Differential Topology Prelim Exam
January 10, 2024, 11:30–1:30

Solve all three problems.

Problem 1. True or false with brief explanations/examples/counterexamples.

(a) Suppose $Y$ is a linear subspace of $\mathbb{R}^n$, $X$ is a smooth manifold, $f : X \to \mathbb{R}^n - \{0\}$ is smooth, and $x \in X$. Then $f$ is transverse to $Y$ at $x$, if and only if $\bar{f}$ is transverse to $PY \subseteq \mathbb{R}P^{n-1}$ at $x$. Here $\bar{f}$ is $f$ followed by the usual map $\mathbb{R}^n - \{0\} \to \mathbb{R}P^{n-1}$.

(b) Suppose $f : X \to Y$ is a smooth map of manifolds, and $y \in Y$. Then $f$ is a regular value of $f$ if and only if $f$ is a local diffeomorphism near each $x \in f^{-1}(y)$.

(c) Every injective immersion is a diffeomorphism onto a submanifold.

Problem 2. Let $\omega$ be the 2-form on $\mathbb{R}^3$ given in standard coordinates by

$$\omega = dx \wedge dz + z \, dx \wedge dy$$

(a) Prove that $\omega$ is neither closed nor translation-invariant.

(b) Nevertheless, if $S$ is an oriented surface embedded in $\mathbb{R}^3$, then $\int_S \omega = \int_{S_v} \omega$ for any vector $v \in \mathbb{R}^3$, where $S_v$ and its orientation are got by translating $S$ by $v$.

Problem 3. Let $\lambda_0, \ldots, \lambda_3 \in \mathbb{R} - \{0\}$, and let $f$ be the following self-map of $\mathbb{R}P^3$, given in homogeneous coordinates by

$$f([x_0 : \cdots : x_3]) = [\lambda_0 x_0 : \cdots : \lambda_3 x_3]$$

(a) Under what conditions is $f$ a Lefschetz map?

(b) Under these conditions, work out the fixed points of $f$, and its local degrees and Lefschetz numbers there.

(c) What are the degree and Lefschetz number of $f$?