Problem 1. Let $n \in \mathbb{Z}$ be an integer and let $f_n(t) = t^3 - t + n \in \mathbb{Q}[t]$.

(a) Suppose $3 \nmid n$. Show that $f_n$ is irreducible.
(b) Suppose that $f_n$ is irreducible. Show that its Galois group is the symmetric group $S_3$.
(c) What isomorphism classes of groups can arise as Galois groups of $f_n$ (for $f_n$ possibly not irreducible)? For each possibility, provide some value of $n$ realizing the specific Galois group.

Problem 2. Suppose $k$ is a field and $f \in k[t]$ is a degree $n$ separable polynomial with splitting field $K$. Let $r_1, \ldots, r_n \in K$ be the roots of $f$.

(a) Show that $K$ is generated (as a $k$-algebra) by $r_1, \ldots, r_{n-1}$.
(b) Suppose $K/k$ has degree $n!$. Show that the subfield of $K$ generated by $r_1, \ldots, r_{n-2}$ is properly contained in $K$.

Problem 3. Let $p$ be an odd prime and let $\zeta_p \in \mathbb{C}$ denote a primitive $p$th root of unity.

There are unique integers $a_1, a_2, \ldots, a_{p-1}$ such that:

- $a_1 = 1$.
- For $G := \sum_{i=1}^{p-1} a_i \zeta_p^i$, $G \notin \mathbb{Q}$ but $G^2 \in \mathbb{Q}$.

Determine the values of $a_i$ and $G^2$.

(Hint: for calculating $G^2$, it helps at one point to use the automorphism of $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$ given by $(i, j) \mapsto (i, \frac{i}{j})$.)