1. Let \( p, q \) be distinct primes and \( G \) a group of order \( p^2q \). Prove that \( G \) has either a normal \( p \)-Sylow subgroup or a normal \( q \)-Sylow subgroup.

Let \( n_p \) be the number of \( p \)-Sylow subgroups of \( G \). Then we know, by Sylow’s theorems that \( n_p \equiv 1 \pmod{p} \) and \( n_p \mid q \). If \( n_p = 1 \), we are done, since the unique \( p \)-Sylow subgroup is normal, by Sylow’s theorems. Otherwise, \( n_p = q \equiv 1 \pmod{p} \), which implies \( q > p \). We also know, by Sylow’s theorems that \( n_q \equiv 1 \pmod{q} \) and \( n_q \mid p^2 \). Again we are done if \( n_q = 1 \). We cannot have \( n_q = p \equiv 1 \pmod{q} \), since \( q > p \). The only other possibility is then \( n_q = p^2 \). The \( q \)-Sylow subgroups have order \( q \) prime so they intersect only at the identity (since the intersection is a subgroup) and every non-identity element of such a subgroup has order \( q \), therefore \( G \) has \( p^2(q-1) \) elements of order \( q \). The other elements of \( G \) number \( p^2q - p^2(q-1) = p^2 \). A \( p \)-Sylow subgroup of \( G \) has \( p^2 \) elements and does not have any element of order \( q \), since \( q \) does not divide \( p^2 \), so it must consist of these remaining \( p^2 \) elements of \( G \) and is thus unique, hence normal, by Sylow’s theorems.
2. Show in detail that a finite abelian group is solvable, that is, has a composition series whose successive quotients are cyclic of prime order.

By the Jordan-Hölder theorem, any finite group has a composition series whose successive quotients are simple. Such a quotient $G/N$ is abelian since $G$ is a subgroup of our group, so abelian and $NaNb = Nab = NbNa$ for any $a, b \in N$, that is $G/N$ is abelian. So we need to show that a simple finite abelian group $A$ is cyclic of prime order. Let $x \in A, x \neq 1$, then $< x >$ is a normal subgroup of $A$, so $< x >/A$. Let $n$ be the order of $x$. If $n$ is not prime $n = ab, a > 1, b > 1$ and $< x^a >$ is a non-trivial proper subgroup of $A$, contradiction, so $n$ is prime and we are done.
3. Let $G$ be a finite group acting on a finite set $S$ and assume that the action is transitive, that is, has only one orbit. Let $H$ be a normal subgroup of $G$ and consider $H$ acting on $S$ by restriction of the action of $G$. Prove that all orbits of $H$ have the same cardinality. Prove also that, for $s \in S$, the number of orbits for the action of $H$ is equal to $|G|/|HG_s|$, where $G_s$ is the stabilizer of $s$ in $G$.

If $s, t \in S$ there exists $g$ in $G$ with $t = gs$, by hypothesis. If $O_s, O_t$ are the $H$-orbits of $s$ and $t$ respectively, I claim that $x \mapsto gx$ is a bijection between $O_s$ and $O_t$. Indeed if $hs \in O_s$ then $ghg^{-1} = h' \in H$, since $H$ is normal in $G$, so $ghs = h'gs = h't \in O_t$, so $x \mapsto gx$ maps $O_s$ to $O_t$. By the same argument, $x \mapsto g^{-1}x$ maps $O_t$ to $O_s$. Clearly, these two maps are inverses of each other and this establishes the bijection.

$S$ is the unique orbit of $G$ so its cardinality is $|S| = |G|/|G_s|$. By the first part, each $H$-orbit has cardinality $|H|/|H_s|$, so the number of orbits is $|G||H_s|/|H||G_s|$. Now, $H_s = \{h \in H \mid hs = s\} = H \cap G_s$ and, since $H$ is normal in $G$ we can apply the diamond isomorphism theorem to conclude that $HG_s/H$ is isomorphic to $G_s/H_s$. In particular, $|HG_s| = |H||G_s|/|H_s|$ and the result now follows from the previous expression for the number of orbits.

For an example that shows that the hypothesis that $H$ is normal in $G$ is necessary, take $G = S_3$ acting on $\{1, 2, 3\}$ as usual, and $H = \{1, (23)\}$, the stabilizer of 1. Then the orbits of $H$ are $\{1\}, \{2, 3\}$ and they don't have the same cardinality.
4. Show that \( \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \) has \( p + 1 \) subgroups of order \( p \), when \( p \) is prime. Show that the group of automorphisms of \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) is isomorphic to \( S_3 \) by considering its action on the subgroups of order 2.

Every element of \( G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \), except the identity, has order \( p \), so it generates a subgroup of order \( p \) of \( G \). Two distinct such subgroup meet only in the identity since \( p \) is prime (since the intersection is a subgroup). Thus the non-zero elements of \( G \) are partitioned into the sets of non-zero elements of the subgroups of order \( p \) and each such set has \( p - 1 \) elements. As \( G \) has \( p^2 \) elements we conclude that there are \( (p^2 - 1)/(p - 1) = p + 1 \) such subgroups.

By the above \( G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) has three subgroups of order 2. If \( \phi \) is an automorphism of \( G \) and \( H \) is a subgroup of order 2, then so is \( \phi(H) \), since \( \phi \) is injective and is a homomorphism. Thus \( \phi \) induces a permutation of these subgroups of order 2. We thus get a homomorphism \( \lambda: Aut(G) \to S_3 \). If \( \phi \) acts trivially on the subgroups, then \( \phi(\{e,x\}) = \{e,x\}, x \in G, x \neq e \), so \( \phi(x) = x \) for all such \( x \) since \( \phi(e) = e \) always, thus \( \phi \) is the identity. Therefore \( \lambda \) is injective. Now \( Aut(G) \) has 6 elements, as can be seen directly, e.g., by noticing that \( Aut(G) = GL_2(\mathbb{Z}/2\mathbb{Z}) \), and \( S_3 \) also has 6 elements so \( \lambda \) is a bijection and thus is an isomorphism.