text{ and } b \neq 1, \quad b > 1. \]

\[ \therefore \text{ by Theorem 4.3.4, there exists some prime number } p \text{ such that } p \mid b. \]

Since \( b^2 n = b(bn) \), \( b \mid b^2 n \) by definition of “divides”.

\[ \therefore p \mid b^2 n, \text{ by transitivity of divisibility. Recall that } b^2 n = a^2. \]

\[ \therefore p \mid a^2, \text{ by substitution.} \]

\[ \therefore \text{ Since } p \text{ is prime and } p \mid a^2, \quad p \mid a, \text{ by Theorem (NIB) 3.} \]

\[ \therefore p \mid a \text{ and } p \mid b, \text{ which contradicts the fact that } a \text{ and } b \text{ have no common prime factor.} \]

\[ \therefore b = 1 \text{ by proof-by-contradiction. [ Considering the initial supposition (***) above ]} \]

\[ \therefore n = \frac{a^2}{b^2} = \frac{a^2}{1} = a^2, \text{ and, therefore, } n \text{ is a perfect square.} \]

\[ \therefore \text{ If } n \text{ is not a perfect square, then } \sqrt{n} \text{ is irrational, by contraposition.} \]

\[ \therefore \text{ For all positive integers } n, \text{ if } n \text{ is not a perfect square, then } \sqrt{n} \text{ is irrational, by Direct Proof.} \]

QED [When applying this result, use the justification, "by Problem #22 of Section 4.6."]