EQUATIONS OF NON-VERTICAL PLANES IN 3-SPACE.

Every plane \( P \) in \( \mathbb{R}^3 \) has an equation of the form:
\[ ax + by + cz = d \]
for some constants \( a, b, c, d \).

Such an equation is called a linear equation of \( P \).
Also, the plane \( P \) is non-vertical \( \iff c \neq 0 \).

In this paper, we show how to find a linear equation of a given non-vertical plane \( P \) when all we know about plane \( P \) is that it passes through three given points, \((x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2)\).

FACT: If we know the plane \( P \) is non-vertical and passes through the given point \((x_0, y_0, z_0)\),
then an equation of \( P \) is
\[ z - z_0 = A(x - x_0) + B(y - y_0) \]
for particular numbers \( A \) and \( B \).

We will know an equation for \( P \) once we have determined what numbers \( A \) and \( B \) are.

When we know two other points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) on plane \( P \), we can use those two points to determine \( A \) and \( B \). Since those two points are also on plane \( P \),
Their coordinates will satisfy the equation
\[ z - z_0 = A(x - x_0) + B(y - y_0) \]
when \((x, y, z) = (x_1, y_1, z_1)\) and
when \((x, y, z) = (x_2, y_2, z_2)\).

We substitute these coordinates into the equation to obtain two linear equations in the two unknowns \(A\) and \(B\). We then solve for \(A\) and \(B\).

Then, finally, we convert the equation
\[ z - z_0 = A(x - x_0) + B(y - y_0) \]
into its linear equation form \(ax + by + cz = d\).

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**Example 1**: It is given that plane \(P\) is non-vertical and passes through the points \((3,5,1)\), \((2,3,0)\), \((1,4,1)\).

Determine a linear equation of Plane \(P\).

**Solution**: It is arbitrary which of these points is to be called \((x_0, y_0, z_0)\) — (it doesn't matter which it is) — so, let's set \((x_0, y_0, z_0) = (3,5,1)\).
So, using \((x_0, y_0, z_0) = (3, 5, 1)\), the equation \(z - z_0 = A(x - x_0) + B(y - y_0)\) becomes:

\[z - 1 = A(x - 3) + B(y - 5)\]  \((*)\)

Substituting in the points \((x_1, y_1, z_1) = (2, 3, 0)\)
and \((x_2, y_2, z_2) = (1, 4, 1)\) into \((*)\),
we get equations \((1)\) and \((2)\) below:

\[\begin{align*}
(1) & \quad 0 - 1 = A(2 - 3) + B(3 - 5) \\
\Rightarrow & \quad -1 = -A - 2B \quad \Rightarrow \quad 1 = A + 2B \\
\end{align*}\]

and

\[\begin{align*}
(2) & \quad 1 - 1 = A(1 - 3) + B(4 - 5) \\
\Rightarrow & \quad 0 = -2A - B \quad \Rightarrow \quad B = -2A \\
\end{align*}\]

According to the final form of equation \((2)\), we can substitute \(-2A\) in for \(B\) in Equation \((1)\), which gives us:

\[1 = A + 2(-2A) \quad \Rightarrow \quad 1 = A - 4A \quad \Rightarrow \quad -3A = 1
\]

\[\Rightarrow A = -\frac{1}{3}
\]

Since \(B = -2A\), \(B = +\frac{2}{3}\).

An equation for \(LP\), then, is \(z - 1 = -\frac{1}{3}(x - 3) + \frac{2}{3}(y - 5)\).

Multiplying through by 3:

\[3z - 3 = -(x - 3) + 2(y - 5)\]

\[\Rightarrow 3z - 3 = -x + 3 + 2y - 10 \quad \Rightarrow \quad 3z - 3 = -x + 2y - 7
\]

\[\Rightarrow x - 2y + 3z = -4
\]

A linear equation of \(LP\) is

\[x - 2y + 3z = -4\]
**Example 2:** It is given that plane \( P \) is non-vertical and passes through the points \((2, -1, 4), (1, 1, -1), (-4, 1, 1)\). Determine a linear equation for \( P \).

Sol'n: Let \((x_0, y_0, z_0) = (2, -1, 4)\).

Then, the equation \(z - z_0 = A(x - x_0) + B(y - y_0)\) becomes \(z - 4 = A(x - 2) + B(y + 1)\) \((\ast)\).

Substituting the points \((x_1, y_1, z_1) = (1, 1, -1)\) and \((x_2, y_2, z_2) = (-4, 1, 1)\) into \((\ast)\), we obtain equations 1 and 2:

1. \(-1 - 4 = A(1 - 2) + B(1 + 1) \Rightarrow -5 = -A + 2B \Rightarrow 2B = A - 5\)

2. \(1 - 4 = A(-4 - 2) + B(1 + 1) \Rightarrow -3 = -6A + 2B \Rightarrow 2B = -3 + 6A\)

Setting the two expressions for \(2B\) equal to each other, we have \(A - 5 = -3 + 6A \Rightarrow -2 = 5A \Rightarrow A = \frac{-2}{5}\).

So, \(2B = -\frac{2}{5} - 5 = -\frac{27}{5} \Rightarrow B = \frac{-27}{10}\).

One equation for \( P \) is \(z - 4 = \left(\frac{-2}{5}\right)(x - 2) + \left(\frac{-27}{10}\right)(y + 1)\).

Multiplying through by 10 gives \(10z - 40 = -4(x - 2) - 27(y + 1)\) \((\ast)\).

This simplifies to \(4x + 27y + 10z = 21\) \((\ast)\).