Adequately Defined Variables vs. Inadequately Defined Variables

(The following discussion concerns some of the conventions used in writing and reading mathematical proofs.)

A variable is a symbol used to represent a particular mathematical or logical entity with a particular (but sometimes unspecified) value.

Examples of variables: \( x, y, z, f, P, m, \text{Temp}, \text{Trigfunc}, p, q, r, s \)

Examples of Mathematical Entities:
  - number, function, circle, set, point, integer, real number, rational number, positive real number, negative integer, non-negative integer, natural number, irrational number

Examples of Logical Entities:
  - statement, predicate, premise, conclusion, existential quantifier, universal quantifier, conditional statement, bi-conditional statement, equivalence, conjunction, disjunction

A Variable Declaration is a statement stating that a certain symbol will be used to represent a particular mathematical or logical entity. A symbol is adequately declared when the reader is unambiguously informed what type of entity is represented when that symbol is used as a variable.

Examples of (Adequate) Declarations of Variables:

  “The symbol \( x \) will represent a real number.”
  “The symbols \( m, n, \) and \( r \) will represent integers.”

Equivalent ways to phrase these two statements are:
  “Let \( x \) be a real number.”
  “Suppose that \( m, n, \) and \( r \) are integers.”

The Definition of a variable (as opposed simply to the declaration of that variable) involves two steps:

1) Adequately declaring the entity-type that it represents, and

2) Giving adequate particularity to its value.

A variable is adequately defined when the entity it represents has been adequately declared and when its particularity has been unambiguously described.

The term particularity here refers to two things:

1) The fact that the value assigned to the variable, which the variable represents in the moment of definition, is the same value that the variable represents when it is used later (unless the variable is redefined or the variable loses its definition (See locally defined variables below)), and

2) The ability to determine whether any given value is one which the variable could not represent or whether that given value is one which the variable could represent. This ability results from the specification of the domain of the variable, where the domain of the variable is the set of values which the variable could represent.
Examples of Adequately Defined Variables (including First-use Definition and Inherited Declarations.)

(It is assumed that the symbols used as variables here have not appeared earlier in the proof.)

“The symbol $x$ will represent the integer 1,736,423.”

(Entity-type is integer and particular value is 1,736,423.)

“Let $x = 1,736,423.$” (Entity-type is integer and particular value is 1,736,423.)

“Let $x$ be a particular positive integer whose value is not specified.”

(Entity type is positive integer and its particular value is one of the following: 1, 2, 3, … )

(Note: The particularity of the value of $x$ is adequately specified because it is unambiguous that, for instance, $x$ cannot represent 0 or $-3$, but $x$ can represent 5 or 3,688,932.)

Shorthand for this statement: “Let $x$ be any positive integer.”

Shorterhand for this statement: “Let $x$ be a positive integer.”

(So: “Let $x$ be … ” means “The symbol $x$ represents …” and “Let $x$ be a …” means “Let $x$ be any …”)

“Let $x$ be a positive integer and let $y = x - 100$.”

(The definition of $y$ here is an example of a First-use Definition. The implied declaration of $y$ as an integer is an example of an Inherited Declaration.)

Here the symbol $y$ is defined to be the integer whose value is 100 less than the value which $x$ represents. Since the value of $y$ is calculated in terms of $x$ and $x$ has been declared already as an integer and the calculation defining $y$ results in another integer, then $y$ inherits the declaration of integer also. Relying on inherited declarations is dangerous and should be avoided because they are easy to state incorrectly or to interpret incorrectly. For instance, without thinking about it carefully, one might assume (incorrectly) that the variable $y$ has been declared as a positive integer because $x$ has been declared as a positive integer. However, when the value of $x$ is 100 or less, the value of $y$ is negative or is 0 and $y$ is not positive. It would be better to state:

“Let $x$ be a positive integer and let $y$ be the integer such that $y = x - 100$.”

This way, the declaration of $y$ as an integer is made explicitly and cannot be incorrectly interpreted.

Example of Inadequately Defined Variables:

(It is assumed that the symbols used as variables here have not appeared earlier in the proof.)

“Let $y = x / 2.$”

The variable $x$ has not been declared earlier and is not adequately declared here so it is not clear whether $x$ is an integer, a real number, or a complex number. Consequently, it is unclear whether $y$ inherits a declaration as a rational number, another real number, or another complex number.

A variable has been adequately declared and defined if, at the point in a proof when the symbol is used, it can be unambiguously determined what entity-type it represents and what is the set of values that the symbol might represent (that is, what the domain of the variable is).
Global Variable and Local Variable Definitions:

A variable is **Globally Defined** if the declaration of its entity-type and the setting of its value is preserved throughout an argument (unless its value is reset by a global redefinition).

A variable is **Locally Defined** if the declaration of its entity-type and the setting of its value is meaningful for only a limited section of an argument, after which the symbol loses the previous definition and must be redefined before it can meaningfully be used again.

The examples of variable definitions presented so far have all been Global Variable Definitions.

Several occasions when variables are locally defined will be encountered during the semester. For now, we look at one of the most common use of Locally Defined Variables:

When a variable is declared or defined within an IF–THEN statement, that declaration or definition applies locally, not globally, only during that statement. That is, its declaration or definition applies only within that particular IF–THEN statement alone. The use of that variable beyond that IF–THEN statement without adequate redefinition of that variable is incorrect and lacks unambiguous meaning.

Example:

In the following argument excerpt, the declaration as an even positive integer applies to the variable \( x \) only in the first sentence which is an IF–THEN statement. After this statement, the symbol \( x \) loses its declaration as a positive integer, or as an integer, or indeed as a number of any kind.

“If \( x \) is an even positive integer, then \( (x + 2) \) is also an even positive integer.
Let \( y \) be such that \( y = (x + 2) / 2 \).
Therefore, \( y \) is an integer because the division of an even integer by 2 results in the calculation of an integer.”

The above argument is meaningless although it might not seem so to you now. Since variable \( x \) is declared as representing an even positive integer in the IF-THEN statement in the first sentence, variable \( x \) loses that declaration as representing an even positive integer when it is used in the second sentence. Since \( x \) loses its declaration as an even positive integer, it is not adequately declared as representing any particular mathematical entity in the second sentence, and so \( y \) is not adequately declared because there is no entity declaration of \( x \) for \( y \) to inherit.

**This point about \( x \) losing its declaration when it is used in the second sentence is subtle, but it is important for you to understand.**

It might be more easily understood after making the following considerations:

In the following, no declaration of variable \( x \) of any kind, globally or locally, is made:

“Let \( y \) be such that \( y = (x + 2) / 2 \).
Therefore, \( y \) is an integer.”

In that case, it is pretty clear that variable \( x \) is used without adequately defining what it represents. In fact, the conclusion that \( y \) is an integer is incorrect when \( x \) is 1, which we might imagine is the case, lacking an adequate definition of \( x \).

In the following, an interesting fact about numbers is added at the start, but still there is no declaration of the variable \( x \):

“If the number 2 is added to any even positive integer, then the result is another even positive integer.
Let \( y \) be such that \( y = (x + 2) / 2 \).
Therefore, \( y \) is an integer.”

In that case also, the variable \( x \) is used without adequately defining what it represents (What if \( x = 1 \)?).
Now, the IF–THEN statement:
“If \( x \) is an even positive integer, then \((x + 2)\) is also an even positive integer.”

is just another way of saying:

“If the number 2 is added to any even positive integer, then the result is another even positive integer.”

Looking now at the first version in this example, you can see why the definition of \( x \) as an even positive integer is local only to the first statement. When it is possible to replace a statement using an internally defined variable with an equivalent generic statement which does not use any variables, it is probably the case that the variable defined within the statement has a definition which is local only to that statement alone and the symbol must be defined again if it is used in a later statement.

The argument excerpt can be corrected by adding a statement redefining the variable \( x \) as a global variable:

“If \( x \) is an even positive integer, then \((x + 2)\) is also an even positive integer.

Let \( x \) represent any even positive integer.

Let \( y \) be such that \( y = (x + 2)/2 \).

Therefore, \( y \) is an integer because the division of an even integer by 2 results in the calculation of an integer.”

Now, even though this argument is now meaningful, it is in poor style. It is confusing and in poor style to use a particular symbol as a locally defined variable in one sentence and to then use the same symbol as a globally defined variable soon thereafter.

It would be good style to use a different symbol, perhaps \( z \) rather than \( x \), in the first sentence:

“If \( z \) is an even positive integer, then \((z + 2)\) is also an even positive integer.

Let \( x \) represent any even positive integer.

Let \( y \) be such that \( y = (x + 2)/2 \).

Therefore, \( y \) is an integer because the division of an even integer by 2 results in the calculation of an integer.”

>> IT IS AN ERROR TO USE AN INADEQUATELY DEFINED VARIABLE IN A PROOF.<<<

Example Exercises: In the following proofs, identify which variables are inadequately defined and explain in each case why the variable is inadequately defined, and then make sufficient changes or additions to the argument so that all the variables are adequately defined and make the corrected argument have good style. If every variable used is adequately defined, then so state:

1) “Let \( x \) be a real number greater than 2.
   If \( y \) is a real number greater than 5, then \((x + y)\) is greater than 7.
   Therefore \( y \) is greater than 3 since 5 is greater than 3.”

2) “If \( x \) is an integer, then \((x + 3)\) is an integer.
   Let \( y \) be a positive integer.
   Let \( z = w + y \).
   Therefore, \( z \) is greater than \( w \).”

3) “Let \( x \) be a real number greater than 2.
   Let \( y \) be a positive integer.
   Let \( z = x + y \).
   Therefore, \( z \) is greater than \( x \).”
Solutions:
1) The variable $y$, when it is used in the third sentence, is not adequately defined. When $y$ is defined in the second sentence, its definition within an IF–THEN statement is local to that sentence and so $y$ is defined only within that statement. When the symbol $y$ is used in the third statement, without $y$ being redefined, it is being used without being adequately defined.
Correction:
   “Let $x$ be a real number greater than 2.
   If $z$ is a real number greater than 5, then $(x + z)$ is greater than 7.
   Let $y$ be a real number greater than 5.
   Therefore $y$ is greater than 3 since 5 is greater than 3.”

(Note: This correction makes it clear that the first and second sentences in the correction are unnecessary to prove the conclusion and they can be omitted.)

2) Both variables $z$ and $w$ are inadequately defined. In the third sentence, variable $w$ is used without previously being defined or declared. The variable $z$ is defined in terms of variable $w$, which has not been declared as representing any particular entity, so $z$ inherits no particular declaration as representing a particular entity.
Correction:
   “If $x$ is an integer, then $(x + 3)$ is an integer.
   Let $y$ be a positive integer.
   Let $w$ be any integer.
   Let $z = w + y$.
   Therefore, $z$ is greater than $w$.”

(Here, the first sentence in the proof is revealed as being unnecessary to prove the conclusion.)

3) Every variable used is adequately defined. There are no places in which an inadequately defined variable is used.

Exercises:
In the following proofs, identify which variables are inadequately defined and explain in each case why each inadequately defined variable is inadequately defined, and then make sufficient changes or additions to the argument so that all the variables are adequately defined and make the corrected argument have good style. Note: Your corrected argument may not be valid in that the conclusion may be false, but your corrected argument must not use inadequately defined variables.
If every variable used is adequately defined, then so state:

1) “Let $x = 4$.
   Let $y = 6$.
   Let $z = x + y$.
   Therefore, $z = 10$.”

2) “Let $x$ be an integer less than 5.
   Let $z = y + 3$.
   Let $w = x + z$.
   Let $q = x + 3$.
   Therefore, $x > 20$.”
Exercises (continued):

3) “Let $y$ be a positive integer.
   If $x$ is 4, then $(x + y) > (y + 3)$.
   If $z$ is 6, then $(z + y) > (y + 5)$.
   Let $w = x + z$.
   Therefore, $w = 10$.”

4) “Suppose $a = 1$, $b = 3$, and $c$ is a negative integer.
   If $d$ is an integer less than $c$, then $d$ is negative.
   Let $y$ be any real number greater than 1,000.
   Let $z = x + y$.
   Let $k = a + c$.
   Let $m = b + c$.
   Let $n = d + c$.
   Therefore, $k + m = 2c + 4$.”

5) “Let $y$ be any real number greater than 1,000.
   Let $x$ be any integer less than $-1,000$.
   Let $z$ be greater than 500 but less than 1,000.
   Therefore, $(y + x - z)$ is a negative number.”

6) “Let $t > 5$.
   Suppose $y$ is an integer such that $y > t$.
   Let $x = t + 10$.
   Therefore, $x > y$.”