To Prove: For all integers $n$, if $n$ is odd, then $n^2$ is odd.

Proof: Let $n$ be any integer.

Suppose $n$ is odd.

By definition of "odd," there exists an integer $k$ such that $n = 2k + 1$.

$n^2 = (2k+1)^2$, by substitution.

$= 4k^2 + 4k + 1$

$= 2(2k^2+2k)+1$, by rules of algebra.

Let $t = 2k^2+2k$, which is an integer since sums and products of integers are integers.

$n^2 = 2t + 1$, by substitution.

$n^2$ is odd, by definition of "odd."

For all integers $n$, if $n$ is odd, then $n^2$ is odd, by Direct Proof.

Q.E.D.
41. In applying the definition of "even" to $m_2$, it was an error to use the variable "k" to represent the integer such that $m_2$ equals 2 times it, because "k" had already been defined as the integer $k$ such that $n = 2k + 1$.

A different variable, such as $l$, should have been used, saying, for instance, "By definition of "even", $m_2 = 2l'$ for some integer $l'".

42. In applying the definition of "even" to $n$, it was an error to use the variable "k" to represent the integer such that $n$ equals 2 times it, because "k" had already been defined as the integer $k$ such that $m = 2k$.

A different variable, such as $l$, should have been used, saying, for instance, "By definition of "even", $m = 2l'$ for some integer $l'"."
To Prove: The product of any even integer and any integer is even.

[Formal restatement: \( \forall m, n \in \mathbb{Z} \), if \( m \) is even, then \( m \cdot n \) is even.]

Proof: Let \( m \) and \( n \) be any integers.

Suppose \( m \) is even. [WTS: \( m \cdot n \) is even]

i. By definition of "even", \( m = 2k \) for some integer \( k \).

\[ m \cdot n = (2k)n \] by substitution.

\[ = 2(kn) \] by rules of algebra.

Let \( t = kn \), and \( t \) is an integer since a product of integers is an integer.

i. \( m \cdot n = 2t \) by substitution.

ii. By definition of "even", \( m \cdot n \) is even.

The product of any even integer and any integer is even by Direct Proof.

Q.E.D.
Sec. 4.11 (NOT ASSIGNED!)

#49. To Prove: The difference of any two odd integers is even.

[Formal restatement: \( \forall m, n \in \mathbb{Z}^{\text{odd}}, m - n \text{ is even} \)]

Proof: Let \( m \) and \( n \) be any two odd integers.

[One could also say: "Let \( m \) and \( n \) be any integers."

Suppose \( m \) and \( n \) are both odd.]

By definition of "odd," there exist integers \( k \) and \( l \) such that \( m = 2k + 1 \) and \( n = 2l + 1 \).

\[
\begin{align*}
m - n &= (2k + 1) - (2l + 1) \quad \text{by substitution} \\
&= 2k + 1 - 2l - 1 \\
&= 2k - 2l \\
&= 2(k - l) \quad \text{all by rules of algebra.}
\end{align*}
\]

Let \( t = k - l \), which is an integer since the difference of integers is an integer.

\[
\begin{align*}
m - n &= 2t \quad \text{by substitution.} \\
m - n \text{ is even, by definition of "even".}
\end{align*}
\]

The difference of any two odd integers is even.

By DIRECT PROOF. Q.E.D.