Homework Set #1
Due: Friday, November 12 by midnight, 2010.

You can submit this homework either in class on Thursday, Nov 11th or by email to me or push it under my door in ENS 439A.

1. (Channel with memory) Consider the following communication channel

\[ Y_k = \sum_{i=0}^{l} H_i X_{k-i} + Z_k \]

where \( H_i, i \in \{0, \ldots, l\} \) are fixed non-zero elements of GF\( (q) \), and all operations are within the field GF\( (q) \). \( Z_k \) is additive independent noise with a p.m.f. \( P_Z \) over \( \mathbb{Z}_q \). The output at instant \( k \) is thus linearly related with the input.

(a) Show that a rate \( R \) arbitrarily close to \( \log q - H(Z) \) is achievable over this channel. Is there a linear code that achieves this rate?

(b) (Optional, need not turn this in) Show that \( \log q - H(Z) \) is the capacity of this channel.

2. (Strong Converse) In class we showed that the average probability of error in decoding for any channel is lower bounded as:

\[ P_e \geq 1 - \frac{4A}{n(R-C)^2} - e^{-n\frac{R-C}{2}} \]

For the binary symmetric channel, show the following result: There exist constants \( K_1, K_2 > 0 \) such that

\[ P_e \geq 1 - K_1 \exp(-K_2 n(R-C)) \]

Hint: Use Chernoff bounds and/or Section 4.6 of Ron Roth.

(Optional) Show this holds true for any DMC.

3. A soccer betting form contains a list of 13 matches. Next to each listed match there are three fill-in boxes which correspond to the following three possible guesses: ”first team wins”, ”second team wins” or ”tied match”. The bettor checks one box for each match. Describe a strategy for filling out the smallest number of forms so that at least one of the forms contains at least 12 correct guesses. How many forms need to be filled out under this strategy?

Hint: A perfect code of length 13 and minimum distance 3 exists over GF\( (3) \).

(a) Verify that this is indeed a perfect code.

(b) The WEF of this code is

\[ A(z) = 1 + 253z^7 + 506z^8 + 1288z^{11} + 1288z^{12} + 506z^{15} + 253z^{16} + z^{23} \]

Find the WEF of the dual code. Comment on the generator matrix of this code. (Optional) show that any code (linear or non-linear) that is [23,12,7] must have \( A(z) \) as its WEF. Thus, this code is unique.