PRELIMINARY EXAMINATION IN ANALYSIS

Part I, Real Analysis

August 15, 2016

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1. Prove that, on a finite measure space, if $f_k \to f$ in measure and $g_k \to g$ in measure, then $f_k g_k \to fg$ in measure.

2. Let f be a locally integrable function on $\mathbb{R}^2$. Assume that, for any given real numbers a and b outside some set of measure zero, $f(x,a) = f(x,b)$ for almost every $x \in \mathbb{R}$ and $f(a,y) = f(b,y)$ for almost every $y \in \mathbb{R}$. Show that f is constant almost everywhere on $\mathbb{R}^2$.

3. Let $f, f_1, f_2, \ldots \in \mathrm{L}^p$ with $1 \le p < \infty$. If $f_k \to f$ pointwise a.e. and $\|f_k\|_p \to \|f\|_p$, show that $\|f - f_k\|_p \to 0$.

4. For a function $f \in \mathrm{L}^1(\mathbb{R}^2)$ let $\widetilde{M}f$ be the unrestricted maximal function

$$\widetilde{M}f(x_0, y_0) = \sup_Q \int_Q \big|f(x,y)\big|\, dxdy\,,$$

where the supremum is over all $Q = [x_0 - k, x_0 + k] \times [y_0 - l, y_0 + l]$ with $k, l > 0$.

(*a*) Show that $\widetilde{M}f(x_0, y_0) \le M_1 M_2 f(x_0, y_0)$, where

$$M_1 f(x_0, y) = \sup_{k>0} \frac{1}{2k} \int\limits_{x_0-k}^{x_0+k} \big|f(x,y)\big|\, dx\,, \quad M_2 f(x, y_0) = \sup_{l>0} \frac{1}{2l} \int\limits_{y_0-l}^{y_0+l} \big|f(x,y)\big|\, dy\,.$$

(*b*) Show that there exists $C > 0$ such that if $f \in \mathrm{L}^2(\mathbb{R}^2)$ then

$$\big\|\widetilde{M}f\big\|_{\mathrm{L}^2(\mathbb{R}^2)} \le C \|f\|_{\mathrm{L}^2(\mathbb{R}^2)}\,.$$