

**PRELIMINARY EXAMINATION:  
APPLIED MATHEMATICS — Part I**

Wednesday, August 21, 2024, 11:30am-1:30pm

*Work all 3 of the following 3 problems.*

1. Let  $X$  and  $Y$  be Banach spaces and  $D$  a linear subspace of  $X$ .
  - (a) State what it means for an operator  $T : D \rightarrow Y$  to be closed.
  - (b) State the Closed Graph Theorem.
  - (c) Let  $T : D \rightarrow Y$  be a closed linear operator. Show that  $T$  is bounded on  $D$  if and only if  $D$  is a closed subspace of  $X$ .
  
2. Let  $X$  be a Banach space,  $Y$  a normed linear space, and  $T_n \in B(X, Y)$  for  $n = 1, 2, \dots$ . Suppose that  $\{T_n x\}_{n=1}^\infty$  is Cauchy in  $Y$  for every  $x \in X$ .
  - (a) State the Uniform Boundedness Theorem.
  - (b) Show that  $\{\|T_n\|\}_{n=1}^\infty$  is bounded.
  - (c) If  $Y$  is a Banach space and  $T$  is defined by  $T_n x \rightarrow Tx$  for  $x \in X$ , show that  $T$  is a well-defined operator and that  $T \in B(X, Y)$ .
  
3. Let  $H$  be a separable Hilbert space with maximal orthonormal basis  $\{u_k\}_{k=1}^\infty$ , let  $H_n = \text{span}\{u_1, \dots, u_n\}$ , and let  $P_n$  denote the orthogonal projection of  $H$  onto  $H_n$ . Suppose that  $A : H \rightarrow H$  is bounded and linear and  $f \in H$ . If for all  $n$

$$P_n A x_n = P_n f$$

has a solution  $x_n \in H_n$  such that

$$\|x_n\| \leq \alpha \|P_n f\|,$$

where  $\alpha > 0$  is independent of  $n$ .

- (a) Recall that  $\|x\|^2 = \sum_{k=1}^\infty |\langle x, u_k \rangle|^2$ . From this fact, prove that  $P_n x \rightarrow x$  as  $n \rightarrow \infty$ .
- (b) Show that there is at least one solution to  $Ax = f$ .