A Few Basic Concepts

Definition 1. A projective plane is defined as a set of elements, called points, together with distinguished sets of points, called lines, as well as a relation between points and lines subject to the following conditions:
1. Every pair of distinct lines is incident with a unique point.
2. Every pair of distinct points is incident with a unique line. (i.e., to every pair of distinct points there is exactly one line which contains both points);
3. There exists four points such that no three of them are incident with a single line.

Remark. If \( \Pi \) is a finite projective plane then:
1. There exist \( m \geq 2 \) such that every point (line) is incident with exactly \( m + 1 \) lines (points) of \( \Pi \).
2. \( \Pi \) consists of exactly \( m^2 + m + 1 \) points (lines).
\( m \) is called the order of the finite projective plane. For every prime power \( q = p^n \), \( p \) prime, there exists a finite projective plane of order \( q \).

Definition 2. An oval in a projective plane of order \( q \) is a set of points which meets every line in at most two points, and has a unique tangent (a line meeting it in one point only) at each of its points. The number points is \( q + 1 \). (For, if \( P \) is a point of the oval, then the \( q \) non-tangent lines through \( P \) each contain one farther point of the oval.)

Definition 3. In a projective plane, a conic is the set of zeros of a non-singular quadratic form.
Any conic is an oval. If \( q \) is odd then the converse statement is a celebrated theorem of Segre.
The Lemma

**Lemma 1.** Every inscribed triangle and its mate are in perspectivity

*Proof.* We may take the inscribed triangle as the reference for a homogenous coordinisation of $\Pi$. Thus the vertices of the triangles are:

$$A_1 = (1, 0, 0), A_2 = (0, 1, 0), A_3 = (0, 0, 1).$$

Let $a_1$, $a_2$, $a_3$ be tangents to the oval at $A_1$, $A_2$, $A_3$ respectively. Hence, the equations of the tangents are of the form

$$a_1 : x_2 = k_1 x_3, \quad a_2 : x_3 = k_2 x_1, \quad a_3 : x_1 = k_3 x_2,$$

where $k_1, k_2, k_3$ are non-zero elements of $F$ (the finite field of order $q$).

Let $B = (c_1, c_2, c_3)$ be any of the $q-2$ points of the oval distinct from $A_1, A_2, A_3$. Then $c_1 c_2 c_3 \neq 0$ because the lines of the triangle already intersect the oval at two points; namely, the vertices of the triangle. Now $A_1 B, A_2 B, A_3 B$ have respective equations

$$x_2 = h_1 x_3, \quad x_3 = h_2 x_1, \quad x_1 = h_3 x_2,$$

where $h_1 = c_2 c_3^{-1}, h_2 = c_3 c_1^{-1}, h_3 = c_1 c_2^{-1}$ and $i \neq k_i, i = 1, 2, 3$. They clearly satisfy

$$h_1 h_2 h_3 = 1.$$

Let $h_1$ be any non-zero element of $F - \{k_1\}$. Now consider the line $x_2 = h_1 x_3$. Since this line is not the tangent at $A_1$ it meets the oval at $A_1$ and another point $B$ (say). So for each $h_1 \in F - \{k_1\}$ we get a line through $A_1$ and some point in the oval. Now for different values of $h_1$ we get different points of the oval because any line intersects the oval at at most two points. So there exists an injection between non-tangent lines through $A_1$ and the points of the oval other than $A_1$. Since both the sets have cardinality $q$, we have a bijection,

$$f_1 : L_{A_1} - \{a_1\} \rightarrow O - \{A_1\}.$$

Similarly we have bijections

$$f_2 : L_{A_2} - \{a_2\} \rightarrow O - \{A_2\},$$

$$f_3 : L_{A_3} - \{a_3\} \rightarrow O - \{A_3\}.$$

So composing these we get bijections,

$$h_2 : L_{A_1} - \{a_1, A_1 A_2, A_1 A_3\} \rightarrow L_{A_2} - \{a_2, A_2 A_1, A_2 A_3\},$$

$$h_3 : L_{A_1} - \{a_1, A_1 A_2, A_1 A_3\} \rightarrow L_{A_3} - \{a_3, A_3 A_2, A_3 A_1\}.$$

So $x_2 = h_1 x_3$ uniquely determines the lines $x_3 = h_2 x_1, x_1 = h_3 x_2,$ and since they all meet at the point $B$ we have

$$h_1 h_2 h_3 = 1.$$

Vary $h_1$ over $F - \{k_1\}$ to get unique $h_2, h_3$ for each value. Further, they satisfy the relationship above. Now multiply these $q-2$ relations to obtain

$$\prod_{h_1 \in F - \{k_1, 0\}} h_1 h_2(h_1) h_3(h_1) = 1.$$

Now, finally we shall use the fact that $q$ is odd. We know that all the non-zero elements of $F$ satisfy $x^{q-1} - 1 = 0$. Because $q$ is odd the product of the roots is $-1$. Let $P$ denote the product of the roots. Then $P^3 = -1$. But relation (5) yields that
(6) \[ k_1 k_2 k_3 = -1 \]

Using equation (6) and the equations of the lines \( A_1 P, A_2 Q \) and \( A_3 R \) (refer to the fig 2. The point B should read K) we can conclude that they intersect at \( K = < 1, k_1 k_2, -k_2 > \) and the lemma is proved.

\( \square \)
A Few Calculations

Without loss of generality assume that K is the point $<(1,1,1)>$ which implies that

$$k_1 = k_2 = k_3 = -1.$$  

If $B =< (c_1,c_2,c_3)>$ is any of the q-2 points of the oval distinct from $A_1,A_2,A_3$. Let $b$ denote, $b_1 x_1 + b_2 x_2 + b_3 x_3 = 0$ the tangent at $B$. Since $B$ lies on $b$ and $A_1,A_2,A_3$ do not lie on $b$ we have

$$(7)\quad b_1 c_1 + b_2 c_2 + b_3 c_3 = 0 \quad \text{and} \quad b_i \neq 0 \quad \text{for} \quad i = 1,2,3.$$  

The direction ratio for PB: $(-c_2 + c_3, c_1 + c_3, -c_1 - c_2)$

The equation of this line is:

$$(-c_2 + c_3)x_1 + (c_1 + c_3)x_2 + (-c_1 - c_2)x_3 = 0.$$  

The direction ratio for RA3: $(b_3 - b_1, -b_2, 0)$

The equation of this line is:

$$(b_3 - b_1)x_1 - b_2 x_2 = 0.$$  

The direction ratio for QA2: $(b_1 - b_2, 0, b_3)$

The equation of this line is:

$$(b_1 - b_2)x_1 + b_3 x_3 = 0.$$  

(refer to the fig 1. Disregard the point K)

Because $\triangle B A_2 A_3$ and $\triangle PQR$ are in perspective (by the lemma) and since we are looking for a non-trivial solution the determinant below must equal 0.

$$\begin{vmatrix} 
-c_2 + c_3 & c_1 + c_3 & -c_1 - c_2 \\
b_3 - b_1 & -b_2 & 0 \\
b_1 - b_2 & 0 & b_3 
\end{vmatrix} = 0,$$

$$\Rightarrow (b_1 - b_2 - b_3)((c_1 + c_3)b_3 - (c_1 + c_2)b_2) = 0.$$  

But since $P$ does not lie on $b$ we have, $b_1 - b_2 - b_3 \neq 0$

$$(8)\quad \Rightarrow (c_1 + c_3)b_3 = (c_1 + c_2)b_2.$$  

A similar consideration for $\triangle B A_3 A_1,\triangle B A_2 A_1$ and their mates we have

$$(9)\quad (c_2 + c_3)b_3 = (c_1 + c_2)b_1 \quad \text{and} \quad (c_1 + c_3)b_1 = (c_3 + c_2)b_2.$$  

But these equations imply that

$$(10)\quad b_1 : b_2 : b_3 = (c_2 + c_3) : (c_1 + c_3) : (c_1 + c_2).$$

Substituting in equation (7) we have

$$(11)\quad 2(c_1 c_2 + c_2 c_3 + c_3 c_2) = 0,$$

which since $q$ is odd, gives

$$(12)\quad c_1 c_2 + c_2 c_3 + c_3 c_2 = 0$$

Since each of the $q-2$ points of the oval together with $A_1,A_2,A_3$ satisfy the equation of the conic

$$x_1 x_2 + x_2 x_3 + x_3 x_2 = 0$$

and since the conic has exactly $q + 1$ points we are done.