M 380 C 54580 First Midterm

Name:

Do three out of the four questions below and please indicate here which questions you chose:

1. Let $p, q$ be distinct primes and $G$ a group of order $p^2q$. Prove that $G$ has either a normal $p$-Sylow subgroup or a normal $q$-Sylow subgroup.

Let $n_p$ be the number of $p$-Sylow subgroups of $G$. Then we know, by Sylow’s theorems that $n_p \equiv 1 \pmod{p}$ and $n_p | q$. If $n_p = 1$, we are done, since the unique $p$-Sylow subgroup is normal, by Sylow’s theorems. Otherwise, $n_p = q \equiv 1 \pmod{p}$, which implies $q > p$. We also know, by Sylow’s theorems that $n_q \equiv 1 \pmod{q}$ and $n_q | p^2$. Again we are done if $n_q = 1$. We cannot have $n_q = p \equiv 1 \pmod{q}$, since $q > p$. The only other possibility is then $n_q = p^2$. The $q$-Sylow subgroups have order $q$ prime so they intersect only at the identity (since the intersection is a subgroup) and every non-identity element of such a subgroup has order $q$, therefore $G$ has $p^2(q-1)$ elements of order $q$. The other elements of $G$ number $p^2q - p^2(q-1) = p^2$. A $p$-Sylow subgroup of $G$ has $p^2$ elements and does not have any element of order $q$, since $q$ does not divide $p^2$, so it must consist of these remaining $p^2$ elements of $G$ and is thus unique, hence normal, by Sylow’s theorems.
2. Show in detail that a finite abelian group is solvable, that is, has a composition series whose successive quotients are cyclic of prime order.

By the Jordan-Hölder theorem, any finite group has a composition series whose successive quotients are simple. Such a quotient $G/N$ is abelian since $G$ is a subgroup of our group, so abelian and $NaNb = Nab = Nba = NbNa$ for any $a, b \in N$, that is $G/N$ is abelian. So we need to show that a simple finite abelian group $A$ is cyclic of prime order. Let $x \in A, x \neq 1$, then $\langle x \rangle$ is a normal subgroup of $A$, so $\langle x \rangle = A$. Let $n$ be the order of $x$. If $n$ is not prime $n = ab, a > 1, b > 1$ and $\langle x^a \rangle$ is a non-trivial proper subgroup of $A$, contradiction, so $n$ is prime and we are done.
3. Let $G$ be a finite group acting on a finite set $S$ and assume that the action is transitive, that is, has only one orbit. Let $H$ be a normal subgroup of $G$ and consider $H$ acting on $S$ by restriction of the action of $G$. Prove that all orbits of $H$ have the same cardinality. Prove also that, for $s \in S$, the number of orbits for the action of $H$ is equal to $|G|/|HG_s|$, where $G_s$ is the stabilizer of $s$ in $G$.

If $s, t \in S$ there exists $g$ in $G$ with $t = gs$, by hypothesis. If $O_s, O_t$ are the $H$-orbits of $s$ and $t$ respectively, I claim that $x \mapsto gx$ is a bijection between $O_s$ and $O_t$. Indeed if $hs \in O_s$ then $ghg^{-1} = h' \in H$, since $H$ is normal in $G$, so $ghs = h'gs = h't \in O_t$, so $x \mapsto gx$ maps $O_s$ to $O_t$. By the same argument, $x \mapsto g^{-1}x$ maps $O_t$ to $O_s$. Clearly, these two maps are inverses of each other and and this establishes the bijection.

$S$ is the unique orbit of $G$ so its cardinality is $|S| = |G|/|G_s|$. By the first part, each $H$-orbit has cardinality $|H|/|H_s|$, so the number of orbits is $|G||H_s|/|H||G_s|$. Now, $H_s = \{h \in H \mid hs = s\} = H \cap G_s$ and, since $H$ is normal in $G$ we can apply the diamond isomorphism theorem to conclude that $HG_s/H$ is isomorphic to $G_s/H_s$. In particular, $|HG_s| = |H||G_s|/|H_s|$ and the result now follows from the previous expression for the number of orbits.

For an example that shows that the hypothesis that $H$ is normal in $G$ is necessary, take $G = S_3$ acting on $\{1, 2, 3\}$ as usual, and $H = \{1, (23)\}$, the stabilizer of 1. Then the orbits of $H$ are $\{1\}, \{2, 3\}$ and they don’t have the same cardinality.
4. Show that $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ has $p + 1$ subgroups of order $p$, when $p$ is prime. Show that the group of automorphisms of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is isomorphic to $S_3$ by considering its action on the subgroups of order 2.

Every element of $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, except the identity, has order $p$, so it generates a subgroup of order $p$ of $G$. Two distinct such subgroup meet only in the identity since $p$ is prime (since the intersection is a subgroup). Thus the non-zero elements of $G$ are partitioned into the sets of non-zero elements of the subgroups of order $p$ and each such set has $p - 1$ elements. As $G$ has $p^2$ elements we conclude that there are $(p^2 - 1)/(p - 1) = p + 1$ such subgroups.

By the above $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has three subgroups of order 2. If $\phi$ is an automorphism of $G$ and $H$ is a subgroup of order 2, then so is $\phi(H)$, since $\phi$ is injective and is a homomorphism. Thus $\phi$ induces a permutation of these subgroups of order 2. We thus get a homomorphism $\lambda : \text{Aut}(G) \to S_3$. If $\phi$ acts trivially on the subgroups, then $\phi(\{e, x\}) = \{e, x\}, x \in G, x \neq e$, so $\phi(x) = x$ for all such $x$ since $\phi(e) = e$ always, thus $\phi$ is the identity. Therefore $\lambda$ is injective. Now $\text{Aut}(G)$ has 6 elements, as can be seen directly, e.g., by noticing that $\text{Aut}(G) = \text{GL}_2(\mathbb{Z}/2\mathbb{Z})$, and $S_3$ also has 6 elements so $\lambda$ is a bijection and thus is an isomorphism.