PRELIMINARY EXAMINATION:
APPLIED MATHEMATICS — Part II
August 18, 2023, 3:00-5:00pm

Work all 3 of the following 3 problems.

1. If $d \geq 1$ and $0 < k < d$ are integers, the restriction map $R : C^0(\mathbb{R}^d) \to C^0(\mathbb{R}^{d-k})$ is defined for $y \in \mathbb{R}^{d-k}$ by $R\varphi(y) = \varphi(y,0)$.
   (a) Define the Sobolev space $H^s(\mathbb{R}^d)$ for any $s \geq 0$ in terms of the Fourier Transform.
   (b) Show that for $\varphi \in C^0(\mathbb{R}^d) \cap L^2(\mathbb{R}^d),
   \hat{R}\varphi(\eta) = (2\pi)^{-k/2} \int_{\mathbb{R}^k} \hat{\varphi}(\eta,\zeta) d\zeta.$
   (c) Show that $R$ extends to a bounded linear map from $H^s(\mathbb{R}^d)$ into $H^{s-k/2}(\mathbb{R}^{d-k})$, provided that $s > k/2$. [You do not need to show that $R$ maps onto, although it does.]

2. For a given bounded domain $\Omega \subset \mathbb{R}^d$ with a smooth boundary and constant $K \geq 0$, consider the problem of finding $u$ solving
   \[ \begin{align*}
   \Delta u - Ku &= f \quad \text{in } \Omega, \\
   \nabla u(x) \cdot \nu &= g(x) & \text{for } x \in \partial \Omega.
   \end{align*} \]
   for a given $f \in H^s(\Omega)$ and $g(x)$.
   (a) To what Sobolev space must $g(x)$ belong, in order that the Neumann boundary value problem has a solution $u \in H^{s+2}(\Omega)$?
   (b) Give estimates for the solution $u$ in the $H^{s+2}(\Omega)$-norm showing the dependence with respect to the data $f$, $g$, and $K$ in terms of their Sobolev norms.
   (c) Do $f$ and $g$ need to satisfy a compatibility condition? If so, give the condition. The answer depends on $K$.
   (d) For what values of $s$ will $u$ be continuous?

3. Show that the Fredholm integral equation
   \[ f(x) = \phi(x) + \lambda \int_a^b K(x,y)f(y) dy \]
   has a unique solution $f \in C([a,b])$ provided that $\lambda$ is sufficiently small, wherein $\phi \in C([a,b])$ and $K \in C([a,b] \times [a,b]))$.