Perturbative approaches for the analysis of the spin–orbit problem

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This paper is dedicated to Jacques Henrard

Abstract

We review some results concerning perturbation techniques applied to the spin–orbit problem. In particular, we start by constructing periodic orbits by means of a combination of the implicit function theorem and of perturbation theory. Next, we focus on rotational and librational invariant tori, whose existence is proved by KAM theory. Applications to the Moon and Mercury are presented.

1 Introduction

We consider the motion of an oblate satellite moving on a Keplerian orbit around a central planet and rotating around an internal spin–axis. Under some simplifying assumptions on the rotation axis, the equation of motion is associated to a one–dimensional, time–dependent Hamiltonian function. We simplify further the model by considering a trigonometric expansion of the perturbing function.

The phase–space structure is composed by periodic orbits surrounded by librational invariant tori; a chaotic separatrix divides the region of librational motion from the region in which rotational invariant tori can be found. Being the phase–space three–dimensional, the invariant tori provide a strong stability property, confining the motion in a given region of the phase–space.

In §3 we will be concerned with the existence of periodic orbits, which correspond to commensurabilities between the rotational and revolutional periods of the satellite. In particular, we concentrate on the synchronous resonance (i.e., when the two periods coincide) and on the 3:2 resonance (such that twice the period of revolution equals three times the period of rotation). Most of the satellites are observed to move in the synchronous resonance, while only Mercury has been observed in the 3:2 spin–orbit resonance. The peculiarity of Mercury may be due to

its large eccentricity in comparison to the other satellites of the solar system (see [2], [6]). Periodic orbits are obtained through an implementation of a constructive version of the implicit function theorem combined with perturbation theory. The existence of the synchronous and 3:2 resonances are obtained for physically relevant values of the parameters (see [5]).
In §4 we concentrate on the invariant rotational tori, with the aim to prove the existence of a pair of surfaces which trap a given periodic orbit from above and below, thus providing a confinement of the motion in the region included by these surfaces. The Kolmogorov–Arnold–Moser theorem ([1], [7], [8]) is applied after computing a high–order initial approximate solution by means of a computer–assisted technique. Invariant surfaces trapping the synchronous and the 3:2 resonances are constructed for the realistic values of the parameters of the Moon and of Mercury (see [2], [4]).
A sketch of the proof of the existence librational invariant tori is presented in §5 (see [3]).

2 The spin–orbit model

We consider a triaxial satellite S, for example the Moon, rotating about an internal spin–axis and orbiting around a central planet P, the Earth. As is well known, the Moon always points the same face to the Earth, due to the fact that its period of rotation coincides with the period of revolution around the Earth: one says that the Moon is trapped in a 1:1 or synchronous spin-orbit resonance. More generally, denoting by T_{rev} and T_{rot} the satellite's revolutional and rotational periods, one has the following

Definition. A spin–orbit resonance of order $p:q$ occurs whenever

$$\frac{T_{rev}}{T_{rot}} = \frac{p}{q} , \qquad p, q \in \mathbf{Z}_+ .$$

Astronomical observations showed that all tidally evolved satellites of the solar system are found in a 1:1 resonance; the only exception is provided by Mercury which is actually observed in a 3:2 spin–orbit resonance, which entails that during two revolutions around the Sun, Mercury makes three rotations about its spin–axis.

The spin–orbit dynamics can be conveniently described using three reference frames: an inertial (or fixed) frame, a reference system oriented according to the principal axes of inertia (body frame) and a reference system whose vertical axis coincides with the direction of the angular momentum (spin frame). In this context we introduce two angles: the non–principal rotation angle J between the spin and body frames, and the obliquity angle K between the spin and fixed frames. We shall be concerned with the simplest spin–orbit model in which $J = K = 0$, so that the rotation axis coincides with the principal axis and it is perpendicular to the fixed frame, which we assume to coincide with the orbital plane. More precisely, we develop our model under the following hypotheses (see [2]):
i) the satellite moves on a Keplerian orbit around the planet;
ii) the spin–axis coincides with the smallest physical axis;
iii) the spin–axis is perpendicular to the orbit plane;
iv) any dissipative force (like the tidal torque due to the internal non–rigidity of the satellite) as well as perturbations due to other bodies are neglected.

In order to write the equation of motion, let us introduce the following notation (see Figure 1). Let $A < B < C$ be the principal moments of inertia, a denotes the semimajor axis, r is the instantaneous orbital radius and f represents the true anomaly. Then, denoting by x the angle between the longest axis of the satellite and the pericentre line, the equation of motion reads as

$$\ddot{x} + \frac{3}{2} \frac{B-A}{C} (\frac{a}{r})^3 \sin(2x - 2f) = 0 \ .$$

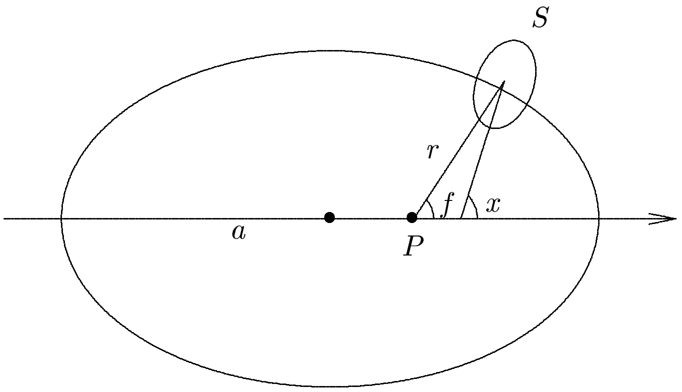

Figure 1: Configuration of the spin–orbit problem

In terms of the previous quantities, we can state that a spin–orbit resonance of order $\frac{p}{q}$ is a periodic orbit $x = x(t)$ such that

$$x(t + 2\pi q) = x(t) + 2\pi p \ ,$$

which implies that after q revolutions around the planet, the satellite makes p rotations about the spin–axis.
Due to the assumption of Keplerian orbit, both r and f are known functions of the time; therefore, one can expand the equation of motion as

$$\ddot{x} + \varepsilon \sum_{m \neq 0, m=-\infty}^{\infty} W(\frac{m}{2}, e) \sin(2x - mt) = 0 \ ,$$

where $\varepsilon \equiv \frac{3}{2} \frac{B-A}{C}$ and the coefficients decay as powers of the orbital eccentricity e as $W(\frac{m}{2}, e) \propto e^{|m-2|}$. According to assumption *iv*), we further simplify the model neglecting those terms whose size is comparable to the most important term we have discarded in the equation of motion, namely the contribution due to the tidal torque. Consistently, one obtains for the Moon the equation of motion

$$\ddot{x} + \varepsilon \ [(-\frac{e}{2} + \frac{e^3}{16}) \ \sin(2x - t) \ + (1 - \frac{5}{2}e^2 + \frac{13}{16}e^4) \sin(2x - 2t) + (\frac{7}{2}e - \frac{123}{16}e^3)$$

$$\sin(2x-3t)+(\frac{17}{2}e^2-\frac{115}{6}e^4)\ \sin(2x-4t)+\ (\frac{845}{48}e^3-\frac{32525}{768}e^5)\ \sin(2x-5t)+$$
$$+\frac{533}{16}e^4\ \sin(2x-6t)\ +\ \frac{228347}{3840}e^5\ \sin(2x-7t)\ ]\ =\ 0\ , \tag{1}$$

which can be derived from Hamilton's equations associated to the one–dimensional, time–dependent Hamiltonian function

$$H(y,x,t)\ \equiv \frac{y^2}{2}\ -\ \frac{\varepsilon}{2}\sum_{m\neq 0,m=1}^{7}\ \tilde{W}(\frac{m}{2},e)\ \cos(2x-mt)\ , \tag{2}$$

where y is the action coordinate conjugated to x and $\tilde{W}(\frac{m}{2},e)$ are defined by comparison with (1).

3 Birkhoff periodic orbits

We want to construct periodic solutions for an equation of the form (1), which we rewrite as $\ddot{x}-\varepsilon g(x,t)=0$ for some trigonometric function $g(x,t)$ (see [5] for further details). Equivalently we study the first–order system of differential equations

$$\begin{aligned}\dot{x}\ &=\ y\\ \dot{y}\ &=\ \varepsilon g(x,t)\ ,\end{aligned} \tag{3}$$

with the aim to construct Birkhoff periodic orbits with frequency $\omega=\frac{p}{q}$, such that

$$x(t+2\pi q)=x(t)+2\pi p\ ,\qquad y(t+2\pi q)=y(t)\ . \tag{4}$$

By the integral form of (3), the periodicity conditions (4) are satisfied whenever

$$\begin{cases} F_1(X,Y;\varepsilon)\equiv\int_0^{2\pi q}y(s;X,Y)ds-2\pi p=0\\ \\ F_2(X,Y;\varepsilon)\equiv\int_0^{2\pi q}g(x(s;X,Y),s)ds=0\ ,\end{cases}$$

for some initial conditions $x(0)=X$, $y(0)=Y$. This equations can be solved through the implicit function theorem applied to the vector function $F\equiv(F_1,F_2)$ as follows (see [5]).

Implicit function theorem: *Let $\rho>0$, $\overline{z}\equiv(\overline{x},\overline{y})\in\mathbf{R}^2$, $\overline{B}_\rho(\overline{z})$ be the close ball around $\overline{z}$ with radius ρ; let $F:(z,\varepsilon)\in\overline{B}_\rho(\overline{z})\times\mathbf{R}\to F(z,\varepsilon)\in\mathbf{R}^2$ be continuous with continuous and invertible jacobian $\frac{\partial F}{\partial z}(\overline{z},\varepsilon)$. Let $M(\varepsilon)\equiv\left(\frac{\partial F}{\partial z}(\overline{z},\varepsilon)\right)^{-1}$ and $m\equiv\sup_\varepsilon\|M\|$; denoting by $\rho\equiv 2m\ \sup_\varepsilon|F(\overline{z},\varepsilon)|$, if the following inequality is satisfied*

$$4m^2\sup|F(\overline{z},\varepsilon)|\ \sup_{\overline{B}_\rho(\overline{z})\times\mathbf{R}}\left\|\frac{\partial^2F}{\partial z^2}\right\|\leq 1\ ,$$

then there exists a unique continuous function $\varepsilon\to z(\varepsilon)\in\overline{B}_\rho(\overline{z})$, such that $F(z(\varepsilon),\varepsilon)\equiv 0$.

As a first attempt we could try to implement the implicit function theorem, starting with a first order approximate solution of the form

$$\begin{cases} x(t)\equiv\overline{x}+\overline{y}t+\varepsilon\xi(t)\\ y(t)\equiv\overline{y}+\varepsilon\eta(t)\ ,\end{cases}$$

where, taking $\xi(0)=\eta(0)=0$, $\xi(t)$, $\eta(t)$ must satisfy

$$\begin{cases} \dot{\xi}=\eta \\ \dot{\eta}=g(\overline{x}+\overline{y}t+\varepsilon\xi(t),t)\ , \end{cases}$$

and where the initial conditions, say $\overline{x}=x_0$, $\overline{y}=y_0$, are defined as

$$y_0=\frac{p}{q}\ , \qquad x_0 \text{ such that } \int_0^{2\pi q} g(x_0+y_0 s,s)ds=0\ .$$

However, the application of the theorem to the functions

$$\begin{cases} F_1(X,Y;\varepsilon)\equiv\int_0^{2\pi q}(Y+\varepsilon\eta(t))\ dt-2\pi p \\ \\ F_2(X,Y;\varepsilon)\equiv\int_0^{2\pi q} g(X+Yt+\varepsilon\xi(t),t)\ dt \end{cases}$$

provides the existence of periodic orbits under too stringent smallness requirements on the perturbing parameter ε. In order to obtain more realistic results, one needs to combine the implicit function theorem with perturbation theory. More precisely, one can start with a second order approximate solution of the form

$$\begin{aligned} x(t) &\equiv \overline{x}+\overline{y}t+\varepsilon x_1(t)+\varepsilon^2\xi(t) \\ y(t) &\equiv \overline{y}+\varepsilon y_1(t)+\varepsilon^2\eta(t)\ , \end{aligned}$$

with initial conditions $x(0)=\overline{x}$, $y(0)=\overline{y}$ and with $x_1(t)$, $y_1(t)$ defined by

$$\begin{cases} x_1(t)\equiv\int_0^t y_1(s)\ ds \\ y_1(t)\equiv\int_0^t g(\overline{x}+\overline{y}s,s)\ ds\ , \end{cases}$$

while $\xi(t)$, $\eta(t)$ are the solutions of the following differential equations with initial conditions $\xi(0)=0$, $\eta(0)=0$:

$$\begin{cases} \dot{\xi}=\eta\ , \\ \dot{\eta}=\frac{1}{\varepsilon}[g(\overline{x}+\overline{y}t+\varepsilon x_1(t)+\varepsilon^2\xi(t),t)-g(\overline{x}+\overline{y}t,t)]\ . \end{cases}$$

We consider the second order initial conditions $\overline{x}=x_0+\varepsilon x_1$, $\overline{y}=y_0+\varepsilon y_1$ with

$$\begin{aligned} y_1 &= -\frac{1}{2\pi q}\int_0^{2\pi q}\int_0^t g(x_0+y_0 s,s)dsdt \\ x_1 &= -\frac{1}{\int_0^{2\pi q} g_x^0(t)dt}\ \left[y_1\int_0^{2\pi q} tg_x^0(t)dt+\int_0^{2\pi q} g_x^0(t)x_1(t)dt\right]\ , \end{aligned}$$

where we denoted by $g_x^0\equiv g_x(x_0+y_0t,t)$ with $\int_0^{2\pi q} g_x^0 dt\neq 0$. Next we define the vector function $F\equiv(F_1,F_2)$ as

$$\begin{aligned} F_1(X,Y;\varepsilon^2) &\equiv \int_0^{2\pi q}(Y+\varepsilon y_1(t)+\varepsilon^2\eta(t))dt-2\pi p \\ F_2(X,Y;\varepsilon^2) &\equiv \int_0^{2\pi q} g(X+Yt+\varepsilon x_1(t)+\varepsilon^2\xi(t),t)dt\ . \end{aligned}$$

By the implicit function theorem, we can find some initial data $(X(\varepsilon), Y(\varepsilon))$ close to $(\overline{x}, \overline{y})$, such that $F(X(\varepsilon), Y(\varepsilon); \varepsilon) = 0$. Performing the necessary (lengthy) "a–priori estimates" on the solution $x(t)$, $y(t)$, one gets the following results (see [5] for further details).

Consider the equation (1) with the Moon's eccentricity $e = 0.0549$. One can prove the existence of a 1:1 spin–orbit resonance provided $\varepsilon \leq 7.1 \cdot 10^{-4}$; the existence of the 3:2 resonance is obtained for any $\varepsilon \leq 7.8 \cdot 10^{-6}$. If one increases the eccentricity up to Mercury's value, say $e = 0.2056$, the existence of the 1:1 spin–orbit resonance is proved whenever $\varepsilon \leq 8.8 \cdot 10^{-5}$, while the 3:2 resonance is obtained for any $\varepsilon \leq 5.7 \cdot 10^{-6}$. We claim that such results can be improved by considering higher order perturbative expansions. Notice that the realistic value of the parameter of the Moon is $\varepsilon_{Moon} = 3.45 \cdot 10^{-4}$, while for Mercury $\varepsilon_{Merc} = 1.5 \cdot 10^{-4}$.

4 Invariant rotational tori

Let us rewrite the Hamiltonian (2) as

$$H(y, x, t) = h(y) + \varepsilon f(x, t) \ , \qquad y \in \mathbf{R}, \ (x, t) \in \mathbf{T}^2 \ , \tag{5}$$

for some analytic functions h, f. In the unperturbed case ($\varepsilon = 0$) the corresponding Hamilton's equations are trivially integrated as

$$y = y_0 \ , \qquad x = x_0 + \omega(y_0) t \ ,$$

with $\omega_0 = \frac{\partial h(y_0)}{\partial y} = \omega(y_0)$ being the *rotation number* associated to the initial condition $y = y_0$. Having fixed a frequency ω_0, KAM theory ensures that for small values of ε, there exists an invariant rotational torus with the same frequency of the unperturbed system; as ε increases such torus becomes more and more distorted and displaced, until the perturbing parameter reaches a critical value, say $\varepsilon = \varepsilon_c$, at which the invariant torus breaks-down and leaves space to chaotic motions. The KAM theorem provides an explicit algorithm to compute a lower estimate of the perturbing parameter, say $\varepsilon = \varepsilon_{KAM}$, ensuring the existence of quasi–periodic motions with frequency ω_0 for any $\varepsilon \leq \varepsilon_{KAM}$.
We stress that the existence of invariant surfaces for (5) provides a strong stability property. Indeed, the phase–space associated to (5) has dimension three; therefore, the two–dimensional KAM tori provide a confinement of the motion. In particular, any trajectory in the phase–space between two invariant tori is forever constrained in such region.

The KAM theory can be applied under the following general assumptions:
i) the unperturbed Hamiltonian must satisfy the non–degeneracy condition:

$$\det \ \frac{\partial^2 h(y)}{\partial y^2} \neq 0 \ ;$$

ii) the frequency ω_0 must satisfy the following *diophantine* condition:

$$|\omega_0 - \frac{m}{n}|^{-1} \leq C|n|^2 \ , \qquad \forall m, n \neq 0, \ C > 0 \ .$$

We stress that a major problem with the applications of KAM theory to concrete models relies on the fact that the estimates provided by the theorem, namely ε_{KAM}, are usually much smaller than the realistic physical value. An improvement of such results is obtained combining refined KAM estimates with a computer assisted approach to perform long series expansions. The rounding–off and propagation errors introduced by the computer can be controlled through "interval arithmetics": roughly speaking, one starts by reducing every computation to a sequence of elementary operations $(+,-,*,/)$, whose precision is guaranteed up to a certain decimal digit; the second step is to replace any number by an interval which certainly contains the exact result, and to perform all subsequent operations on intervals, rather than on real numbers.

We proceed now to give a sketch of the proof of the KAM theorem (see [2], [4] for further details), starting from equation (1) that we rewrite as

$$\ddot{x} - \varepsilon g(x,t) \ = \ 0 \ . \tag{6}$$

Definition. A rotational KAM torus with frequency ω_0 is an invariant surface described parametrically by

$$\begin{aligned} x &= \theta + u(\theta,t) \qquad\qquad (\theta,t) \in \mathbf{T}^2 \\ y &= v(\theta,t) \ , \end{aligned} \tag{7}$$

where u, v are periodic continuous functions depending on ω_0 and ε, such that the flow in the coordinates (θ,t) is linear, namely after a time s: $(\theta,t) \to (\theta+\omega_0 s, t+s)$.

Notice that (7) is a diffeomorphism provided $1+\frac{\partial u}{\partial \theta} \neq 0$, $\forall \theta, t \in \mathbf{T}^2$. Inserting (7) in (6), one obtains that the function u must satisfy:

$$D^2 u(\theta,t) - \varepsilon g(\theta + u(\theta,t),t) = 0 \ , \qquad D \equiv \omega_0 \frac{\partial}{\partial \theta} + \frac{\partial}{\partial t} \ . \tag{8}$$

We remark that if we expand u in Fourier series as $u = \sum_{n,m} \hat{u}_{nm} e^{i(n\theta+mt)}$, then the inverse of D is given by

$$D^{-1} u = \sum_{n,m} \frac{\hat{u}_{nm}}{i(n\omega_0 + m)} \ e^{i(n\theta+mt)} \ ,$$

where small divisors of the form $n\omega_0 + m$ do appear.
In order to solve (8) we implement a Newton iteration procedure as follows. Let u_0 be an approximate solution of (8) up to an error term e_0, namely

$$D^2 u_0 - \varepsilon g(\theta + u_0, t) = e_0 \ .$$

We construct a sequence of approximate solutions $\{u_j\}$, such that they satisfy the same equation with a quadratically smaller error term:

$$D^2 u_j - \varepsilon g(\theta + u_j, t) = e_j \ ,$$

where $|e_j| = O(|e_{j-1}|^2) = O(|e_0|^{2^j})$. A set of accurate estimates on the quantities involved yields a complete control on the parameters, so to ensure the convergence of the method provided that

$$\lim_{j\to\infty} |e_j| \ = \ 0 \ ,$$

which is in turn a smallness requirement on the perturbing parameter.

We stress that the good quality of the results strongly depends on the choice of the initial approximation; to this end, we expand the function u in Taylor series around $\varepsilon = 0$ as

$$u(\theta, t; \varepsilon) = \sum_{j=1}^{\infty} u^{(j)}(\theta, t)\varepsilon^j \ , \tag{9}$$

where the coefficients $u^{(j)}$ can be iteratively constructed inserting (9) in (8) and equating same orders of ε. As initial approximation we select a truncation, up to a certain order N, of the series expansion (9), i.e. we define

$$u_0(\theta, t; \varepsilon) = \sum_{j=1}^{N} u^{(j)}(\theta, t)\varepsilon^j \ .$$

In the case of the equation (1) we compute the initial approximation up to the order $N = 89$ and for a given $p : q$ resonance (with $p, q \in \mathbf{Z}$), we define a pair of diophantine numbers, trapping the $p : q$ resonance from above and below, as

$$\Gamma(\frac{p}{q}) = \frac{p}{q} - \frac{1}{2+\gamma} \ , \qquad \Delta(\frac{p}{q}) = \frac{p}{q} + \frac{1}{2+\gamma} \ ,$$

where γ is the golden ratio: $\gamma = \frac{\sqrt{5}-1}{2}$. We recall that the astronomical value of the perturbing parameter for the Moon and Mercury amounts, respectively, to $\varepsilon_{Moon} = 3.45 \cdot 10^{-4}$, $\varepsilon_{Merc} = 1.5 \cdot 10^{-4}$. Then we have the following results, whose thresholds are consistent with the astronomical observed values:

Moon–Earth system: Let $e = 0.0549$; there exist the invariant tori with frequencies $\Gamma(\frac{1}{1})$ and $\Delta(\frac{1}{1})$ (trapping the 1:1 resonance) for any $\varepsilon \leq 0.02$. Moreover, there exist the invariant tori with frequencies $\Gamma(\frac{3}{2})$ and $\Delta(\frac{3}{2})$ (trapping the 3:2 resonance) for any $\varepsilon \leq 0.01$.

Mercury–Sun system: Let $e = 0.2056$; there exist the invariant tori with frequencies $\Gamma(\frac{1}{1})$ and $\Delta(\frac{1}{1})$ (trapping the 1:1 resonance) for any $\varepsilon \leq 0.01$. Moreover, there exist the invariant tori with frequencies $\Gamma(\frac{3}{2})$ and $\Delta(\frac{3}{2})$ (trapping the 3:2 resonance) for any $\varepsilon \leq 0.008$.

5 Invariant librational tori

We briefly sketch the strategy adopted in [3] to prove the existence of librational invariant surfaces surrounding the synchronous resonance. Let us start by writing the spin–orbit Hamiltonian (2) as

$$H(y, x, t) = \Big[\frac{y^2}{2} - \varepsilon a \cos(x - t)\Big] - \varepsilon e f_0(x, t) \ .$$

We perform a change of variables, $(y, x) \to (\tilde{y}, \tilde{x})$, in order to center around the 1:1 periodic orbit, obtaining the Hamiltonian function

$$\tilde{H}(\tilde{y}, \tilde{x}, t) = 2\tilde{y}^2 - \varepsilon a \cos \tilde{x} - \varepsilon e \tilde{f}(\tilde{x}, t)$$

for some function $\tilde{f} \equiv \tilde{f}(\tilde{x}, t)$. Next, we expand in Taylor series around the unperturbed equilibrium position and we diagonalize the time–independent quadratic terms, obtaining a harmonic oscillator perturbed by higher order terms. We compute action–angle variables (I,φ) for the harmonic oscillator, obtaining the Hamiltonian

$$H(I,\varphi,t)\equiv\omega I+\varepsilon\overline{h}(I)+\varepsilon R(I,\varphi,t)$$

for some functions $\overline{h}$ and R. Finally, we perform a Birkhoff normal form up to the order k to reduce the Hamiltonian to the form:

$$H_k(I',\varphi',t)=\omega I'+\varepsilon h_k(I')+\varepsilon^{k+1}R_k(I',\varphi',t)$$

for some functions h_k and R_k and where (I',φ') denotes the new set of variables.
For the Moon–Earth system such procedure leads to the construction of librational tori surrounding the synchronous resonance for values of the perturbing parameter less than the real physical value by a factor 5.26; realistic results consistent with the observations are obtained for the Rhea–Saturn system, taking the true eccentricity $e_{Rhea}=0.00098$ and the true oblateness parameter $\varepsilon_{Rhea}=3.45\cdot10^{-3}$.

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