1. A code $C : \mathbb{F}_2^k \rightarrow \mathbb{F}_2^n$ is $(r,\delta,\epsilon)$-locally decodable if there exists a randomized decoding algorithm $A$ that, on input a message coordinate $i \in [k]$ and a received word $y$ with $\Delta(y, C(x)) \leq \delta n$:

- $A$ queries at most $r$ bits of $y$;
- $A$ outputs a guess $z$ for $x_i$ such that $\Pr[z = x_i] \geq 1 - \epsilon$.

Fix $m$ and let $C_r$ be the subcode of the binary Reed-Muller code $\text{RM}(r,m)$ where the messages are restricted to be homogenous polynomials of degree $r$. Show that for any $\delta > 0$, $C_r$ is $(2^r,\delta,2^r\delta)$-locally decodable. (Hint: do it for $r = 1$ first.)

2. Let $C_1$ and $C_2$ be $[n_1,k_1,d_1]$ and $[n_2,k_2,d_2]$ binary linear codes, respectively. Define $C \subseteq \mathbb{F}_2^{n_1 \times n_2}$ to be the subset of $n_1 \times n_2$ matrices whose columns belong to $C_1$ and whose rows belong to $C_2$.

(a) If $E_1 : \mathbb{F}_2^{k_1} \rightarrow \mathbb{F}_2^{n_1}$ is an efficient encoder for $C_1$ and $E_2 : \mathbb{F}_2^{k_2} \rightarrow \mathbb{F}_2^{n_2}$ is an efficient encoder for $C_2$, describe an efficient encoder for $C$ which takes as input a $k_1 \times k_2$ matrix as the message.

(b) Prove that $C$ is an $[n_1n_2,k_1k_2,d_1d_2]$ binary linear code. Compare to concatenated codes.

3. Show that there exists constants $c,\delta > 0$ such that the following game has an efficient (polynomial time) solution. Alice has input $x \in \{0,1\}^n$ and Bob has input $y \in \{0,1\}^n$. In addition, Alice and Bob have access to the same random string $r \in \{0,1\}^\ell$, where $\ell = \lceil \log_2 n \rceil + c$. Alice sends one bit $b = b(x,r)$ to Bob, and Bob must accept or reject based on $y, r$, and $b$. The goal is that if $x = y$, Bob always accepts, and if $x \neq y$, Bob rejects with probability at least $\delta$. Here the probability is over the choice of the random string $r$. 


4. A vector of random variables is called \( k \)-wise independent if any \( k \) coordinates are independent.

   (a) Show that a random vector from an \([n, k]_q\) Reed Solomon code is \( k \)-wise independent.

   (b) Show that a random vector from a linear code \( C \) is \( k \)-wise independent if and only if \( C^\perp \) has minimum distance bigger than \( k \).

5. A group of \( n \) players first agree on a strategy and then play the following game. A hat is placed on each player’s head that is red or blue, each with probability half and independent of the other hats. Each player can see the hats of all other players, but not his own. Next, each player is given the option to guess the color of his hat; he may either make no guess or guess “red” or “blue.” When making his guess, a player may not see other players’ guesses. The players win if at least one player guesses and all guesses are correct.

Show that for infinitely many values of \( n \), there is a strategy for the players which achieves success probability \( 1 - 1/(n + 1) \).

Hint: Think about the \( n \)-dimensional hypercube. It’s helpful to first show how to achieve probability \( 3/4 \) with \( n = 3 \).