Please provide as-complete-as-possible proofs for all of the following 4 problems.

(1) Let $1 < p < \infty$ and $f_n \in L^p([0,1], \lambda)$ ($n = 1, 2, \ldots$) and $f \in L^p([0,1], \lambda)$ (where $\lambda$ is Lebesgue measure). Let $M > 0$ be a constant and suppose
(a) $f_n$ converges to $f$ pointwise a.e.;
(b) $\| f_n \|_p \leq M$ for all $n$.
Prove that for every $g \in L^q([0,1], \lambda)$ ($\frac{1}{p} + \frac{1}{q} = 1$),
$$\lim_{n \to \infty} \int f_n g \, d\lambda = \int fg \, d\lambda.$$ Warning: do not assume $f_n$ converges to $f$ in $L^p([0,1], \lambda)$.

(2) Let $\nu$ be a finite Borel measure on $[0,1]$. Define $f : [0,1] \to \mathbb{R}$ by
$$f(x) = \nu([0,x)).$$
Prove $\nu$ is absolutely continuous to Lebesgue measure if and only if $f$ is absolutely continuous.

(3) Given a function $\phi : \mathbb{R} \to [0,\infty)$ and $\epsilon > 0$, define $\phi_\epsilon : \mathbb{R} \to \mathbb{R}$ by $\phi_\epsilon(x) = \epsilon^{-1} \phi(x/\epsilon)$. Let $f \in L^\infty(\mathbb{R}, \lambda)$ (where $\lambda$ is Lebesgue measure).
(a) Fix $0 < a < 1$ and let $\psi = (1/2a) 1_{[-a,a]}$ (where $1_{[-a,a]}$ is the characteristic function of $[-a,a]$). Explain why the convolution $f * \psi_\epsilon$ converges to $f$ pointwise a.e. as $\epsilon \searrow 0$.
(b) Suppose $\phi : \mathbb{R} \to [0,\infty)$ be a continuous function which is even (so $\phi(x) = \phi(-x)$ for all $x$), has $\phi(x) = 0$ for all $|x| \geq 1$ and has $\| \phi \|_1 = 1$.
Prove $f * \phi_\epsilon \to f$ pointwise a.e. as $\epsilon \searrow 0$.

(4) Let $\mu_n$ be a sequence of Borel probability measures on $[0,1]$ which converges to a probability measure $\mu$ in the weak* topology. This means: for every continuous function $f$ on $[0,1]$, $\int f \, d\mu = \lim_{n \to \infty} \int f \, d\mu_n$.
Let $C \subset [0,1]$ be closed. Prove $\limsup_{n \to \infty} \mu_n(C) \leq \mu(C)$. 