LAST TIME - DOUBLE RIEMANN Sums

of a function \( z = f(x,y) \)

over a rectangle \( R = [a, b] \times [c, d] \):

RIEMANN Sum \( = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{i}^{*}, y_{j}^{*}) \Delta A_{ij} \)

Then we define the number

\[
\iint_{R} f(x,y) \, dA = \lim_{m \to \infty} \lim_{n \to \infty} \left( \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{i}^{*}, y_{j}^{*}) \Delta A_{ij} \right)
\]

How we find this number?

By using ITERATED Integrals

For Example: Let \( R = [0, 3] \times [1, 2] \)

Let \( f(x,y) = y^2 + 8xy^3 + x^3 \).

How do we determine the number

\[
\iint_{R} (y^2 + 8xy^3 + x^3) \, dA ?
\]

Here, it is the Volume of the solid region above Rectangle \( R \) and below the surface graph.

Use Iterated Integrals:

Let \( x \) be fixed at a value in \([a, b]\), \( a \leq x \leq b \), and, while \( x \) is held constant, let \( y \) vary in \([c, d]\), \( c \leq y \leq d \).

Then \( g_{x}(y) = f(x,y) \) defines a function of \( y \) for each \( x \), \( a \leq x \leq b \).
Here, for $x$ with $0 \leq x \leq 3$

define $g_x(y) = y^2 + 8xy^3 + x^3$,

that is, $g_x(y) = f(x,y)$.

Here, $g_0(y) = y^2$

$g_1(y) = y^2 + 8y^3 + 1$

$g_{1.5}(y) = y^2 + 12y^3 + 3.375$

$g_2(y) = y^2 + 16y^3 + 8$

$g_3(y) = y^2 + 24y^3 + 27$

Fix $x$ at any number with $a = 0 \leq x \leq 3 = b$, and define the number $A(x)$ as follows:

$$A(x) = \int_1^2 (y^2 + 8xy^3 + x^3) \, dy = \int_c^d g_x(y) \, dy$$

$$= \left. \left( \frac{1}{3}y^3 + 2xy^4 + x^3y \right) \right|_{y=1}^{y=2}$$

$$= \left( \frac{8}{3} + 32x + 2x^3 \right) - \left( \frac{1}{3} + 2x + x^3 \right)$$

$$A(x) = x^3 + 30x + \frac{7}{3}, \quad 0 \leq x \leq 3 .$$

$A(x)$ is here the cross-sectional area of the solid $S$

cut by the vertical plane through $(x,0,0)$ and perpendicular to the $x$-axis.

The volume of the solid $S$ is determined by

$$V = \int_0^3 A(x) \, dx = \int_0^3 (x^3 + 32x + \frac{7}{3}) \, dx = 162.25$$

Also,

$$V = \iint_R (y^2 + 8xy^3 + x^3) \, dA = 162.25$$
By Fubini's Theorem (p. 984)

\[ \int_0^3 \left( \int_0^2 (y^2 + 8xy^3 + x^3) \, dx \right) \, dy = 162.25 \]

Also, by Fubini's Theorem, you can

\[ \int \int_R (y^2 + 8xy^3 + x^3) \, dA \]

\[ = \int_0^2 \left( \int_0^3 (y^2 + 8xy^3 + x^3) \, dx \right) \, dy \]

\[ = 162.25 \]

Problem: Determine the volume \( V \) of the solid \( S \) below

the surface graph of \( z = 16 - x^2 - y^2 \) and above

the rectangle \([0, 1] \times [0, 1]\).

Solution: \( V = \int \int_R (16 - x^2 - y^2) \, dA \), so,

\[ V = \int_0^1 \int_0^1 (16 - x^2 - y^2) \, dy \, dx \]

\[ = \int_0^1 \left( \int_0^1 (16 - x^2 - y^2) \, dy \right) \, dx \]

\[ = \int_0^1 \left( \left[ 16y - x^2y - \frac{1}{3}y^3 \right]_{y=0}^{y=1} \right) \, dx \]

\[ = \int_0^1 (16 - x^2 - \frac{1}{3}) \, dx \]
\[
V = \int_0^1 \left( (16 - x^2 - \frac{1}{3}) - (0) \right) \, dx \\
= \int_0^1 (16 - x^2 - \frac{1}{3}) \, dx = \int_0^1 \left( \frac{47}{3} - x^2 \right) \, dx \\
= \left[ \frac{47}{3} x - \frac{1}{3} x^3 \right]_0^1 = \left( \frac{47}{3} - \frac{1}{3} \right) - (0-0) \\
V = \frac{46}{3} = 15 \frac{1}{3} \text{ cubic units}.
\]

Look ahead and be careful which order you choose to do the integrating in.

If you end up with a difficult integral, consider switching the order of integration.

Ex:  \( R = [1, 2] \times [0, \pi] \).

Determine \( \iint_R y \sin(xy) \, dA \).

You have two choices. Which order should you choose?

\[
\int_0^1 \int_0^\pi y \sin(xy) \, dy \, dx \quad \text{or} \quad \int_0^\pi \int_0^2 y \sin(xy) \, dx \, dy
\]

A PRODUCT OF A FUNCTION OF TWO FUNCTIONS OF \( y \) TO INTEGRATE “\( dy \)”

A PRODUCT OF A CONSTANT TIMES A FUNCTION OF \( x \) TO INTEGRATE “\( dx \)”

You can use u-substitution here.

Choose this one!
Compare these functions:

\[ f_1(x, y) = xy \sin y \quad \text{and} \quad f_2(x) = x^4 \sin (xy) \]

\[ f_1(x, y) = g(x) h(y) \]

When \( f(x, y) = g(x) h(y) \),

\[
\int_a^b \int_c^d (g(x) h(y)) \, dy \, dx
\]

\[
= \left( \int_a^b g(x) \, dx \right) \left( \int_c^d h(y) \, dy \right)
\]

because

\[
\int_a^b \left( \int_c^d g(x) h(y) \, dy \right) \, dx
\]

\[
= \int_a^b \left( g(x) \int_c^d h(y) \, dy \right) \, dx
\]

\[
= \int_a^b \left( g(x) \left( \int_c^d h(y) \, dy \right) \right) \, dx
\]

\[
= \left( \int_c^d h(y) \, dy \right) \int_a^b g(x) \, dx = \int_a^b g(x) \, dx \int_c^d h(y) \, dy
\]

so,

\[
\int_0^{\pi/2} \int_0^1 x^4 \sin y \, dy \, dx = \left( \int_0^1 x^4 \, dx \right) \left( \int_0^{\pi/2} \sin y \, dy \right) = \left( \frac{1}{5} \right) (1) = \frac{1}{5}
\]