Worksheet: The Mean Value Theorem

1. State the mean value theorem and illustrate the theorem in a sketch.

2. Suppose that \( g \) is differentiable for all \( x \) and that \(-5 \leq g'(x) \leq 2\) for all \( x \). Assume also that \( g(0) = 2 \). Based on this information, is it possible that \( g(2) = 8 \)?

3. Section 4.2 in the text contains the following important corollary which you should commit to memory:

   \[
   \text{Corollary 7, p. 284: If } f'(x) = g'(x) \text{ for all } x \text{ in an interval } (a, b) \text{ then } f(x) = g(x) + c \text{ for some constant } c. 
   \]

   Use this result to answer the following questions:

   (a) If \( f'(x) = \sin(x) \) and \( f(0) = 15 \) what is \( f(x) \)?
   (b) If \( f'(x) = \sqrt{x} \) and \( f(4) = 5 \) what is \( f(x) \)?
   (c) If \( f'(x) = k \) where \( k \) is a constant, show that \( f(x) = kx + d \) for some other constant \( d \).

4. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers \( c \) that satisfy the conclusion of the Mean Value Theorem.

   (a) \( f(x) = e^{-2x}, [0,3] \)
   (b) \( f(x) = \frac{x}{x+2}, [1,4] \)

5. A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 miles on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

6. If \( f(1) = 10 \) and \( f'(x) \geq 2 \) for \( 1 \leq x \leq 4 \), how small can \( f(4) \) possibly be?

7. For what values of \( a, m, \) and \( b \) does the function

   \[
   f(x) = \begin{cases} 
   3 & \text{if } x = 0 \\
   -x^2 + 3x + a & \text{if } 0 < x < 1 \\
   mx + b & \text{if } 1 \leq x \leq 2
   \end{cases}
   \]

   satisfy the hypotheses of the Mean Value Theorem on the interval \([0,2]\)?

8. Determine whether the following statements are true or false. If the statement is false, provide a counterexample.

   (a) If \( f \) is differentiable on the open interval \((a, b)\), \( f(a) = 1 \), and \( f(b) = 1 \), then \( f'(c) = 0 \) for some \( c \) in \((a, b)\).
   (b) If \( f \) is differentiable on the open interval \((a, b)\), continuous on the closed interval \([a, b]\), and \( f'(x) \neq 0 \) for all \( x \) in \((a, b)\), then we have \( f(a) \neq f(b) \).
   (c) Suppose \( f \) is a continuous function on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\). If \( f(a) = f(b) \), then \( f'\left(\frac{a+b}{2}\right) = 0 \).
   (d) If \( f \) is differentiable everywhere and \( f(-1) = f(1) \), then there is a number \( c \) such that \(|c| < 1\) and \( f'(c) = 0 \).