Problem 1. Let $T : [0, 1] \to [0, 1]$ be given by

$$T(0) = 0 \text{ and } T(x) = \{1/x\} \text{ for } x \in (0, 1],$$

where $\{a\} := \sup\{a - m : m \in \mathbb{Z} \text{ and } m \leq a\} = a - \lfloor a \rfloor$ is the fractional part of $a$. Show that $T_*\mu = \mu$, where $T_*\mu$ is the pushforward of $\mu$ via $T$ and

$$\mu(B) = \int_B \frac{1}{1 + x} \, dx \text{ for } B \in \mathcal{B}([0, 1]).$$

(Note: $T$ is (one of several things) known as the Gauss map.)

Problem 2. Let $\mu$ be a probability measure on $\mathbb{R}$, and let $\varphi$ be its characteristic function. Show that $\mu$ is diffuse (has no atoms) if

$$\lim_{t \to \infty} |\varphi(t)| = \lim_{t \to -\infty} |\varphi(t)| = 0.$$

(Hint: For $a \in \mathbb{R}$, compute $\lim_{T \to \infty} \int_{-T}^{T} e^{-ita} \varphi(t) \, dt$.)

Problem 3. Given $X \in L^2(\mathcal{F})$ and two sub-$\sigma$-algebras $\mathcal{G}, \mathcal{H}$ of $\mathcal{F}$ such that $\mathcal{G} \subseteq \mathcal{H}$, show that

$$\mathbb{E}[\text{Var}[X | G]] \geq \mathbb{E}[\text{Var}[X | H]],$$

where $\text{Var}[X | \mathcal{K}] := \mathbb{E}[(X - \mathbb{E}[X | \mathcal{K}])^2 | \mathcal{K}]$ for $\mathcal{K} \subseteq \mathcal{F}$. When does the equality hold?

(Note: $\text{Var}[X | \mathcal{K}]$ is called the conditional variance of $X$ given $\mathcal{K}$.)