1. Let the field be real and $\Omega \subset \mathbb{R}^d$ be a domain with a Lipschitz boundary. For $w \in L^\infty(\Omega)$, define
$$H_w(\Omega) = \{ f \in L^2(\Omega) : \nabla(w f) \in (L^2(\Omega))^d \}.$$

(a) Give reasonable conditions on $w$ so that $H_w(\Omega) = H^1(\Omega)$.
(b) Prove that $H_w(\Omega)$ is a Hilbert space. What is the inner-product?
(c) Suppose that $\Omega$ is bounded. Prove that there is a constant $C > 0$ such that for all $f \in H_w(\Omega)$ satisfying $\int_\Omega w(x) f(x) \, dx = 0$,
$$\|f\|_{L^2(\Omega)} \leq C \left\{ \|\nabla(w f)\|_{L^2(\Omega)} + \|(1 - w) f\|_{L^2(\Omega)} \right\}.$$

[Hint: use the usual Poincaré inequality for functions with zero average value.]

2. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, $a \in (L^\infty(\Omega))^d$, and $f \in L^p(\Omega)$ (for some $p$). Consider the boundary value problem (BVP)
$$-\Delta u + a \cdot \nabla u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial \Omega.$$

(a) Pose the BVP as a variational problem (VP) in $H^1_0(\Omega) \times H^1_0(\Omega)$. [You do not need to justify the equivalence.]
(b) Use the Sobolev Embedding Theorem to find the range of $p \geq 1$ such that your VP is well posed.
(c) Determine a bound on $\|a\|_{(L^\infty(\Omega))^d}$ (which will depend on the Poincaré constant $C_P$) in $\|v\|_{L^2(\Omega)} \leq C_P \|
abla v\|_{L^2(\Omega)}$ so that you have coercivity, and then apply the Lax-Milgram Theorem to show existence and uniqueness of a solution.

3. Let $X$ be a Banach space and $g : X \to X$ be a nonlinear mapping that is $C^1$ and has $g(0) = 0$ and $Dg(0) = 0$. For $f \in X$, we want to solve
$$F(u) = u + g(u) = f.$$ 

We consider the map $G(u) = u + \alpha(F(u) - f)$ for some $\alpha \in \mathbb{R}$.

(a) Show that $G(u)$ is a contractive map for small enough $u$ and properly chosen $\alpha$.
(b) Use the Banach contraction mapping theorem to show that there is a solution to $F(u) = f$, provided $f$ is sufficiently small.
(c) Solve $F(u) = f$ using the inverse function theorem, provided $f$ is sufficiently small.