1. Let \( R \) be a domain and \( I \) and \( J \) ideals of \( R \). Prove that \( I \) and \( J \) are isomorphic as \( R \)-modules if and only if there exists \( a, b \in R \) such that \( aI = bJ \).

First, a correction to the statement. We need to assume \( a, b \) are non-zero.

Now, if \( aI = bJ \), given \( x \in I \) we have \( ax \in aI = bJ \) so there exists \( y \in J, ax = by \). Since \( R \) is a domain, \( y \) is uniquely determined by \( x \) and so we can let \( \phi(x) = y \) and this defines \( \phi : I \to J \). It is easy to show that \( \phi(x) \) is an \( R \)-module homomorphism. E.g.

\[
b\phi(x_1 + x_2) = a(x_1 + x_2) = ax_1 + ax_2 = b\phi(x_1) + b\phi(x_2)
\]

and since \( R \) is a domain, \( \phi(x_1 + x_2) = \phi(x_1) + \phi(x_2) \) and so on. If \( ax = b0 = 0 \) then \( x = 0 \) so \( \phi \) is injective and the same construction with \( a, b \) reversed gives an inverse for \( \phi \) so it is also surjective.

Now suppose there exists a \( R \)-module isomorphism \( \phi : I \to J \). As the result is obvious if \( I = (0) \), we may assume there exists \( b \neq 0, b \in I \) and we let \( a = \phi(b) \in J, a \neq 0 \). Now let’s prove that \( aI = bJ \). Given \( x \in I \) we have \( ax = \phi(b)x = x\phi(b) = \phi(bx) = b\phi(x) \) using that \( \phi(x) \) is an \( R \)-module homomorphism and that \( x, b \in I \subset R \). It follows that \( aI \subset bJ \), since \( \phi(x) \in J \). Using the inverse of \( \phi(x) \) in the same manner, we get \( bJ \subset aI \) and we are done.