Problem 1. Let $\Omega = [0, 1)$, $\mathcal{F} = \mathcal{B}[0, 1)$, and $\mathbb{P} = \lambda$, where $\lambda$ denotes the Lebesgue measure on $[0, 1)$. For $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2^n - 1\}$, we define

$$I_{k,n} = [k2^{-n}, (k + 1)2^{-n}), \quad \mathcal{F}_n = \sigma(I_{0,n}, I_{1,n}, \ldots, I_{2^n-1,n}).$$

In words, $\mathcal{F}_n$ is generated by the $n$-th dyadic partition of $[0, 1)$. For $x \in [0, 1)$, let $k_n(x)$ be the (ordinal) number of the partition element which contains $x$, i.e., unique number in $\{0, 1, \ldots, 2^n - 1\}$ such that $x \in I_{k_n(x),n}$. For a function $f : [0, 1) \to \mathbb{R}$ we define the process $\{X^f_n\}_{n \in \mathbb{N}_0}$ by

$$X^f_n(x) = 2^n \left( f((k_n(x) + 1)2^{-n}) - f(k_n(x)2^{-n}) \right), \quad x \in [0, 1).$$

(1) Show that $\{X^f_n\}_{n \in \mathbb{N}_0}$ is a martingale.

(2) Assume that the function $f$ is Lipschitz, i.e., that there exists $K > 0$ such that $|f(y) - f(x)| \leq K|y - x|$, for all $x, y \in [0, 1)$. Show that the limit $X^f = \lim_n X^f_n$ exists a.s.

(3) Show that, for $f$ Lipschitz, $X^f$ has the property that

$$f(y) - f(x) = \int_x^y X^f(\xi) \, d\xi, \quad \text{for all } 0 \leq x < y < 1.$$  

(Note: This problem gives an alternative proof of the fact that Lipschitz functions are absolutely continuous.)

Problem 2. Let $\{(B^1_t, B^2_t)\}_{t \in [0,T]}$ be a two-dimensional Brownian motion. Given $(a_1, a_2, b) \in \mathbb{R}^3$ find the distribution of the random time $T$ given by

$$T = \inf\{t \geq 0 : a_1 B^1_t + a_2 B^2_t = b\}$$

(Note: Any of the following will be accepted: pdf, cdf, Laplace transform, characteristic function.)

Problem 3. Let $\{B_t\}_{t \in [0,1]}$ be a Brownian motion on $[0,1]$ and let $\{H_t\}_{t \in [0,1]}$ be a deterministic process with $\int_0^1 H_u^2 \, du < \infty$. Show that the random variable $\int_0^1 H_u \, dB_u$ is normally distributed.

For extra credit give an example of a non-deterministic, progressive, and $B$-integrable process $H$ such that $\int_0^1 H_u \, dB_u$ is normal.