In multi-part problems, you may use earlier parts even if you have not done them.

**Problem 1**

Let \( k \) be a field, \( n \) a positive integer, and \( T \) the linear transformation on \( k^n \) defined by

\[
T(x_1, x_2, \ldots, x_n) = (x_n, x_1, x_2, \ldots, x_{n-1}).
\]

We view \( k^n \) as a \( k[x] \)-module with \( x \) acting as \( T \).

a) Show that the \( k[x] \)-module \( k^n \) is isomorphic to \( k[x]/(x^n - 1) \).

b) Let \( V \) be a linear subspace of \( k^n \) satisfying \( T(V) \subseteq V \). Prove that there exists a monic polynomial \( g(x) \in k[x] \) such that \( V \) corresponds to

\[
\{g(x)a(x) \mid a(x) \in k[x], \deg a(x) < n - \deg g(x)\}
\]

under the above isomorphism.

c) Take \( k = \mathbb{R} \), the real numbers, and \( n = 3 \). Describe explicitly all subspaces \( V \) of \( \mathbb{R}^3 \) satisfying \( T(V) \subseteq V \).

**Problem 2**

Let \( R \) be an integral domain and \( I \) be an ideal of \( R \). Fix \( x \in R \) and define

\[
(I : x) = \{r \in R \mid rx \in I\}.
\]

a) Prove \((I : x)\) is an ideal of \( R \).

b) Show that if \((I : x) = I\), then \((I : x^2) = I\).

c) Show that if \((I : x) \subseteq xR\), but \((I : x) \not\subseteq \bigcap_{n=1}^{\infty} x^nR\), then \( I \neq (I : x)\).

d) If \( R \) is not a principal ideal domain, show that \( R \) has an ideal maximal with respect to the property of not being principal.

e) If \( R \) is a unique factorization domain with the property that every maximal ideal is principal, show that \( R \) is a principal ideal domain.

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Problem 3

Assume $G$ is a group of order $456 = 2 \cdot 3 \cdot 7 \cdot 13$.

a) Show $G$ has a normal Sylow subgroup of order either 7 or 13.

b) Show that $G$ is a semidirect product of a cyclic group of order 91 by a group of order 6.

c) Let $K$ be a group of order 42. Then $K$ is the semidirect product of a cyclic group of order 7 by a group of order 6. (Prove this only if you are unable to do part (b).) How many non-isomorphic groups of order 42 are there? Roughly describe them.

d) More challenging – do not spend too much time.
   (i) Let $H$ be the direct product of two characteristic subgroups $H_1$ and $H_2$. Prove $\text{Aut } H \cong \text{Aut } H_1 \times \text{Aut } H_2$.
   (ii) Noting the similarity between $42 = 7 \cdot 6$ and $78 = 13 \cdot 6$, how many non-isomorphic groups of order 456 are there?