1. Let $G = \langle a, b \mid a^2, b^2, (ab)^4 \rangle$ and $D = \langle x, y \mid x^4, y^2, xyxy \rangle$. Prove that $G$ is isomorphic to $D$.

Let $F$ be the free group on generators $a, b$ and define $\phi : F \to D$ by $\phi(a) = y, \phi(b) = xy$ (there are other choices). There is a unique homomorphism $\phi : F \to D$ with these properties. Now

$$
\phi(a^2) = y^2 = e, \phi(b^2) = (xy)^2 = e, \phi(((ab)^4) = (yx)^4 = (x^{-1})^4 = (x^4)^{-1} = e
$$

so ker $\phi$ contains the relations defining $G$ and by the universal property of presentations $\phi$ induces a homomorphism $\overline{\phi} : G \to D$.

Similarly we can define $\overline{\psi} : D \to G$ by first defining $\psi$ on on the free group on generators $x, y$ by $\psi(x) = ba, \psi(y) = a$ and showing that it induces $\overline{\psi} : D \to G$.

Finally $\overline{\psi}(\overline{\phi}(a)) = \overline{\psi}(y) = a$ and $\overline{\psi}(\overline{\phi}(b) = \overline{\psi}(xy) = baa = b$ so $\overline{\psi} \circ \overline{\phi}$ is the identity on $G$. Likewise $\overline{\phi} \circ \overline{\psi}$ is the identity on $D$ so they are inverses of each other and $G$ is isomorphic to $D$. 
2. Let $G$ be a finite group, $H$ a normal subgroup of $G$, $p$ a prime number and $S$ a $p$-Sylow subgroup of $G$. Show that $H \cap S$ is a $p$-Sylow subgroup of $H$ and that $HS/H$ is a $p$-Sylow subgroup of $G/H$. Give a counterexample to the first conclusion if $H$ is no longer assumed to be normal in $G$.

We know that there exists $g \in G$ such that $H \cap gSg^{-1}$ is a $p$-Sylow subgroup of $H$. As $H$ is a normal subgroup of $G$, $H = gHg^{-1}$ so $H \cap gSg^{-1} = g(H \cap S)g^{-1}$ which is isomorphic to $H \cap S$, hence $H \cap S$ has the correct cardinality to be a $p$-Sylow subgroup of $H$ and since it is a subgroup of $H$, it has to be a $p$-Sylow subgroup of $H$.

By the isomorphism theorem $HS/H$ is isomorphic to $S/H \cap S$, so its cardinality is the correct cardinality to be a $p$-Sylow subgroup of $G/H$ (since $|G/H| = |G|/|H|$, the highest power of $p$ dividing $|G/H|$ is the quotient of the highest powers of $p$ dividing $|G|$ and $|H|$) since $S$ is a $p$-Sylow subgroup of $G$ and $H \cap S$ is a $p$-Sylow subgroup of $H$. As $HS/H$ is a subgroup of $G/H$ we get the conclusion.

There are many examples. The simplest is $G = S_3, H = \langle (12) \rangle, S = \langle (13) \rangle$. $S$ is a 2-Sylow subgroup of $G$ and $H$ is the 2-Sylow subgroup of $H$ but $H \cap S = \{e\}$. 
3. Let $S$ be a $p$-Sylow subgroup of $GL_3(\mathbb{Z}/p)$. Show that $S$ is a solvable non-abelian group of order $p^3$ and, if $p \neq 2$, that every element of $S$ has order dividing $p$.

We know that $GL_3(\mathbb{Z}/p)$ has $(p^3 - 1)(p^3 - p)(p^3 - p^2)$ elements so the highest power of $p$ dividing this number is $p^3$ and so a $p$-Sylow subgroup of $GL_3(\mathbb{Z}/p)$ has order $p^3$. We also know that $p$-groups are solvable so a $p$-Sylow subgroup of $GL_3(\mathbb{Z}/p)$ is solvable.

The group $U$ consisting of the matrices

$$
\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{pmatrix}
$$

with $a, b, c$ running through $\mathbb{Z}/p$ is a subgroup of $GL_3(\mathbb{Z}/p)$ of order $p^3$. By Sylow’s theorems any $p$-Sylow subgroup of $GL_3(\mathbb{Z}/p)$ is conjugate, in particular isomorphic, to $U$. So, to finish the proof we need to show that $U$ is non-abelian and all its elements have order dividing $p$.

Let, for example,

$$
g = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, h = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
$$

Then

$$
gh = \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}, hg = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
$$

so $gh \neq hg$ and $U$ is non-abelian. Also, by induction it is easy to check that

$$
\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{pmatrix}^n = \begin{pmatrix}
1 & na & nb + n(n-1)ac/2 \\
0 & 1 & nc \\
0 & 0 & 1
\end{pmatrix}
$$

and, when $n = p > 0$, we get $na = nb = nc = n(n-1)ac/2 = 0$ for all $a, b, c \in \mathbb{Z}/p$, so $g^p = 1$ for all $g \in U$. (A fancier proof can be given using that the characteristic polynomial of any element of $U$ is $(x - 1)^3$ and applying the Cayley-Hamilton theorem to get $(g - 1)^3 = 0, g \in U$ and conclude that $g^p - 1 = (g - 1)^p = 0, p > 2.$)

3