1. Let $X$ be a normed linear space and $X^*$ its dual space. Let $\{f_n\}_{n=1}^{\infty} \subset X^*$ and $f \in X^*$.
   (a) Define what it means for $f_n$ to converge weak-* to $f$.
   (b) Prove that if $f_n$ converges weak-* to $f$, then $f$ is unique.
   (c) State the Uniform Boundedness Principle.
   (d) Suppose that $X$ is a Banach space. Prove that $\{\|f_n\|_{X^*}\}_{n=1}^{\infty}$ is bounded.

2. Let $X$ be a normed linear space and $Y$ a finite dimensional subspace of $X$. For $x \in X$ and $S \subset X$, let
   \[ d(x, S) = \inf_{z \in S} \|x - z\| \]
   denote the distance from $x$ to $S$. Fix $x_0 \in X$ and let $B = \{y \in Y : \|y\| \leq 3\|x_0\|\}$.
   (a) Show that $d(x_0, Y) = d(x_0, B)$.
      [Hint: first show that $\|x_0\| \geq d(x_0, B) \geq d(x_0, Y)$.]
   (b) Show that there is some $y_0 \in B \subset Y$ such that
       \[ d(x_0, y_0) = d(x_0, Y). \]
       We say that $y_0$ is a best approximation to $x_0$. [Hint: why is $B$ compact?]
   (c) Show by example that a best approximation may not be unique. [Hint: try $X = (\mathbb{R}^2, \| \cdot \|_1)$ and $Y = \text{span}(1, 1)$.]

3. Let $X$ be a Banach space, $S, T \in B(X, X)$, and $I$ be the identity map on $X$. Suppose further that $T$ is compact.
   (a) Prove that $TS$ and $ST$ are compact.
   (b) Describe the spectrum of a compact operator.
   (c) If $S$ is invertible and $S + T$ is injective, show that $S + T$ is invertible on all of $X$. 