A Theorem and The Parity Corollary

Theorem: Every integer \( n \) can be written as \( n = 2k \) or as \( n = 2k+1 \), for some integer \( k \).

\[ \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z} \text{ such that } n = 2k \text{ or } n = 2k+1. \]

Proof: Let \( n \) be any integer. Let \( d = 2 \).

[we apply the Quotient-Remainder Theorem to \( n \) and \( d \).]

By the Quotient-Remainder Theorem, there exist unique integers \( q \) and \( r \) such that \( n = 2q + r \) and \( 0 \leq r < 2 \).

\[ r = 0 \text{ or } r = 1. \] [These are the cases to consider.]

CASE 1: \( r = 0 \)

Suppose \( r = 0 \). Then, \( n = 2q + 0 = 2q \).

\[ \therefore n = 2q. \]

\[ \therefore n = 2q \text{ or } n = 2q + 1, \text{ by generalization.} \]

\[ \therefore \text{There exists an integer } k \text{ [here, } k = q \text{]} \text{ such that } \]

\[ n = 2k \text{ or } n = 2k + 1, \text{ in CASE 1.} \] [End of CASE 1]

CASE 2: \( r = 1 \)

Suppose \( r = 1 \). Then, \( n = 2q + 1 \).

\[ \therefore n = 2q \text{ or } n = 2q + 1, \text{ by generalization.} \]

\[ \therefore \text{There exists an integer } k \text{ [here, } k = q \text{]} \text{ such that } \]

\[ n = 2k \text{ or } n = 2k + 1, \text{ in CASE 2.} \] [End of Case 2]

\[ \therefore \text{In General, there exists an integer } k \text{ such that } n = 2k \]

or \( n = 2k+1 \). QED, by Direct Proof.

The Parity Corollary:

For every integer \( n \), \( n \) is even or \( n \) is odd.

Proof: Exercise, using the Theorem above.