1. Let $\Omega$ be a compact set in $\mathbb{R}^d$ and let $K : \Omega \times \Omega \to \mathbb{R}$ be continuous and symmetric (i.e., $K(x, y) = K(y, x)$). Suppose that $K \geq 0$, and let the operator $T$ be defined by $Tf(x) = \int_{\Omega} K(x, y) f(y) \, dy$.

(a) State the spectral theorem for a compact, self-adjoint operator.

(b) Show Mercer’s Theorem: there is an ON base for $L^2(\Omega)$ consisting of eigenfunctions $\{e_j\}_{j=1}^{\infty}$ of $T$ with corresponding eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ such that each $\lambda_j \geq 0$ and

$$K(x, y) = \sum_{j=1}^{\infty} \lambda_j e_j(x) e_j(y).$$

[The sum is absolutely and uniformly convergent in $L^2(\Omega \times \Omega)$, but you need not show this fact.]

(c) Define $\text{Trace}(T) = \int_{\Omega} K(x, x) \, dx$ and show that

$$\text{Trace}(T) = \sum_{j=1}^{\infty} \lambda_j.$$

2. Let $H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

(a) Prove the parallelogram law: For all $x, y \in H$,

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

(b) Prove the Best Approximation Theorem. That is, if $M \subset H$ is nonempty, convex, and closed, and if $x \in H$, then there is a unique $y \in M$ such that

$$\text{dist}(x, M) = \inf_{z \in M} \|x - z\| = \|x - y\|.$$

3. Let $X$ and $Y$ be NLS’s.

(a) Show that if a linear operator $S : X^* \to Y^*$ is weakly-* sequentially continuous, that is,

$$f_n \xrightarrow{\text{weak-*}} f \text{ in } X^* \implies S(f_n) \xrightarrow{\text{weak-*}} S(f) \text{ in } Y^*,$$

then $S$ is bounded.

(b) Given a linear operator $T : X \to Y$, assume that the dual (or conjugate or adjoint) $T^* : Y^* \to X^*$ is defined. Show that $T^*$ is weakly-* sequentially continuous.

(c) Show that whenever $T^* : Y^* \to X^*$ is defined, $T^*$ is bounded.
1. Let $\Omega \subset \mathbb{R}^2$ be an open, connected, and bounded domain containing 0. Let 
\[ X = \{ f \in W^{1,3}(\Omega) : f(0) = 0 \} . \]

(a) Use the Sobolev Embedding Theorem to conclude that $X$ is a Banach space, and $X \neq W^{1,3}(\Omega)$.

(b) Prove the Poincaré-like inequality 
\[ \| f \|_{L^3(\Omega)} \leq C \| \nabla f \|_{L^3(\Omega)} , \]
for some constant $C$ independent of $f \in X$.

2. Suppose that $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with Lipschitz boundary and \{\(u_k\)\}_{k=1}^{\infty} \subset H^{2+\varepsilon}(\Omega)$ is a bounded sequence, where $\varepsilon > 0$.

(a) State the Rellich-Kondrachov Theorem. [For the rest of the problem, assume that it holds with nonintegral values for the number of derivatives.]

(b) Show that there is $u \in H^{2+\varepsilon}(\Omega)$ such that, for a subsequence, $u_{kj} \to u$ in $H^2(\Omega)$.

(c) Find all $q$ and $s \geq 0$ such that, for a subsequence, $u_{kj} \to u$ in $W^{s,q}(\Omega)$.

3. Let $\Omega$ be a domain with a smooth boundary. Consider the differential problem
\[
\begin{align*}
p - \nabla \cdot a \nabla p - \nabla \cdot b \nabla q + d(p - q) &= 0 \quad \text{in } \Omega, \\
-\nabla \cdot c \nabla q + d(q - p) &= f \quad \text{in } \Omega, \\
-(a \nabla p + b \nabla q) \cdot \nu &= g \quad \text{on } \partial \Omega, \\
q &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
where $a$, $b$, $c$, and $d \geq 0$ are bounded, smooth functions, $f \in H^{-1}(\Omega)$, and $g \in H^{-1/2}(\partial \Omega)$. Moreover, assume that there is some $\gamma > 0$ such that $a \geq \gamma$, $c \geq \gamma$, and $|b| \leq \gamma$.

(a) Define a suitable variational problem for the differential equations. Be sure to identify your function spaces for $p$, $q$, and the test functions.

(b) Show that there is a unique solution to the variational problem.