PRELIMINARY EXAMINATION:
APPLIED MATHEMATICS — Part I

January 9, 2023

Work all 3 of the following 3 problems.

1. Let X and Y be normed linear spaces and $T \in B(X, Y)$.

(**a**) Define the dual operator $T^* : Y^* \to X^*$. Be sure to justify that $T^*(g) \in X^*$ for each $g \in Y^*$.

(**b**) Prove that $T^* \in B(Y^*, X^*)$.

(**c**) Prove that $\|T^*\|_{B(Y^*,X^*)} = \|T\|_{B(X,Y)}$. [Hint: recall that the Hahn-Banach Theorem implies that or any $y_0 \in Y$, there exists $g_0 \in Y^*$ such that $\|g_0\| = 1$ and $\|y_0\| = g_0(y_0)$.]

2. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space.

(**a**) If X is a nonempty subset of H, prove that $X^\perp$ is a closed subspace of H.

(**b**) Let $T : H \to H$ be a bounded linear operator. Let $N = N(T)$ be the null space of T and $R(T)$ be the range or image of T. Let $P : H \to N$ be orthogonal projection onto N. Prove that $S = T \circ P^\perp$ is a one-to-one mapping when restricted to $N^\perp$ and that $R(S) = R(T)$.

3. Let $\Omega \subset \mathbb{R}^d$ be a domain and recall that for $\phi \in \mathcal{D}(\Omega)$,

$$\|\phi\|_{m,\infty,\Omega} = \sum_{|\alpha| \le m} \|D^\alpha \phi\|_{L^\infty(\Omega)}.$$

(**a**) For ϕ_j and ϕ in $\mathcal{D}(\Omega)$, explain what it means for $\phi_j \to \phi$ as $j \to \infty$.

(**b**) Suppose that $T : \mathcal{D}(\Omega) \to \mathbb{F}$ is linear. Prove that $T \in \mathcal{D}'(\Omega)$, i.e., T is (sequentially) continuous, if and only if for every $K \subset\subset \Omega$, there are $n \ge 0$ and $C > 0$, depending on K, such that

$$|T(\phi)| \le C\|\phi\|_{n,\infty,\Omega}$$

for every $\phi \in \mathcal{D}_K = \{f \in C_0^\infty(\Omega) : \text{supp}(f) \subset K\}$.