Work 4 of the following 5 problems.

1. If $f$ is a map from a Banach space $X$ to a Banach space $Y$, such that $\psi \circ f$ belongs to the dual of $X$ for every $\psi$ in the dual of $Y$, show that $f$ is linear and continuous.

2. Let $P$ be a linear operator on a Banach space, satisfying $P^2 = P$. Show that the operator $P$ is continuous if and only if its null space and range are both closed.

3. Show that a compact linear operator maps weakly convergent sequences to strongly convergent sequences.

4. If $U$ is an unitary operator on a Hilbert space, show that $n^{-1}[I+U+U^2+\ldots+U^{n-1}]$ converges strongly to an orthogonal projection, as $n \to \infty$.

5. For $m = 1, 2, 3, \ldots$, define $H_m$ to be the regular distribution on $\mathbb{R}$ associated with the function $h_m(x) = m^2 \sin(mx)$. Show that $H_m \to 0$ in $\mathcal{D}(\mathbb{R})$ as $m \to \infty$. 