PDE I – HOMEWORK ASSIGNMENT 4

Due Monday, September 27, 2010. Please write clearly, and staple your work!

1. Problem

Let \( U \subset \mathbb{R}^n \) be an open subset. Find the characteristic ODE of the Hamilton-Jacobi equation,

\[
    u_t + H(D_x u, x, t) = 0
\]

for \( u \in C^2(U \times \mathbb{R}_+) \) as a function of \((x, t) \in U \times \mathbb{R}_+\), with \( H \) being \( C^2 \) in all its arguments. Show that it has the form

\[
    x_t(t) = D_p H(p, x, t), \quad u_t(t) = (p \cdot D_p H)(p, x, t), \quad p_t(t) = -(D_x H)(p, x, t),
\]

where \( u(t) = u(x(t), t) \), and \( p(t) = D_x u(x(t), t) \).

**Hint:** You might want to consider \( y := (x, t) \) and \( D_y u \), to bring the above equation into the form \( F(D_y u, y) = 0 \) which has been discussed in class. Show that the characteristic curve \( y(s) \) is such that in fact, \( t(s) = s \) (which is why the above characteristic ODE is parametrized by \( t \), not \( s \)).

**Solution:** Let \( y := (x, t) \), so that

\[
    F(D_y u, y) = u_{y_2} + H(u_{y_1}, y) = 0,
\]

and \( u = g \) on \( \Gamma \).

2. Problem

Consider the Hamilton-Jacobi equation

\[
    u_t + \frac{1}{2}(D_x u)^2 - x = 0
\]

for \( x \in \mathbb{R} \) and \( t \in \mathbb{R}_+ \), with

\[
    u(x, 0) = x.
\]

Determine the initial conditions for the characteristic ODE, and solve the characteristic ODE. Subsequently, find the solution \( u(x, t) \).

**Hint:** Again, consider \( y := (x, t) \), \( D_y u \), and \( F(D_y u, y) = 0 \). Then, the initial condition at \( t = 0 \) becomes a boundary condition for \( y \in \mathbb{R} \times \mathbb{R}_+ \). Determine the initial conditions for the characteristic ODE as discussed in class.

**Solution:** The characteristic system of ODE’s for \((p(t), u(t), x(t))\) are obtained from problem 1, and have the form

\[
    \dot{x} = p \quad \dot{p} = -(\partial_x H) = 1 \quad \dot{u} = D_p H \cdot p - F = \frac{1}{2} p^2 + x.
\]

Notice that the system without the variable \( z(s) \) is closed.
The initial conditions are given by
\[ p(0) = 1 \]
\[ x(0) = x_0 \]
\[ u(0) = g(x_0) = x_0. \]

Then, we obtain
\[ p_1(t) = 1 + t \]
\[ x(t) = \frac{1}{2} t^2 + t + x_0. \]

Thus,
\[ \dot{u}(t) = \frac{1}{2} (1 + t)^2 + x_0 + \frac{1}{2} t^2 = t^2 + t + \frac{1}{2} + x_0 \]
so that
\[ u(t) = \frac{1}{3} t^3 + \frac{1}{2} t^2 + x_0 t + x_0. \]

Tracing the projected characteristic to the initial point, we have \( x_0 = x - t - \frac{1}{2} t^2 \), and therefore get
\[ u(x, t) = -\frac{1}{6} t^3 - \frac{1}{2} t^2 - \frac{1}{2} t + tx + x. \]