PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part II

August 18, 2023, 3:00-5:00pm

Work all 3 of the following 3 problems.

1. If $d \geq 1$ and $0 < k < d$ are integers, the restriction map $R : C^0(\mathbb{R}^d) \to C^0(\mathbb{R}^{d-k})$ is defined for $y \in \mathbb{R}^{d-k}$ by $R\varphi(y) = \varphi(y, 0)$.

(**a**) Define the Sobolev space $H^s(\mathbb{R}^d)$ for any $s \geq 0$ in terms of the Fourier Transform.

(**b**) Show that for $\varphi \in C^0(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$,

$$\widehat{R\varphi}(\eta) = (2\pi)^{-k/2} \int_{\mathbb{R}^k} \hat{\varphi}(\eta, \zeta)\, d\zeta.$$

(**c**) Show that R extends to a bounded linear map from $H^s(\mathbb{R}^d)$ into $H^{s-k/2}(\mathbb{R}^{d-k})$, provided that $s > k/2$. [You do not need to show that R maps onto, although it does.]

2. For a given bounded domain $\Omega \subset \mathbb{R}^d$ with a smooth boundary and constant $K \geq 0$, consider the problem of finding u solving

$$\begin{aligned} \Delta u - K\, u &= f && \text{in } \Omega, \\ \nabla u(x) \cdot \nu &= g(x) && \text{for } x \in \partial\Omega. \end{aligned}$$

for a given $f \in H^s(\Omega)$ and $g(x)$.

(**a**) To what Sobolev space must $g(x)$ belong, in order that the Neumann boundary value problem has a solution $u \in H^{s+2}(\Omega)$?

(**b**) Give estimates for the solution u in the $H^{s+2}(\Omega)$-norm showing the dependence with respect to the data f, g, and K in terms of their Sobolev norms.

(**c**) Do f and g need to satisfy a compatibility condition? If so, give the condition. The answer depends on K.

(**d**) For what values of s will u be continuous?

3. Show that the Fredholm integral equation

$$f(x) = \phi(x) + \lambda \int_a^b K(x, y) f(y)\, dy$$

has a unique solution $f \in C([a, b])$ provided that λ is sufficiently small, wherein $\phi \in C([a, b])$ and $K \in C([a, b] \times [a, b]))$