PRELIMINARY EXAMINATION:
APPLIED MATHEMATICS — Part I

January 15, 2016, 1:00-2:30

Work all 3 of the following 3 problems.

1. Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be closed and bounded (i.e., compact). Let $T : C(A) \to C(B)$ be a linear map taking continuous functions on A to continuous functions on B. Suppose that T is positive ($T \geq 0$) in the sense that $Tf(y) \geq 0$ for all $y \in B$ whenever $f(x) \geq 0$ for all $x \in A$.

(**a**) Prove that the map T is continuous and that $\|T\| = \|T(1)\|_{L^\infty(B)}$.

(**b**) Let $T_n : C(A) \to C(B)$ be an increasing family of maps ($T_{n+1} - T_n \geq 0$ for all n). Prove that T_n converges in the operator norm if and only if $T_n(1)$ converges in the norm of $C(B)$.

2. Let X be an NLS.

(**a**) State what it means for a sequence $\{x_n\}_{n=1}^{\infty}$ in X to converge weakly to x, and show that in this case, $\|x_n\|$ is uniformly bounded.

(**b**) Define the weak topology on X and describe a base for the weak topology at 0.

(**c**) Let H be a separable Hilbert space and $T : H \to H$ a bounded linear operator. Suppose that $f \in H$ and there is a sequence $f_n \in H$ such that $f_n \xrightarrow{w} f$ and there is a solution x_n so that $Tx_n = f_n$ for all n. Suppose that x_n is bounded, and prove that there is $x \in H$ such that $Tx = f$.

3. Let H be a Hilbert space.

(**a**) If M is a nonempty subset of H, show that the span of M is dense in H if and only if $M^\perp = \{0\}$.

(**b**) Let $T : H \to H$ be a bounded linear operator. Let $N = N(T)$ be the null space of T and $R(T)$ be the range or image of T. Let $P : H \to N$ be orthogonal projection onto N. Show that $S = T \circ P^\perp$ is a one-to-one mapping when restricted to $N^\perp$ and that $R(S) = R(T)$.