#2 \quad A = \{0, 1, 2, 3\}

\[ R_2 = \{ (0,0), (0,1), (1,1), (1,2), (2,2), (2,3) \} \]

a) The directed graph of \( R_2 \):

\[
\begin{array}{c}
0 \quad \rightarrow \quad 1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\end{array}
\]

b) \( R_2 \) is not reflexive since \((3,3) \notin R_2\) and so \( 3 \notin R_2 \).

c) \( R_2 \) is not symmetric because, for example, whereas \((1,2) \in R_2\), \((2,1) \notin R_2\), so \( 1 \notin R_2 \) but \( 2 \in R_2 \).

d) \( R_2 \) is not transitive:

1. \( R_2 \) and \( 2 \in R_2 \) since \((1,2) \in R_2 \) and \((2,1) \in R_2 \), but \( 1 \notin R_2 \).
Sec 8.2

10. Define relation $C$ on $\mathbb{R}$ as follows:
   
   For all $x, y \in \mathbb{R}$, $x Cy \iff x^2 + y^2 = 1$.

   $C$ is not reflexive:
   
   Proof: Let $x = 2$. [For example]
   
   $2^2 + 2^2 = 8 \neq 1 \iff 2 \notin 2$.
   
   $\therefore$ $C$ is not reflexive, by proof by counterexample. $\blacksquare$.

   $C$ is symmetric
   
   Proof: [WTS: For all $x, y \in \mathbb{R}$, if $x Cy$, then $y Cx$.]
   
   Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$ be given.
   
   Suppose $x Cy$.
   
   $\therefore x^2 + y^2 = 1$, by defn of relation $C$.
   
   $\therefore y^2 + x^2 = 1$, by the Commutative Property of Addition.
   
   $\therefore y Cx$, by defn of relation $C$.
   
   $\therefore C$ is symmetric, by direct proof. $\blacksquare$.

   $C$ is not transitive
   
   Proof: Let $x = 0$, $y = 1$ and $z = 0$. [For example]
   
   $0^2 + 1^2 = 1 \iff 0C1$, by defn of $C$.
   
   $1^2 + 0^2 = 1 \iff 1C0$, by defn of $C$.
   
   But, $0^2 + 0^2 = 0 \neq 1$. $\therefore 0 \notin 0$.
   
   $\therefore C$ is not transitive, by proof by counterexample. $\blacksquare$. 
Sec. 8.2

\#13 Def'n: Define relation \( F \) on \( \mathbb{Z} \) as follows:
For all \( m, n \in \mathbb{Z} \), \( mF n \iff \exists t \in \mathbb{Z} \) such that \( m - n = 5t \).

The relation \( F \) is called the "congruence modulo 5" relation.

\[ F \text{ is reflexive.} \]

\[ \text{Proof:} \quad \begin{align*}
\text{[TST]: For all } n \in \mathbb{Z}, nF n. \\
\text{Let } n \in \mathbb{Z} \text{ be given.} \\
n - n = 0 \quad \text{and} \quad 0 = 0 \cdot 5, \\
\implies 5 \mid (n-n) \\
\implies nF n \quad \text{by def'n of } F, \\
\therefore F \text{ is reflexive, by direct proof} \\
\text{QED.}
\end{align*} \]

\[ F \text{ is symmetric.} \]

\[ \text{Proof:} \quad \begin{align*}
\text{[TST]: } & F \text{ for all } m, n \in \mathbb{Z}, \\
& \text{if } mF n, \text{ then } nF m. \\
\text{Let } m \in \mathbb{Z} \text{ and } n \in \mathbb{Z} \text{ be given.} \\
\text{Suppose } mF n. \\
\implies 5 \mid (m-n) \quad \text{by def'n of } F, \\
\implies m-n = 5t \quad \text{for some integer } t, \\
\implies -(m-n) = -5t \\
\implies n-m = 5 \cdot (-t), \text{ and } -t \text{ is an integer,} \\
\implies 5 \mid (n-m) \\
\implies nF m \quad \text{by def'n of } F. \\
\therefore F \text{ is symmetric, by direct proof} \\
\text{QED.}
\end{align*} \]
Sec 8.2
#13 (cont.)

F is transitive

Proof:

[NTS: For all m, n, p ∈ ℤ]

if mFn and nFp, then mFp.

Let m, n, and p be any elements in ℤ (i.e., integers).

Suppose mFn and nFp.

Then, 5 | (m-n) and 5 | (n-p) by defn of F.

\[ m-n=5k \text{ and } n-p=5l \text{ for some integers } k \text{ and } l. \]

\[ (m-n)+(n-p)=5k+5l. \]

\[ m+(m-n)-p=5(k+l). \]

\[ m+p=5(k+l) \text{ and } k+l \text{ is an integer}. \]

\[ 5 | (m-p) \]

\[ mFp \text{ by defn of F}. \]

F is transitive, by direct proof.

QED.

#26 The Relation R on A = \{all strings of 0's, 1's and 2's of length 4\} is defined as follows: For all s, t ∈ A,

\[ sRt \iff \text{Resum of the character in } s = \text{Resum of the character in } t. \]

R is reflexive, symmetric, and transitive.
Sec 8.2

#33  \[ A = \mathbb{R} \times \mathbb{R} - \{(0,0)\} \]

For all \( p_1, p_2 \in A \),
\[ p_1 R p_2 \iff p_1 \text{ and } p_2 \text{ are on the same half-line emanating from } (0,0). \]

\( R \) is reflexive: For any \( p_2 \in A \), \( p_1 \) and \( p_1 \) are (is?) on the same half-line emanating from \( (0,0) \).

\( R \) is symmetric: Suppose \( p_1 R p_2 \), then \( p_1 \) and \( p_2 \) are on the same half-line, so \( p_2 \) and \( p_1 \) are on the same half-line, so \( p_2 R p_1 \).

\( R \) is transitive: Given any \( p_1, p_2, p_3 \in A \), there is only one unique half-line emanating from \( (0,0) \) which passes through \( p_2 \).

If \( p_1 R p_2 \) and \( p_2 R p_3 \), then \( p_1, p_2, \) and \( p_3 \) are on the same half-line, so \( p_1 \) and \( p_3 \) are on the same half-line, so \( p_1 R p_3 \).

#33  \[ A = \{ \text{all lines in the plane } \mathbb{R} \times \mathbb{R}, \] 
\[ l_1 R l_2 \iff l_1 \perp l_2. \]

\( R \) is not reflexive: No line \( l \) is perpendicular to itself.

\( R \) is symmetric: If \( l_1 \perp l_2 \), then \( l_2 \perp l_1 \), so if \( l_1 R l_2 \), then \( l_2 R l_1 \).

\( R \) is not transitive: Let \( l_1 = x\text{-axis} \), \( l_2 = y\text{-axis} \), \( l_3 = x\text{-axis} \). So \( l_1 R l_3 \), \( l_1 R l_2 \), \( l_2 R l_3 \), \( l_1 R l_3 \).