A SECOND EIGENVALUE BOUND FOR THE DIRICHLET SCHRÖDINGER OPERATOR

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Abstract. Let $\lambda_i(\Omega, V)$ be the ith eigenvalue of the Schrödinger operator with Dirichlet boundary conditions on a bounded domain $\Omega \subset \mathbb{R}^n$ and with the positive potential V. Following the spirit of the Payne-Pólya-Weinberger conjecture and under some convexity assumptions on the spherically rearranged potential $V_\star$, we prove that $\lambda_2(\Omega, V) \leq \lambda_2(S_1, V_\star)$. Here S_1 denotes the ball, centered at the origin, that satisfies the condition $\lambda_1(\Omega, V) = \lambda_1(S_1, V_\star)$.

Further we prove under the same convexity assumptions on a spherically symmetric potential V, that $\lambda_2(B_R, V)/\lambda_1(B_R, V)$ decreases when the radius R of the ball B_R increases.

We conclude with several results about the first two eigenvalues of the Laplace operator with respect to a measure of Gaussian or inverted Gaussian density.

1. Introduction

In an earlier publication [3], Ashbaugh and one of us have proven the Payne-Pólya-Weinberger (PPW) conjecture, which states that the first two eigenvalues λ_1, λ_2 of the Dirichlet-Laplacian on a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ obey the bound

$$\lambda_2/\lambda_1 \leq j_{n/2,1}^2/j_{n/2-1,1}^2. \tag{1}$$

Here $j_{\nu,k}$ stands for the kth positive zero of the Bessel function J_ν. Thus the right hand side of (1) is just the ratio of the first two eigenvalues of the Dirichlet-Laplacian on an n-dimensional ball of arbitrary radius. This result is optimal in the sense that equality holds in (1) if and only if Ω is a ball.

The proof of the PPW conjecture has been generalized in several ways. In [2] a corresponding theorem has been established for the Laplacian operator on a domain Ω that is contained in a hemisphere of the n-dimensional sphere $\mathbb{S}^n$. More precisely, it has been shown that $\lambda_2(\Omega) \leq \lambda_2(S_1)$, where S_1 is the n-dimensional geodesic ball in $\mathbb{S}^n$ that has $\lambda_1(\Omega)$ as its first Dirichlet eigenvalue.

A further variant of the PPW conjecture has been considered by Haile. In [11] he compares the second eigenvalue $\lambda_2(\Omega, kr^\alpha)$ of the

R.B. was supported by FONDECYT project # 102-0844. H.L. gratefully acknowledges financial support from DIPUC of the Pontifícia Universidad Católica de Chile and from CONICYT.

Schrödinger operator with the potential $V = kr^\alpha$ $(k > 0, \alpha \geq 2)$ with $\lambda_2(S_1, kr^\alpha)$, where S_1 is the ball, centered at the origin, that satisfies the condition $\lambda_1(\Omega, kr^\alpha) = \lambda_1(S_1, kr^\alpha)$. Here and in the following we denote by $\lambda_i(\Omega, V)$ the ith eigenvalue of the Schrödinger operator $-\Delta + V(\vec{r})$ with Dirichlet boundary conditions on a bounded domain $\Omega \subset \mathbb{R}^n$.

We have to mention a gap in [11], which occurs in the proof of Lemma 3.2. The author claims (and uses) that all derivatives of the function $Z(\theta)$ (which is equal to $T'(\theta)$ where $T(\theta) = 0$) coincide with the derivatives of $T'(\theta)$ in the points where $T(\theta) = 0$. This is not proven and there seems to be no reason why it should be true. The same problem occurs in the proof of Lemma 3.3. In the present paper we will prove a theorem that includes Haile's theorem as a special case and thus remedies the situation.

One very important difference between the original PPW conjecture and the extended problems in [2, 11] is that in the later cases the ratio λ_2/λ_1 is not scaling invariant anymore. While λ_2/λ_1 is the same for any ball in $\mathbb{R}^n$, it is an increasing function of the radius for balls in $\mathbb{S}^n$ [2]. On the other hand, we will see that $\lambda_2(B_R, V)/\lambda_1(B_R, V)$ on the ball B_R is a decreasing function of the radius R, if V has certain convexity properties. This rises the question which is the 'right size' of the comparison ball in the PPW estimate. We will make some remarks on this problem below.

The main objective of the present work is to prove a PPW type result for a Schrödinger operator with a positive potential. We will state the corresponding theorem in the following section. In Section 3 we will transfer our results to the case of a Laplacian operator with respect to a metric of Gaussian or inverted Gaussian measure, the two cases of which are closely related to the harmonic oscillator. The rest of the article will be devoted to the proofs of our results.

2. Main Results

Let $\Omega \subset \mathbb{R}^n$ (with $n \geq 2$) be some bounded domain and $V : \Omega \to \mathbb{R}^+$ some positive potential such that the Schrödinger operator $-\Delta + V$ (subject to Dirichlet boundary conditions) is self-adjoint in $L^2(\Omega)$. We call $\lambda_i(\Omega, V)$ its ith eigenvalue. Further, we denote by $V_\star$ the radially increasing rearrangement of V. Then the following PPW type estimate holds:

Theorem 2.1. *Let $S_1 \subset \mathbb{R}^n$ be a ball centered at the origin and of radius R_1 and let $\tilde{V} : S_1 \to \mathbb{R}^+$ be some radially symmetric positive potential such that $\tilde{V}(r) \leq V_\star(r)$ for all $0 \leq r \leq R_1$ and $\lambda_1(\Omega, V) = \lambda_1(S_1, \tilde{V})$. If $\tilde{V}(r)$ satisfies the conditions*

a) $\tilde{V}(0) = \tilde{V}'(0) = 0$ *and*

b) $\tilde{V}'(r)$ *exists and is increasing and convex,* *then*

$$\lambda_2(\Omega, V) \leq \lambda_2(S_1, \tilde{V}). \tag{2}$$

If V is such that $V_\star$ satisfies the convexity conditions stated in the theorem, the best bound is obtained by choosing $\tilde{V} = V_\star$. In this case the theorem is a typical PPW result and optimal in the sense that equality holds in (2) if Ω is a ball and $V = V_\star$. For a general potential V we still get a non-trivial bound on $\lambda_2(\Omega, V)$ though it is not sharp anymore. To show that our Theorem 2.1 contains Haile's result [11] as a special case, we state the following corollary:

Corollary 2.1. *Let $\tilde{V} : \mathbb{R}^n \to \mathbb{R}^+$ be a radially symmetric positive potential that satisfies the conditions a) and b) of Theorem 2.1 and let $S_1 \subset \mathbb{R}^n$ be the ball (centered at the origin) such that $\lambda_1(\Omega, \tilde{V}) = \lambda_1(S_1, \tilde{V})$. Then*

$$\lambda_2(\Omega, \tilde{V}) \leq \lambda_2(S_1, \tilde{V}).$$

The proof of Theorem 2.1 follows the lines of the proof in [3] and will be presented in Section 5. Let us make a few remarks on the conditions that $\tilde{V}$ has to satisfy. Condition a) is not a very serious restriction, because any bounded potential can be shifted such that $V_\star(0) = 0$. Also $V_\star'(0) = 0$ holds if V is somewhat regular where it takes the value zero. Moreover, our method relies heavily on the fact that

$$\lambda_2(B_R, \tilde{V}) \geq \left(1 + \frac{2}{n}\right) \lambda_1(B_R, \tilde{V}), \tag{3}$$

which is a byproduct of our proof and holds for any ball B_R and any potential $\tilde{V}$ that satisfies the conditions of Theorem 2.1. The conditions a) and b) will be needed to show the above inequality, which is equivalent to $q''(0) \leq 0$ for a function q to be defined in the proof. Numerical studies indicate that b) is somewhat sharp in the sense that, for example, a potential $r^{2-\epsilon}$ (which violates b) only 'slightly') does not satisfy (3) for every R. In this case the statement of Theorem 2.1 may still be true, but the typical scheme of the PPW proof will fail. Furthermore, condition a) and b) will allow us to employ the crucial Baumgartner-Grosse-Martin (BGM) inequality [7, 4]: From a) and b) we see that $V(r) + rV'(r)$ is increasing. Consequently $rV(r)$ is convex, which is just the condition needed to apply the BGM inequality.

As mentioned above, one has to chose carefully the size of the comparison ball in a PPW estimate if λ_2/λ_1 is a non-constant function of the ball's radius. In the case of the Laplacian on $\mathbb{S}^n$, one compares the second eigenvalues on Ω and S_1, the ball that has the same first eigenvalue as Ω. By the Rayleigh-Faber-Krahn (RFK) inequality for $\mathbb{S}^n$ it is clear that $S_1 \subset \Omega^\star$, where $\Omega^\star$ is the spherically symmetric rearrangement of Ω. It has also be shown in [2] that λ_2/λ_1 on a geodesic ball in $\mathbb{S}^n$ is an increasing function of the ball's radius. One can conclude from these two facts that in $\mathbb{S}^n$ an estimate of the type (2) is stronger than the inequality

$$\lambda_2(\Omega)/\lambda_1(\Omega) \leq \lambda_2(\Omega^\star)/\lambda_1(\Omega^\star). \tag{4}$$

It has also been argued in [4] why the situation is different in the hyperbolic space $\mathbb{H}^n$. Here an estimate of the type (4) is not possible, for the following reason: One can show that λ_2/λ_1 on geodesic balls in $\mathbb{H}^n$ is a decreasing function of the radius. Now suppose, for example, that Ω is the ball B_R with very long and thin tentacles attached to it. Then the first and the second eigenvalue of the Laplacian on Ω and B_R are almost the same, while the ratio λ_2/λ_1 on $\Omega^\star$ can be considerably less than on B_R (and thus on Ω). We will prove a PPW inequality of the type $\lambda_2(\Omega) \leq \lambda_2(S_1)$ for $\mathbb{H}^n$ and the monotonicity of λ_2/λ_1 on geodesic balls in a future publication.

To shed light on the question which is the right type of PPW inequality for the Schrödinger operator on Ω, we state

Theorem 2.2. *Let $V : \mathbb{R}^n \to \mathbb{R}^+$ be a spherically symmetric potential that satisfies the conditions of Theorem 2.1, i.e.*

a) $V(0) = V'(0) = 0$ *and*
b) $V'(r)$ *exists and is increasing and convex.*

Then the ratio

$$\frac{\lambda_2(B_R, V)}{\lambda_1(B_R, V)}$$

is a decreasing function of R.

This theorem shows that one can not replace equation (2) in our Theorem 2.1 by an inequality of the type (4), following the same reasoning as in the case of the Laplacian on $\mathbb{H}^n$. Theorem 2.2 will be proven in Section 6.

3. Connection to the Laplacian operator in Gaussian space

Recently, there has been some interest in isoperimetric inequalities in $\mathbb{R}^n$ endowed with a measure of Gaussian ($\mathrm{d}\mu_- = e^{-r^2/2}\,\mathrm{d}^n r$) or inverted Gaussian ($\mathrm{d}\mu_+ = e^{+r^2/2}\,\mathrm{d}^n r$) density. For the Gaussian space it has been known for several years that a classical isoperimetric inequality holds. Yet the ratio of Gaussian perimeter and Gaussian measure is minimized by half-spaces instead of spherical domains [9]. The 'inverted Gaussian' case, i.e., $\mathbb{R}^n$ with the measure μ_+, is more similar to the Euclidean case: It has been shown recently that a classical isoperimetric inequality holds and that the minimizers are balls centered at the origin [15].

We consider the Dirichlet-Laplacians $-\Delta_\pm$ on $L^2(\Omega, \mathrm{d}\mu_\pm)$, where $\Omega \subsetneq \mathbb{R}^n$ is a domain of finite measure $\mathrm{d}\mu_\pm(\Omega)$. These two operators are defined by their quadratic forms

$$h_\pm[\Psi] = \int_\Omega |\nabla\Psi(\vec{r})|^2 \,\mathrm{d}\mu_\pm, \quad \Psi \in W_0^{1,2}(\Omega,\, \mathrm{d}\mu_\pm). \tag{5}$$

The eigenfunctions $\Psi_i^\pm$ and eigenvalues $\lambda_i^\pm(\Omega)$ in question are determined by the the differential equation

$$-\sum_{k=1}^n \frac{\partial}{\partial r_k}\left(e^{\pm r^2}\frac{\partial \Psi_i^\pm}{\partial r_k}\right) = \lambda_i^\pm(\Omega) e^{\pm r^2}\Psi_i^\pm(\vec{r}). \tag{6}$$

There is a tight connection between the operators $-\Delta_\pm$ on a domain Ω and the harmonic oscillator $-\Delta + r^2$ restricted to Ω. Their eigenfunctions and eigenvalues are related by [6]

$$\begin{aligned} \Psi_i^\pm(\vec{r}) &= \Psi_i(\vec{r}) \cdot e^{\mp r^2/2} \quad \text{and} \\ \lambda_i^\pm(\Omega) &= \lambda_i(\Omega, r^2) \pm n, \end{aligned} \tag{7}$$

denoting by Ψ_i the Dirichlet eigenfunctions of $-\Delta + r^2$ on Ω.

There is an equivalent of the RFK inequality in Gaussian space [6] stating that $\lambda_1^-(\Omega)$ is minimized for given $\mu_-(\Omega)$ if Ω is a half-space. The corresponding fact for the 'inverted' Gaussian space is that $\lambda_1^+(\Omega)$ is minimized for given $\mu_+(\Omega)$ by the ball centered at the origin. It can be seen by the RFK inequality for Schrödinger operators [14] in combination with (7).

Concerning the second eigenvalue, we will now show what our results from Section 2 imply for the operators $-\Delta_\pm$. We state

Theorem 3.1. *For the operator $-\Delta_+$ on a ball B_R of radius R (centered at the origin) the ratio $\lambda_2^+(B_R)/\lambda_1^+(B_R)$ is a strictly decreasing function of R.*

In Section 7 we will derive Theorem 3.1 from Theorem 2.2 in a purely algebraic way using only the relation (7). Repeating the argument for $\mathbb{H}^n$ from the previous section, we see that by Theorem 3.1 the best PPW result we can expect to get is

Theorem 3.2. *Be S_1 the ball (centered at the origin) that satisfies the condition $\lambda_1^+(S_1) = \lambda_1^+(\Omega)$. Then*

$$\lambda_2^+(\Omega) \leq \lambda_2^+(S_1).$$

Theorem 3.2 follows immediately from Theorem 2.1 and (7). In the same way we easily get the corresponding version for $-\Delta_-$:

Theorem 3.3. *Be S_1 the ball (centered at the origin) that satisfies the condition $\lambda_1^-(S_1) = \lambda_1^-(\Omega)$. Then*

$$\lambda_2^-(\Omega) \leq \lambda_2^-(S_1).$$

Yet in this case it is not clear anymore whether S_1 is the optimal comparison ball: First, in contrast to the 'inverted' Gaussian case the ratio $\lambda_2^-(B_R)/\lambda_1^-(B_R)$ is not a decreasing function of R anymore. This can be seen by comparing the values of $\lambda_2^-(B_R)/\lambda_1^-(B_R)$ for $R \to 0$ and $R \to \infty$: For small R the ratio is close to the Euclidean value (≈ 2.539) while for large R it approaches infinity (by (7)). Second, the RFK inequality in Gaussian space states that $\lambda_1^-(\Omega)$ is minimized by half-spaces, not circles. This means that for general Ω we don't know whether $\Omega^\star$ is bigger or smaller than S_1. For these differences it remains unclear what is the most natural way to generalize the PPW conjecture to Gaussian space.

4. A monotonicity lemma

In our proof of Theorem 2.1 we will need

Lemma 4.1 (Monotonicity of g and B)**.** *Let $\tilde{V}$, S_1 and R_1 be as in Theorem 2.1 and call $z_1(r)$ and $z_2(r)$ the radial parts (both chosen positive) of the first two Dirichlet eigenfunctions of $-\Delta + \tilde{V}$ on S_1. Set*

$$\begin{aligned} g(r) &= \frac{z_2(r)}{z_1(r)} \quad \textit{and} \\ B(r) &= g'(r)^2 + (n-1)\frac{g(r)^2}{r^2} \end{aligned}$$

for $0 < r < R_1$. Then $g(r)$ is increasing on $(0, R_1)$ and $B(r)$ is decreasing on $(0, R_1)$.

Proof. [11, 1] In this section we abbreviate $\lambda_i = \lambda_i(S_1, \tilde{V})$. The functions z_1 and z_2 are solutions of the differential equations

$$\begin{aligned} -z_1'' - \frac{n-1}{r}z_1' + \left(\tilde{V} - \lambda_1\right) z_1 &= 0, \\ -z_2'' - \frac{n-1}{r}z_2' + \left(\frac{n-1}{r^2} + \tilde{V} - \lambda_2\right) z_2 &= 0 \end{aligned} \tag{8}$$

with the boundary conditions

$$z_1'(0) = 0, \quad z_1(R_1) = 0, \quad z_2(0) = 0, \quad z_2(R_1) = 0. \tag{9}$$

This is assured by the BGM inequality [1, 7], which is applicable because $r\tilde{V}$ is convex. As in [1] we define the function

$$q(r) := \frac{rg'(r)}{g(r)}.$$

Proving the lemma is thus reduced to showing that $0 < q(r) < 1$ and $q'(r) < 0$ for $r \in [0, R]$. Using the definition of g and the equations (8), one can show that $q(r)$ is a solution of the Riccati differential equation

$$q' = (\lambda_1 - \lambda_2)r + \frac{(1-q)(q+n-1)}{r} - 2q\frac{z_1'}{z_1}. \tag{10}$$

It is straightforward to establish the boundary behavior

$$q(0) = 1, \quad q'(0) = 0, \quad q''(0) = \frac{2}{n}\left(\left(1 + \frac{2}{n}\right)\lambda_1 - \lambda_2\right)$$

and

$$q(R_1) = 0.$$

Fact 4.1. *For $0 \leq r \leq R$ we have $q(r) \geq 0$.*

Proof. Assume the contrary. Then there exist two points $0 < r_1 < r_2 \leq R_1$ such that $q(r_1) = q(r_2) = 0$ but $q'(r_1) \leq 0$ and $q'(r_2) \geq 0$. If $r_2 < R_1$ then the Riccati equation (10) yields

$$0 \geq q'(r_1) = (\lambda_1 - \lambda_2) r_1 + \frac{n-1}{r_1} > (\lambda_1 - \lambda_2) r_2 + \frac{n-1}{r_2} = q'(r_2) \geq 0,$$

which is a contradiction. If $r_2 = R_1$ then we get a contradiction in a similar way by

$$0 \geq q'(r_1) = (\lambda_1 - \lambda_2) r_1 + \frac{n-1}{r_1} > (\lambda_1 - \lambda_2) R_1 + \frac{n-1}{R_1} = 3q'(R_1) \geq 0.$$

□

In the following we will analyze the behavior of q' according to (10), considering r and q as two independent variables. For the sake of a compact notation we will make use of the following abbreviations:

$$\begin{array}{rclrcl} p(r) & = & z_1'(r)/z_1(r) & N_y & = & y^2 - n + 1 \\ \nu & = & n-2 & M_y & = & N_y^2/(2y) - \nu^2 y/2 \\ E & = & \lambda_2 - \lambda_1 & Q_y & = & 2y\lambda_1 + EN_y y^{-1} - 2E \end{array}$$

We further define the function

$$T(r,y) := -2p(r)y - \frac{\nu y + N_y}{r} - Er. \tag{11}$$

Then we can write (10) as

$$q'(r) = T(r, q(r))$$

The definition of $T(r,y)$ allows us to analyze the Riccati equation for q' considering r and $q(r)$ as independent variables. For r going to zero, p is $\mathcal{O}(r)$ and thus

$$T(r,y) = \frac{1}{r}\left((\nu + 1 + y)(1-y)\right) + \mathcal{O}(r) \quad \text{for } y \text{ fixed.}$$

Consequently,

$$\begin{array}{rcll} \lim_{r\to 0} T(r,y) & = & +\infty & \text{for } 0 \leq y < 1 \text{ fixed,} \\ \lim_{r\to 0} T(r,y) & = & 0 & \text{for } y = 1 \text{ and} \\ \lim_{r\to 0} T(r,y) & = & -\infty & \text{for } y > 1 \text{ fixed.} \end{array}$$

For r approaching R_1, the function $p(r)$ goes to minus infinity, while all other terms in (11) are bounded. Therefore

$$\lim_{r\to R_1} T(r,y) = +\infty \quad \text{for } y > 0 \text{ fixed.}$$

The partial derivative of $T(r,y)$ with respect to r is given by

$$T' = \frac{\partial}{\partial r} T(r,y) = -2yp' + \frac{\nu y}{r^2} + \frac{N_y}{r^2} - E. \tag{12}$$

In the points (r,y) where $T(r,y)=0$ we have, by (11),

$$p|_{T=0} = -\frac{\nu}{2r} - \frac{N_y}{2yr} - \frac{Er}{2y}. \tag{13}$$

From (8) we get the Riccati equation

$$p' + p^2 + \frac{\nu+1}{r}p + \lambda_1 - \tilde{V} = 0. \tag{14}$$

Putting (13) into (14) and the result into (12) yields

$$T'|_{T=0} = \frac{M_y}{r^2} + \frac{E^2r^2}{2y} + Q_y - 2y\tilde{V}. \tag{15}$$

If we define the function

$$Z_y(r) := \frac{M_y}{r^2} + \frac{E^2r^2}{2y} + Q_y - 2y\tilde{V},$$

it is clear that $T'(r,y) = Z_y(r)$ for any r, y with $T(r,y)=0$. The behavior of $Z_y(r)$ at $r=0$ is determined by M_y. From the definition of M_y we get

$$yM_y = \frac{1}{2}(y^2-1)\cdot[(y-1)-(n-2)]\cdot[(y+1)+(n-2)]. \tag{16}$$

This implies that

$$\begin{aligned} M_y &> 0 \quad \text{for } 0<y<1, \\ M_1 &= 0. \end{aligned}$$

and therefore

$$\lim_{r\to 0} Z_y(r) = \infty \quad \text{for } 0<y<1.$$

Fact 4.2. *There is some $r_0>0$ such that $q(r)\leq 1$ for $0<r<r_0$ and $q(r_0)<1$.*

Proof. Suppose the contrary, i.e., $q(r)$ first increases away from $r=0$. Then, because $q(0)=1$ and $q(R)=0$ and because q is continuous and differentiable, we can find two points $r_1<r_2$ such that $\hat{q}:=q(r_1)=q(r_2)>1$ and $q'(r_1)>0>q'(r_2)$. Even more, we can chose r_1 and r_2 such that $\hat{q}$ is arbitrarily close to one. Writing $\hat{q}=1+\epsilon$ with $\epsilon>0$, we can calculate from the definition of Q_y that

$$Q_{1+\epsilon} = Q_1 + \epsilon n\left(\lambda_2 - (1-2/n)\,\lambda_1\right) + \mathcal{O}(\epsilon^2).$$

The term in brackets can be estimated by

$$\lambda_2 - (1-2/n)\lambda_1 > \lambda_2 - \lambda_1 > 0.$$

We can also assume that $Q_1\geq 0$, because otherwise $q''(0)=\frac{2}{n^2}Q_1<0$ and Fact 4.2 is immediately true. Thus, choosing r_1 and r_2 such that ϵ is sufficiently small, we can make sure that $Q_{\hat{q}}>0$. We further note, that in view of (16) the constant $M_{\hat{q}}$ can be positive or negative (depending on n), but not zero because $1<\hat{q}<2$.

Now consider the function $T(r,\hat{q})$. We have $T(r_1,\hat{q}) > 0 > T(r_2,\hat{q})$ and the boundary behavior $T(0,\hat{q}) = -\infty$ and $T(R_1,\hat{q}) = +\infty$. Thus $T(r,\hat{q})$ changes its sign at least thrice on $[0,R_1]$. Consequently, we can find three points $0 < \hat{r}_1 < \hat{r}_2 < \hat{r}_3 < R_1$ such that

$$Z_{\hat{q}}(\hat{r}_1) \geq 0, \quad Z_{\hat{q}}(\hat{r}_2) \leq 0, \quad Z_{\hat{q}}(\hat{r}_3) \geq 0. \tag{17}$$

Let us define

$$h(r) = \frac{E^2 r^2}{2\hat{q}} - 2\hat{q}\tilde{V}(r).$$

Then

$$Z_{\hat{q}}(r) = \frac{M_{\hat{q}}}{r^2} + Q_{\hat{q}} + h(r). \tag{18}$$

By condition b) on $\tilde{V}$, the function $h'(r)$ is concave. Also $h(0) = h'(0) = 0$. We conclude that if $h'(r_0) < 0$ or $h(r_0) < 0$ for some $r_0 > 0$, then $h'(r)$ is negative and decreasing for all $r > r_0$. We will now show that $Z_{\hat{q}}$ cannot have the properties (17), a contradiction that proves Fact 4.2:

Case 1: Assume $M_{\hat{q}} > 0$. Then from $Z_{\hat{q}}(\hat{r}_2) \leq 0$ we see that

$$-h(\hat{r}_2) \geq \frac{M_{\hat{q}}}{\hat{r}_2^2} + Q_{\hat{q}} > 0.$$

By what has been said above about $h(r)$, we conclude that $-h(r)$ is a strictly increasing function on $[\hat{r}_2,\hat{r}_3]$. Therefore

$$-h(\hat{r}_3) > -h(\hat{r}_2) \geq \frac{M_{\hat{q}}}{\hat{r}_2^2} + Q_{\hat{q}} > \frac{M_{\hat{q}}}{\hat{r}_3^2} + Q_{\hat{q}},$$

such that $Z_{\hat{q}}(\hat{r}_3) < 0$, contradicting (17).

Case 2: Assume $M_{\hat{q}} < 0$. Then from $Z_{\hat{q}}(\hat{r}_1) \geq 0 \geq Z_{\hat{q}}(\hat{r}_2)$ follows that $Z'_{\hat{q}}(\hat{r}) \leq 0$ for some $\hat{r} \in [\hat{r}_1,\hat{r}_2]$. In view of (18) we have $h'(\hat{r}) < 0$. But this means by our above concavity argument that $h'(r)$ is decreasing and thus $h'(r) < 0$ for all $r > \hat{r}$. Then $Z'_{\hat{q}}$ is strictly decreasing for $r \geq \hat{r}$. Together with $Z_{\hat{q}}(\hat{r}_2) \leq 0$ and $Z'_{\hat{q}}(\hat{r}) \leq 0$ this implies that $Z_{\hat{q}}(\hat{r}_3) < 0$, a contradiction to (17). $\square$

Fact 4.3. *For all $0 \leq r \leq R_1$ the inequality $q'(r) \leq 0$ holds.*

Proof. Assume the contrary. Then there are three points $r_1 < r_2 < r_3$ in $(0,R_1)$ with $0 < \hat{q} := q(r_1) = q(r_2) = q(r_3) < 1$ and $q'(r_1) < 0$, $q'(r_2) > 0$, $q'(r_3) < 0$. Consider the function $T(r,\hat{q})$, which is equal to $q'(r)$ at r_1, r_2, r_3. Taking into account its boundary behavior at $r = 0$ and $r = R_1$, it is clear that $T(r,\hat{q})$ must have at least the sign changes positive-negative-positive-negative-positive. Thus $T(r,\hat{q})$ has at least four zeros $\hat{r}_1 < \hat{r}_2 < \hat{r}_3 < \hat{r}_4$ with the properties

$$Z_{\hat{q}}(\hat{r}_1) \leq 0, \quad Z_{\hat{q}}(\hat{r}_2) \geq 0, \quad Z_{\hat{q}}(\hat{r}_3) \leq 0, \quad Z_{\hat{q}}(\hat{r}_4) \geq 0.$$

We also know that $Z_{\hat{q}}(0) = +\infty$. To satisfy all these requirements, $Z_{\hat{q}}$ must either have at least three extremal points where $Z'_{\hat{q}}$ crosses zero or $Z_{\hat{q}}$ must vanish on a finite interval. But we have

$$Z'_{\hat{q}}(r) = -\frac{2M_{\hat{q}}}{r^3} + \frac{E^2 r}{\hat{q}} - 2\hat{q}\tilde{V}'(r),$$

which is a strictly concave function (recall $M_{\hat{q}} > 0$ for $0 < \hat{q} < 1$). A strictly concave function can only cross zero twice and not be zero on a finite interval, which is a contradiction that proves Fact 4.3. $\square$

Altogether we have shown that $0 < q(r) < 1$ and $q'(r) \leq 0$ for all $r \in [0, R]$, proving Lemma 4.1. $\square$

5. Proof of Theorem 2.1

Proof of Theorem 2.1. We start from the basic gap inequality

$$\lambda_2(\Omega, V) - \lambda_1(\Omega, V) \leq \frac{\int_\Omega |\nabla P|^2 u_1^2 \, \mathrm{d}^n r}{\int_\Omega P^2 u_1^2 \, \mathrm{d}^n r}, \tag{19}$$

where u_1 is the first Dirichlet eigenfunction of $-\Delta + V$ on Ω and P is a suitable test function that satisfies the condition $\int_\Omega P u_1^2 \, \mathrm{d}^n r = 0$. We set

$$P_i(\vec{r}) = g(r)\frac{r_i}{r} \quad \text{for } i = 1, 2, ..., n, \tag{20}$$

where

$$g(r) = \begin{cases} \frac{z_2(r)}{z_1(r)} & \text{for } r < R_1 \\ \lim\limits_{t \uparrow R_1} g(t) & \text{for } r \geq R_1. \end{cases} \tag{21}$$

Here z_1 and z_2 are the radial parts (both chosen positive) of the first two eigenfunctions of $-\Delta + \tilde{V}$ on S_1. More precisely, $z_2(r) r_i r^{-1}$ for $i = 1, \ldots, n$ is a basis of the space of second eigenfunctions. It follows from the convexity of $r\tilde{V}$ and the BGM inequality [1, 7] that the second eigenfunctions can be written in that way.

According to an argument in [3] one can always chose the origin of the coordinate system such that $\int_\Omega P_i u_1^2 \, \mathrm{d}^n r = 0$ is satisfied for all i. Putting the functions P_i into (19) and summing over all i yields

$$\lambda_2(\Omega, V) - \lambda_1(\Omega, V) \leq \frac{\int_\Omega B(r) u_1^2 \, \mathrm{d}^n r}{\int_\Omega g(r)^2 u_1^2 \, \mathrm{d}^n r} \tag{22}$$

with

$$B(r) = g'(r)^2 + (n-1)\frac{g(r)^2}{r^2}.$$

By Lemma 4.1 we know that B is a decreasing and g an increasing function of r. Thus, denoting by $u_1^\star$ the spherically decreasing rearrangement of u_1 with respect to the origin, we have

$$\begin{aligned}(23)\qquad \int_\Omega B(r)u_1^2\,\mathrm{d}^n r &\leq \int_{\Omega^\star} B^\star(r)\,u_1^{\star 2}\,\mathrm{d}^n r \\ &\leq \int_{\Omega^\star} B(r)\,u_1^{\star 2}\,\mathrm{d}^n r \leq \int_{S_1} B(r)\,z_1^2\,\mathrm{d}^n r\end{aligned}$$

and

$$\begin{aligned}(24)\qquad \int_\Omega g(r)^2 u_1^2\,\mathrm{d}^n r &\geq \int_{\Omega^\star} g_\star(r)^2\,u_1^{\star 2}\,\mathrm{d}^n r \\ &\geq \int_{\Omega^\star} g(r)^2\,u_1^{\star 2}\,\mathrm{d}^n r \geq \int_{S_1} g(r)^2\,z_1^2\,\mathrm{d}^n r\end{aligned}$$

In each of the above chains of inequalities the first step follows from general properties of rearrangements and the second from the monotonicity properties of g and B. The third step is justified by a comparison result that we state below and the monotonicity of g and B again. Putting (23) and (24) into (22) we get

$$\lambda_2(\Omega,V)-\lambda_1(\Omega,V)\leq\frac{\int_{S_1}B(r)\,z^2\,\mathrm{d}^n r}{\int_{S_1}g(r)^2\,z^2\,\mathrm{d}^n r}=\lambda_2(S_1,\tilde V)-\lambda_1(S_1,\tilde V).$$

Keeping in mind that $\lambda_1(\Omega,V)=\lambda_1(S_1,\tilde V)$, Theorem 2.1 is proven by this last inequality. □

Lemma 5.1 (Chiti Comparison result)**.** *Let $u_1^\star$ be the radially decreasing rearrangement of the first eigenfunction of $-\Delta+V$ on Ω and z_1 the first eigenfunction of $-\Delta+\tilde V$ on S_1. Assume both functions to be positive and normalized in $L^2(\Omega^\star)$. Then there exists an r_0 such that*

$$\begin{aligned} u_1^\star(r) &\leq z_1(r) \quad \text{for } r\leq r_0 \text{ and}\\ u_1^\star(r) &\geq z_1(r) \quad \text{for } r_0< r\leq R_1.\end{aligned}$$

Proof. By a version of the RFK inequality for Schrödinger operators [14] and by domain monotonicity of the first eigenvalue it is clear that $S_1\subset\Omega^\star$. This is why we can view $z_1(r)$ as a function in $L^2(\Omega^\star)$, setting $z_1(r)=0$ for $r>R_1$.

Both $u_1^\star$ and z_1 are positive and spherically symmetric. Moreover, $u_1^\star(r)$ and $z_1(r)$ are decreasing functions of r. For $u_1^\star$ this is clear by definition of the rearrangement. For z_1 it follows from a simple comparison argument using $z_1^\star$ as a test function in the Rayleigh quotient for λ_1. (Here and in the sequel we write short-hand $\lambda_1=\lambda_1(\Omega,V)=\lambda_1(S_1,\tilde V)$.)

We introduce a change of variables via $s=C_n r^n$ and write $u_1^\#(s)\equiv u_1^\star(r)$, $z_1^\#(s)\equiv z_1(r)$ and $\tilde V_\#(s)\equiv\tilde V(r)$.

Fact 5.1. *For the functions $u_1^\#(s)$ and $z_1^\#(s)$ we have*

$$-\frac{\mathrm{d}u_1^\#}{\mathrm{d}s} \leq n^{-2}C_n^{-2/n}s^{n/2-2}\int_0^s (\lambda_1 - \tilde{V}_\#(w))\,u_1^\#(w)\,\mathrm{d}w, \tag{25}$$

$$-\frac{\mathrm{d}z_1^\#}{\mathrm{d}s} = n^{-2}C_n^{-2/n}s^{n/2-2}\int_0^s (\lambda_1 - \tilde{V}_\#(w))\,z_1^\#(w)\,\mathrm{d}w. \tag{26}$$

Proof. We integrate both sides of $-\Delta u_1 + Vu_1 = \lambda_1 u_1$ over the level set $\Omega_t := \{\vec{r} \in \Omega : u_1(\vec{r}) > t\}$ and use Gauss' Divergence Theorem to obtain

$$\int_{\partial\Omega_t} |\nabla u_1| H_{n-1}(\mathrm{d}r) = \int_{\Omega_t} (\lambda_1 - V(\vec{r}))\,u_1(\vec{r})\,\mathrm{d}^n r, \tag{27}$$

where $\partial\Omega_t = \{\vec{r} \in \Omega : u_1(\vec{r}) = t\}$. Now we define the distribution function $\mu(t) = |\Omega_t|$. Using the coarea formula, the Cauchy-Schwarz inequality and the classical isoperimetric inequality, Talenti derives ([18], p.709, eq. (32))

$$\int_{\partial\Omega_t} |\nabla u_1| H_{n-1}(\mathrm{d}r) \geq -n^2 C_n^{2/n}\frac{\mu(t)^{2-2/n}}{\mu'(t)}. \tag{28}$$

The left sides of (27) and (28) are the same, thus

$$\begin{aligned}
-n^2 C_n^{2/n}\frac{\mu(t)^{2-2/n}}{\mu'(t)} &\leq \int_{\Omega_t} (\lambda_1 - V(\vec{r}))\,u_1(\vec{r})\,\mathrm{d}^n r \\
&\leq \int_{\Omega_t^\star} (\lambda_1 - V_\star(\vec{r}))\,u_1^\star(\vec{r})\,\mathrm{d}^n r \\
&\leq \int_{\Omega_t^\star} (\lambda_1 - \tilde{V}(\vec{r}))\,u_1^\star(\vec{r})\,\mathrm{d}^n r \\
&= \int_0^{(\mu(t)/C_n)^{1/n}} nC_n r^{n-1}(\lambda_1 - \tilde{V}(r))u_1^\star(r)\,\mathrm{d}r.
\end{aligned}$$

Now we perform the change of variables $r \to s$ on the right hand side of the above chain of inequalities. We also chose t to be $u_1^\#(s)$. Using the fact that $u_1^\#$ and μ are essentially inverse functions to one another, this means that $\mu(t) = s$ and $\mu'(t)^{-1} = (u_1^\#)'(s)$. The result is (25). Equation (26) is proven analogously. □

Fact 5.1 enables us to prove Lemma 5.1. We have $u_1^\#(|S_1|) > z_1^\#(|S_1|) = 0$. Being equally normalized, $u_1^\star$ and z_1 must have at least one intersection on $[0, R]$. Thus $u_1^\#$ and $z_1^\#$ have at least one intersection on $[0, |S_1|]$. Now assume that they intersect at least twice. Then there is an interval $[s_1, s_2] \subset [0, |S_1|]$ such that $u_1^\#(s) > z^\#(s)$ for $s \in (s_1, s_2)$, $u_1^\#(s_2) = z_1^\#(s_2)$ and either $u_1^\#(s_1) = z_1^\#(s_1)$ or $s_1 = 0$. There is also an interval $[s_3, s_4] \subset [s_2, |S_1|]$ with $u_1^\#(s) < z_1^\#(s)$ for $s \in (s_3, s_4)$, $u_1^\#(s_3) = z_1^\#(s_3)$ and $u_1^\#(s_4) = z_1^\#(s_4)$. Be further $\tilde{s}$ the point where $\tilde{V}_{\#}(s) - \lambda_1(S_1, \tilde{V})$ crosses zero (Set $\tilde{s} = |S_1|$ if $\tilde{V}_{\#}(s) - \lambda_1$ doesn't cross zero on $[0, |S_1|]$). To keep our notation compact we will write

$$I_a^b[u] = \int_a^b (\lambda_1 - \tilde{V}_{\#}(w))\, u(w)\, \mathrm{d}w.$$

Case 1: Assume $\tilde{s} \geq s_2$. Then $\tilde{V}_{\#}(s) - \lambda_1(S_1, \tilde{V})$ is negative for $s < s_2$. Set

$$v(s) = \begin{cases} u_1^{\#}(s) & \text{on } [0, s_1] \text{ if } I_0^{s_1}[u_1^{\#}] > I_0^{s_1}[z_1^{\#}], \\ z_1^{\#}(s) & \text{on } [0, s_1] \text{ if } I_0^{s_1}[u_1^{\#}] \leq I_0^{s_1}[z_1^{\#}], \\ u_1^{\#}(s) & \text{on } [s_1, s_2] \\ z_1^{\#}(s) & \text{on } [s_2, |S_1|] \end{cases}$$

Using Fact 5.1, one can check that then $v(s)$ fulfills the inequality

$$-\frac{\mathrm{d}v}{\mathrm{d}s} \leq n^{-2} C_n^{-2/n} s^{n/2-2} \int_0^s (\lambda_1 - \tilde{V}_{\#}(s)) v(w)\, \mathrm{d}w. \tag{29}$$

Case 2: Assume $\tilde{s} < s_2$. Then $\tilde{V}_{\#}(s) - \lambda_1(S_1, \tilde{V})$ is positive for $s \geq s_3$. Set

$$v(s) = \begin{cases} u_1^{\#}(s) & \text{on } [0, s_3] \text{ if } I_0^{s_3}[u_1^{\#}] > I_0^{s_3}[z_1^{\#}], \\ z_1^{\#}(s) & \text{on } [0, s_3] \text{ if } I_0^{s_3}[u_1^{\#}] \leq I_0^{s_3}[z_1^{\#}], \\ u_1^{\#}(s) & \text{on } [s_3, s_4] \\ z_1^{\#}(s) & \text{on } [s_4, |S_1|] \end{cases}$$

Again using Fact 5.1, one can check that also in this case $v(s)$ fulfills the inequality (29).

Now define the test function

$$\Psi(\vec{r}) = v(C_n r^n) = v(s).$$

Then we use the Rayleigh characterization of λ_1, equation (29) and integration by parts to calculate

$$\begin{aligned} \lambda_1 \int_{S_1} \Psi(\vec{r})^2\, \mathrm{d}^n r &< \int_{S_1} \left(|\nabla \Psi|^2 + \tilde{V}(\vec{r}) \Psi^2 \right) \mathrm{d}^n r \\ &= \int_0^{|S_1|} \left(v'(s)^2 n^2 s^{2-2/n} C_n^{2/n} + \tilde{V}_{\#}(s) v^2(s) \right) \mathrm{d}s \\ &\leq \int_0^{|S_1|} \left(-v'(s) \int_0^s (\lambda_1 - \tilde{V}_{\#}(w)) v(w)\, \mathrm{d}w + \tilde{V}_{\#}(s) v^2(s) \right) \mathrm{d}s \\ &= \int_0^{|S_1|} \left(v(s)(\lambda_1 - \tilde{V}_{\#}(s)) v(s) + \tilde{V}_{\#}(s) v^2(s) \right) \mathrm{d}s \\ &= \lambda_1 \int_{S_1} \Psi(\vec{r})^2\, \mathrm{d}^n r. \end{aligned}$$

This is a contradiction to our original assumption that $u_1^{\#}(r)$ and $z_1^{\#}(r)$ have more than one intersection, thus proving Lemma 5.1. $\square$

6. Proof of Theorem 2.2

Proof of Theorem 2.2. The first eigenfunction of $-\Delta+V$ on B_R is radially symmetric and will be called $z_1(r)$. Further, a standard separation of variables and the Baumgartner-Grosse-Martin [7, 4] inequality imply that we can write a basis of the space of second eigenfunctions in the form $z_2(r)\cdot r_i\cdot r^{-1}$. The radial parts z_1 and z_2 of the first and the second eigenfunction, which we assume to be positive, solve the differential equations

$$\begin{aligned} -z_1''(r)-\frac{n-1}{r}z_1'(r)+(V(r)-\lambda_1)\,z_1(r) &= 0, \\ -z_2''(r)-\frac{n-1}{r}z_2'(r)+\left(\frac{n-1}{r^2}+V(r)-\lambda_2\right)z_2(r) &= 0 \end{aligned} \tag{30}$$

with the boundary conditions

$$z_1'(0)=0, \quad z_1(R)=0, \quad z_2(0)=0, \quad z_2(R)=0. \tag{31}$$

We define the rescaled functions $\tilde z_{1/2}(r)=z_{1/2}(\beta r)$. Putting βr (with $\beta>0$) instead of r into the equations (30) and multiplying by β^2 yields the rescaled equations

$$\begin{aligned} -\tilde z_1''(r)-\frac{n-1}{r}\tilde z_1'(r)+\left(\beta^2V(\beta r)-\beta^2\lambda_1\right)\tilde z_1(r) &= 0, \\ -\tilde z_2''(r)-\frac{n-1}{r}\tilde z_2'(r)+\left(\frac{n-1}{r^2}+\beta^2V(\beta r)-\beta^2\lambda_2\right)\tilde z_2(r) &= 0. \end{aligned}$$

We conclude that $\tilde z_1$ and $\tilde z_2$ are the radial parts of the first two eigenfunctions of $-\Delta+\beta^2V(\beta r)$ on $B_{R/\beta}$ to the eigenvalues $\beta^2\lambda_1$ and $\beta^2\lambda_2$. Consequently, if we replace R by R/β and $V(r)$ by $\beta^2V(\beta r)$, then the ratio λ_2/λ_1 doesn't change.

For the rest of this section we shall write $\lambda_{1/2}(R,V)$ instead of $\lambda_{1/2}(B_R,V)$. We also fix two radii $0<R_1<R_2$ and let $\rho(\beta)$ for $\beta>1$ be the function defined implicitly by

$$\lambda_1(\rho(\beta),V(r))=\lambda_1(R_2/\beta,\beta^2V(\beta r)). \tag{32}$$

Then we have $\rho(1)=R_2$. By domain monotonicity of λ_1 and because $V(r)$ is increasing and positive we see that the right hand side of (32) is increasing in β. Therefore, again by domain monotonicity, $\rho(\beta)$ must be decreasing in β. One can also check that $\rho(\beta)$ is a continuous function and that $\rho(\beta)$ goes to zero for $\beta\to\infty$. Thus we can find $\beta_0>1$ such that $\rho(\beta_0)=R_1$. Then we can apply Theorem 2.1, with B_{R_2/β_0} for Ω and $B_{\rho(\beta_0)}$ for S_1, as well as $\beta_0^2V(\beta_0 r)$ for V and $V(r)$ for $\tilde V$, to get

$$\lambda_2(R_2/\beta_0,\beta_0^2V(\beta_0 r)\le\lambda_2(\rho(\beta_0),V(r))=\lambda_2(R_1,V(r)). \tag{33}$$

But by what has been said above about the scaling properties of the problem, we have

$$\frac{\lambda_2(R_2/\beta_0, \beta_0^2 V(\beta_0 r))}{\lambda_1(R_2/\beta_0, \beta_0^2 V(\beta_0 r))} = \frac{\lambda_2(R_2, V(r))}{\lambda_1(R_2, V(r))}. \tag{34}$$

Combining (32) for $\beta = \beta_0$, (33) and(34), we get

$$\frac{\lambda_2(R_1, V(r))}{\lambda_1(R_1, V(r))} \geq \frac{\lambda_2(R_2, V(r))}{\lambda_1(R_2, V(r))}. \tag{35}$$

Because R_1 and R_2 were chosen arbitrarily, this proves Theorem 2.2. □

7. Proof of Theorem 3.1

Before we prove Theorem 3.1 we need to state the following technical Lemma:

Lemma 7.1. *Be $a, b, c, d > 0$ with $a \geq b$, $d \geq b$ and $\frac{a}{b} < \frac{c}{d}$. Then*

$$\frac{a+x}{b+x} < \frac{c+x}{d+x} \tag{36}$$

holds for any $x > 0$.

Proof. Define the function

$$f(x) := \frac{c+x}{d+x} - \frac{a+x}{b+x}.$$

then $f(0) > 0$. A straightforward calculation shows that f has exactly one zero at

$$x_0 = -\frac{bc - ad}{b + c - a - d}.$$

The numerator $bc - ad$ in the expression for x_0 is positive because of the condition $\frac{a}{b} < \frac{c}{d}$. For the denominator we get

$$b + c - a - d > c + b - \frac{bc}{d} - d = \frac{(d-b)(c-d)}{d} \geq 0.$$

This means that $x_0 < 0$, such that $f(x) > 0$ for all $x > 0$. □

Proof of Theorem 3.1. Choose some $x > 0$. From Theorem 2.2 we know that

$$\frac{\lambda_2(B_{R+x}, r^2)}{\lambda_1(B_{R+x}, r^2)} < \frac{\lambda_2(B_R, r^2)}{\lambda_1(B_R, r^2)} \quad \text{for } x > 0.$$

Moreover, $\lambda_1(B_R, r^2) \geq \lambda_1(B_{R+x}, r^2)$ and $\lambda_2(B_{R+x}, r^2) > \lambda_1(B_{R+x}, r^2)$. Thus we can apply first (7), then Lemma 7.1 and then (7) again, to get

$$\frac{\lambda_2^+(B_{R+x})}{\lambda_1^+(B_{R+x})} = \frac{\lambda_2(B_{R+x}, r^2) + n}{\lambda_1(B_{R+x}, r^2) + n} < \frac{\lambda_2(B_R, r^2) + n}{\lambda_1(B_R, r^2) + n} = \frac{\lambda_2^+(B_R)}{\lambda_1^+(B_R)}.$$

□

References

[1] M.S. Ashbaugh, R.D. Benguria: *A second proof of the Payne-Pólya-Weinberger conjeture*, Commun. Math. Phys. **147** (1992) 181–190

[2] M.S. Ashbaugh, R.D. Benguria: *A sharp bound for the ratio of the first two Dirichlet eigenvalues of a domain in a hemisphere of* $\mathbb{S}^n$, Transactions of the AMS **353** No. 3 (2000), 1055–1087

[3] M.S. Ashbaugh, R.D. Benguria: *A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions*, Annals of Mathematics **135** (1992), 601–628

[4] M.S. Ashbaugh, R.D. Benguria: *Log-concavity of the ground state of Schrödinger operators: A new proof of the Baumgartner-Grosse-Martin inequality*, Physical Letters A **131** No. 4,5 (1988) 273–276

[5] M. S. Ashbaugh, and R. D. Benguria, *Isoperimetric inequalities for eigenvalue ratios*, **Partial Differential Equations of Elliptic Type, Cortona, 1992**, A. Alvino, E. Fabes, and G. Talenti, editors, Symposia Mathematica, vol. **35**, Cambridge University Press, Cambridge, United Kingdom, 1994, pp. 1–36.

[6] M.F. Betta, F. Chiacchio, A. Ferone: *Isoperimetric estimates for the first eigenfunction of a class of linear elliptic problems*, to be published in ZAMP

[7] B. Baumgartner, H. Grosse, A. Martin: *The Laplacian of the potential and the order of energy levels*, Physics Letters **146B** No. 5 (1984), 363–366

[8] B. Baumgartner, H. Grosse and A. Martin, *Order of levels in potential models*, Nucl. Phys. B **254** (1985), 528–542.

[9] C. Borell: *The Brunn-Minkowski inequality in Gauss space*, Invent. Math. **30** (1975), 207-211

[10] G. Faber, *Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt*, Sitzungberichte der mathematisch-physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München Jahrgang, 1923, pp. 169–172.

[11] C. Haile: *A second eigenvalue bound for the Dirichlet Schrödinger equation with a radially symmetric potential*, Electronic Journal of Differential Equations **2000** No. 10 (2000), 1–19.

[12] E. Krahn, *Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises*, Math. Ann. **94** (1925), 97–100.

[13] E. Krahn, *Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen*, Acta Comm. Univ. Tartu (Dorpat) **A9** (1926), 1–44. [English translation: *Minimal properties of the sphere in three and more dimensions*, **Edgar Krahn 1894–1961: A Centenary Volume**, Ü. Lumiste and J. Peetre, editors, IOS Press, Amsterdam, The Netherlands, 1994, pp. 139–174.]

[14] J.M. Luttinger: *Generalized isoperimetric inequalities*, Proc. Nat. Acad. Sci. USA **70** (1973), 1005-1006

[15] A. Mercaldo, M.R. Posteraro, F. Brock: *On Schwarz and Steiner symmetrization with respect to a measure* (preprint)

[16] L. E. Payne, G. Pólya, and H. F. Weinberger, *Sur le quotient de deux fréquences propres consécutives*, Comptes Rendus Acad. Sci. Paris **241** (1955), 917–919.

[17] L. E. Payne, G. Pólya, and H. F. Weinberger, *On the ratio of consecutive eigenvalues*, J. Math. and Phys. **35** (1956), 289–298.

[18] G. Talenti: *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa (4) **3** (1976), 697–718

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