Tests for the Convergence or Divergence of Infinite Series

Three useful sequence limits:

\[
\begin{align*}
\text{If } |x| < 1, & \text{ then } x^n \to 0 \text{ as } n \to \infty \\
\text{If } x = 1, & \text{ then } 1^n \to 1 \text{ as } n \to \infty \\
\text{If } x > 1, & \text{ then } x^n \to \infty \text{ as } n \to \infty
\end{align*}
\]

Series that we know Converge:

A. For any fixed \( p > 1 \): \[\sum_{k=1}^{\infty} \frac{1}{k^p}\] converges. (These type series are called p-series.)

Examples: With \( p = 2 \), \[\sum_{k=1}^{\infty} \frac{1}{k^2}\] converges.

With \( p = 3/2 \), \[\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}\] converges.

B. The Geometric Series when \( |r| < 1 \) will converge to \( 1 / (1 - r) \).

In this case: \[1 + r + r^2 + r^3 + \cdots = \sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}\]

Series that we know Diverge:

A) The Harmonic Series: \[\sum_{k=1}^{\infty} \frac{1}{k} \to \infty\]

B) The p-series when \( p \leq 1 \): \[\sum_{k=1}^{\infty} \frac{1}{k^p} \to \infty\]

C) The Geometric Series when \( |r| \geq 1 \) will diverge.

In this case: \[1 + r + r^2 + r^3 + \cdots = \sum_{k=0}^{\infty} r^k \text{ diverges and if } r \geq 1, \sum_{k=0}^{\infty} r^k \to \infty\]
The Major Tests for Convergence and Divergence for Infinite Series

1) **The Basic Divergence Test:** If \( a_k \xrightarrow{\text{Does NOT}} 0 \), then \( \sum_{k=0}^{\infty} a_k \) diverges.

2) **The Integral Test:** If \( f \) is continuous, decreasing, and positive on \([1, \infty)\),

then with \( a_k = f(k) \):

\[
\begin{align*}
\sum a_k & = \sum f(k) \text{ Converges, if } \int_{1}^{\infty} f(x) \, dx \text{ Converges.} \\
\sum a_k & = \sum f(k) \text{ Diverges, if } \int_{1}^{\infty} f(x) \, dx \text{ Diverges.}
\end{align*}
\]

3) **The Basic (Direct) Comparison Test:** Let \( \sum a_k \) and \( \sum b_k \) be series with all non-negative terms such that for ALL BIG \( k \), \( a_k \leq b_k \), then:

\[
\begin{align*}
\text{If } \sum b_k \text{ converges, then } \sum a_k \text{ converges, too.} \\
\text{If } \sum a_k \text{ diverges, then } \sum b_k \text{ diverges, too.}
\end{align*}
\]

4) **The Limit Comparison Test:** Let \( \sum a_k \) be a series with all non-negative terms and let \( \sum b_k \) be series with all positive terms.

If the sequence \( \frac{a_k}{b_k} \to L > 0 \), where \( L \) is a real number, then

either both of the series converge or both of the series diverge.
5) The Root Test:

Let \( \sum a_k \) be a series with all non-negative terms such that

\[
\sqrt[k]{a_k} \to p \quad \text{as} \quad k \to \infty.
\]

\[
\left\{
\begin{array}{l}
\text{a) If } p < 1 \text{, then the series } \sum a_k \text{ converges.}
\end{array}
\right.
\]

\[
\left\{
\begin{array}{l}
\text{b) If } p > 1 \text{, then the series } \sum a_k \text{ diverges.}
\end{array}
\right.
\]

\[
\left\{
\begin{array}{l}
\text{c) If } p = 1 \text{, then the results of the test are inconclusive.}
\end{array}
\right.
\]

6) The Ratio Test:

Let \( \sum a_k \) be a series with all positive terms such that

\[
\frac{a_{k+1}}{a_k} \to L \quad \text{as} \quad k \to \infty.
\]

\[
\left\{
\begin{array}{l}
\text{a) If } L < 1 \text{, then the series } \sum a_k \text{ converges.}
\end{array}
\right.
\]

\[
\left\{
\begin{array}{l}
\text{b) If } L > 1 \text{, then the series } \sum a_k \text{ diverges.}
\end{array}
\right.
\]

\[
\left\{
\begin{array}{l}
\text{c) If } L = 1 \text{, then the results of the test are inconclusive.}
\end{array}
\right.
\]