Real Analysis Prelim Spring 2023  - January 10, 2023

Solve all three problems:

1. Consider the Hardy-Littlewood maximal function (in balls)

   \[ Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_{B} |f|, \quad \text{for } f \text{ a continuous, compactly supported function in } \mathbb{R}^n, \]

   where the supremum is taken over all balls \( B \subset \mathbb{R}^n \) containing \( x \).

   Prove that \( Mf(x) \) belongs to \( L^1_{\text{weak}}(\mathbb{R}^n) \).

2. Let \( f \in L^1(\mathbb{R}) \) and \( \varphi_\varepsilon \) be a mollifier, that is \( \varphi_\varepsilon = \varepsilon^{-1} \varphi(x/\varepsilon), \varphi : \mathbb{R} \to \mathbb{R} \) is a function satisfying \( \varphi \geq 0 \), the support of \( \varphi \) is compact and \( \int \varphi = 1 \).

   Let \( f_\varepsilon = f \ast \varphi_\varepsilon \) be the convolution. Show that

   \[ \int_{\mathbb{R}} \liminf_{\varepsilon \to 0} |f_\varepsilon| \leq \int_{\mathbb{R}} |f|. \]

3. Let \( 1 \leq p \leq \infty \), with \( \frac{1}{p} + \frac{1}{q} = 1 \). Show that if \( f \in L^p(\mathbb{R}^n) \) and \( g \in L^q(\mathbb{R}^n) \), then

   (a) the convolution \( f \ast g \) is bounded and continuous on \( \mathbb{R}^n \).

   (b) In addition, prove that if either \( f \in (L^p \cap C^m)(\mathbb{R}^n) \), or \( g \in (L^q \cap C^m)(\mathbb{R}^n) \), then \( f \ast g \in C^m(\mathbb{R}^n) \).