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The Absolute Continuity of the Integrated Density of States for Magnetic Schrödinger Operators with Certain Unbounded Random Potentials

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Abstract: The object of the present study is the integrated density of states of a quantum particle in multi-dimensional Euclidean space which is characterized by a Schrödinger operator with magnetic field and a random potential which may be unbounded from above and below. In case that the magnetic field is constant and the random potential is ergodic and admits a so-called one-parameter decomposition, we prove the absolute continuity of the integrated density of states and provide explicit upper bounds on its derivative, the density of states. This local Lipschitz continuity of the integrated density of states is derived by establishing a Wegner estimate for finite-volume Schrödinger operators which holds for rather general magnetic fields and different boundary conditions. Examples of random potentials to which the results apply are certain alloy-type and Gaussian random potentials. Besides we show a diamagnetic inequality for Schrödinger operators with Neumann boundary conditions.

Key words. Random Magnetic Schrödinger Operators, Density of States, Wegner estimate, Diamagnetic inequality.

Contents

1. Introduction

The integrated density of states is a quantity of primary interest in the theory [34, 10, 49] and application [54, 7, 40, 2, 37] of Schrödinger operators for a particle in d-dimensional Euclidean space $\mathbb{R}^d$ ($d = 1, 2, 3, \ldots$) subject to a random potential. Its knowledge allows one to compute the free energy and hence all basic thermostatic quantities of the corresponding non-interacting many-particle system. It also enters formulae for transport coefficients.

The main goal of the present paper is to prove the absolute continuity of the integrated density of states N for certain unbounded random potentials, thereby generalizing a result in [23] for zero magnetic field to the case of a constant magnetic field. Examples of random potentials to which our result applies are certain alloy-type and Gaussian random potentials. In particular, we consider the situation of two space dimensions and a perpendicular constant magnetic field where N is not absolutely continuous without random potential.

For the proof of absolute continuity of N, we use the abstract one-parameter spectral-averaging estimate of [11] to derive what is called a Wegner estimate [65]. Such estimates provide upper bounds on the averaged number of eigenvalues of finite-volume random Schrödinger operators in a given energy regime. They play a major rôle in proofs of Anderson localization for multi-dimensional random Schrödinger operators [10, 49, 11, 24, 61]. In contrast to the Wegner estimates with magnetic fields which are available so far, we are neither restricted to the case of a constant magnetic field [12, 5, 64] nor to the existence of gaps in the spectrum of the magnetic Schrödinger operator without random potential [4]. In fact, the Wegner estimate in the present paper holds for magnetic vector potentials whose components are locally square integrable. Its proof involves techniques for (non-random) magnetic Neumann Schrödinger operators among them Dirichlet-Neumann bracketing and a diamagnetic inequality. Appendix A provides the definition of these operators and proofs of the latter techniques in greater generality than actually needed for the main body of the present paper.

2. Random Schrödinger Operators with Magnetic Fields

2.1. Basic notation. As usual, let $\mathbb{N} := \{1, 2, 3, \ldots\}$ denote the set of natural numbers. Let $\mathbb{R}$, respectively $\mathbb{C}$, denote the algebraic field of real, respectively complex numbers and let $\mathbb{Z}^d$ be the simple cubic lattice in d dimensions, $d \in \mathbb{N}$. An open cube Λ in d-dimensional Euclidean space $\mathbb{R}^d$ is a translate of the d-fold Cartesian product $I \times \cdots \times I$ of an open interval $I \subseteq \mathbb{R}$. The open unit cube in $\mathbb{R}^d$ which is centered at site $y \in \mathbb{R}^d$

and whose edges are oriented parallel to the co-ordinate axes is denoted by $\Lambda(y)$. The Euclidean norm of $x \in \mathbb{R}^d$ is $|x| := \big(\sum_{j=1}^{d} x_j^2\big)^{1/2}$.

The volume of a Borel subset $\Lambda \subseteq \mathbb{R}^d$ with respect to the d-dimensional Lebesgue measure is $|\Lambda| := \int_\Lambda \mathrm{d}^d x = \int_{\mathbb{R}^d} \mathrm{d}^d x\, \chi_\Lambda(x)$, where χ_Λ is the indicator function of Λ. In particular, if Λ is the strictly positive half-line, $\Theta := \chi_{]0,\infty[}$ is the left-continuous Heaviside unit-step function. The Banach space $\mathrm{L}^p(\Lambda)$, $p \in [1,\infty]$, consists of the Borel-measurable complex-valued functions $f : \Lambda \to \mathbb{C}$ which are identified if their values differ only on a set of Lebesgue measure zero and which obey $\int_\Lambda \mathrm{d}^d x\, |f(x)|^p < \infty$ if $p < \infty$ and $\|f\|_\infty := \operatorname{ess\,sup}_{x\in\Lambda} |f(x)| < \infty$ if $p = \infty$. We recall that $\mathrm{L}^2(\Lambda)$ is a separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ given by $\langle f, g\rangle = \int_\Lambda \mathrm{d}^d x\, \overline{f(x)}\, g(x)$. Here the overbar denotes complex conjugation. We write $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$, if $f\chi_\Lambda \in \mathrm{L}^p(\mathbb{R}^d)$ for any bounded Borel set $\Lambda \subset \mathbb{R}^d$. Finally, $\mathscr{C}_0^\infty(\Lambda)$ is the vector space of functions $f : \Lambda \to \mathbb{C}$ which are arbitrarily often differentiable and have compact supports.

2.2. Basic assumptions. Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a complete probability space and $\mathbb{E}\{\cdot\} := \int_\Omega \mathbb{P}(\mathrm{d}\omega)(\cdot)$ be the expectation induced by the probability measure $\mathbb{P}$. By a *random potential* we mean a (scalar) random field $V : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $(\omega, x) \mapsto V^{(\omega)}(x)$ which is assumed to be jointly measurable with respect to the product of the sigma-algebra $\mathscr{A}$ of event sets in Ω and the sigma-algebra $\mathscr{B}(\mathbb{R}^d)$ of Borel sets in $\mathbb{R}^d$. We will always assume $d \geq 2$, because magnetic fields in one space dimension may be "gauged away" and are therefore of no physical relevance. Furthermore, for $d = 1$ far more is known [10, 49] thanks to methods which only work for one dimension.

We list four properties which V may have or not:

(F) There exists some real $p \in]1,\infty[$ with $p > 1$ if $d = 2$ and $p \geq d/2$ if $d \geq 3$ such that for $\mathbb{P}$-almost each $\omega \in \Omega$ the realization $V^{(\omega)} : x \mapsto V^{(\omega)}(x)$ of V belongs to $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$.

(S) There exists some pair of reals $p_1 > p(d)$ and $p_2 > p_1 d/\big[2(p_1 - p(d))\big]$ such that

$$\sup_{y\in\mathbb{Z}^d} \mathbb{E}\Big\{\Big[\int_{\Lambda(y)} \mathrm{d}^d x\, |V(x)|^{p_1}\Big]^{p_2/p_1}\Big\} < \infty. \tag{2.1}$$

Here $p(d)$ is defined as follows: $p(d) := 2$ if $d \leq 3$, $p(d) := d/2$ if $d \geq 5$ and $p(4) > 2$, otherwise arbitrary.

(E) V is $\mathbb{Z}^d$-ergodic or $\mathbb{R}^d$-ergodic.

(I) The finiteness condition

$$\sup_{y\in\mathbb{Z}^d} \mathbb{E}\Big[\int_{\Lambda(y)} \mathrm{d}^d x\, |V(x)|^{2\vartheta+1}\Big] < \infty \tag{2.2}$$

holds, where $\vartheta \in \mathbb{N}$ is the smallest integer with $\vartheta > d/4$.

Remark 2.1. (i) Property (E) requires the existence of a group $\mathscr{T}_x$, $x \in \mathbb{Z}^d$ or $\mathbb{R}^d$, of probability-preserving and ergodic transformations on Ω such that V is $\mathbb{Z}^d$- or $\mathbb{R}^d$-homogeneous in the sense that $V^{(\mathscr{T}_x\omega)}(y) = V^{(\omega)}(y - x)$ for all $x \in \mathbb{Z}^d$ or $\mathbb{R}^d$, all $y \in \mathbb{R}^d$ and all $\omega \in \Omega$.

(ii) Since property (S) assures that the realization $V^{(\omega)}$ belongs to $\mathrm{L}^{p(d)}_{\mathrm{loc}}(\mathbb{R}^d)$ for $\mathbb{P}$-almost each $\omega \in \Omega$, property (S) implies property (F). Property (I) also implies property (F).

We proceed by listing two properties either of which a random potential may additionally have or not and which characterize two examples of random potentials, which we will consider in the present paper.

(A) V is an *alloy-type random field*, that is, a random field with realizations given by

$$V^{(\omega)}(x) = \sum_{j \in \mathbb{Z}^d} \lambda_j^{(\omega)} u_0(x - j). \tag{2.3}$$

The *coupling strengths* $\{\lambda_j\}$ form a family of random variables which are $\mathbb{P}$-independent and identically distributed according to the common probability measure $\mathscr{B}(\mathbb{R}) \ni I \mapsto \mathbb{P}\{\lambda_0 \in I\}$. Moreover, we suppose that the *single-site potential* $u_0 : \mathbb{R}^d \to \mathbb{R}$ satisfies the Birman-Solomyak condition $\sum_{j \in \mathbb{Z}^d} \left(\int_{\Lambda(j)} \mathrm{d}^d x \, |u_0(x)|^{p_1} \right)^{1/p_1} < \infty$ with some real $p_1 \geq 2\vartheta + 1$ and that $\mathbb{E}\left(|\lambda_0|^{p_2}\right) < \infty$ for some real p_2 satisfying $p_2 \geq 2\vartheta + 1$ and $p_2 > p_1 d/[2(p_1 - p(d))]$. [The constants $p(d)$ and ϑ are defined in properties (S) and (I).]

(G) V is a *Gaussian random field* [1, 41] which is $\mathbb{R}^d$-homogeneous. It has zero mean, $\mathbb{E}[V(0)] = 0$, and its covariance function $x \mapsto C(x) := \mathbb{E}[V(x)V(0)]$ is continuous at the origin where it obeys $0 < C(0) < \infty$.

Remark 2.2. (i) Consider an *alloy-type random potential* V, that is, a random potential with property (A). Then V has properties (E), (I), (S) and (F), see, for example [29].

(ii) Consider a random field with the Gaussian property (G). Then its covariance function C is bounded and uniformly continuous on $\mathbb{R}^d$. Consequently, [22, Thm. 3.2.2] implies the existence of a separable version V of this field which is jointly measurable. Speaking about a *Gaussian random potential*, we tacitly assume that only this version will be dealt with. By the Bochner-Khintchine theorem [51, Thm. IX.9] there is a one-to-one correspondence between finite positive (and even) Borel measures on $\mathbb{R}^d$ and Gaussian random potentials. An explicit calculation shows that a Gaussian random potential enjoys properties (I), (S) and (F). A simple sufficient criterion for the ergodicity property (E) is the mixing condition $\lim_{|x|\to\infty} C(x) = 0$.

By a *vector potential* we mean a (non-random) Borel-measurable vector field $A : \mathbb{R}^d \to \mathbb{R}^d$, $x \mapsto A(x)$ which we assume to possess either the property

(B) $|A|^2$ belongs to $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$,

or the property

(C) A has continuous partial derivatives which give rise to a magnetic field (tensor) with *constant* components given by $B_{jk} := \partial_j A_k - \partial_k A_j$, where $j, k \in \{1, \ldots, d\}$.

Remark 2.3. (i) Property (C) implies property (B).

(ii) Given property (C), we may exploit the gauge freedom to choose the vector potential in the *symmetric gauge* in which the components of A are given by $A_k(x) = \sum_{j=1}^{d} x_j B_{jk}/2$, where $k \in \{1, \ldots, d\}$.

2.3. Definition of the operators. We are now prepared to precisely define magnetic Schrödinger operators with random potentials on the Hilbert spaces $\mathrm{L}^2(\Lambda)$ and $\mathrm{L}^2(\mathbb{R}^d)$. The finite-volume case is treated in

Proposition 2.1. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded open cube, A be a vector potential with the property* (B) *and V be a random potential with the property* (F). *Then*

(i) the sesquilinear form

$$h_{\Lambda,\mathrm{N}}^{A,0}(\varphi,\psi) := \frac{1}{2}\sum_{j=1}^{d} \langle (\mathrm{i}\nabla + A)_j\,\varphi\,,\,(\mathrm{i}\nabla + A)_j\,\psi\rangle\,, \tag{2.4}$$

with φ, ψ in the form domain $\mathscr{Q}\big(h_{\Lambda,\mathrm{N}}^{A,0}\big) := \big\{\phi \in \mathrm{L}^2(\Lambda) : (\mathrm{i}\nabla + A)\,\phi \in (\mathrm{L}^2(\Lambda))^d\big\}$ and $\nabla - \mathrm{i}A$ denoting the gauge-covariant gradient in the sense of distributions on $\mathscr{C}_0^\infty(\Lambda)$, uniquely defines a self-adjoint positive operator on $\mathrm{L}^2(\Lambda)$, which we denote by $H_{\Lambda,\mathrm{N}}(A,0)$. The closure $h_{\Lambda,\mathrm{D}}^{A,0}$ of the restriction of $h_{\Lambda,\mathrm{N}}^{A,0}$ to the domain $\mathscr{C}_0^\infty(\Lambda)$ uniquely defines another self-adjoint positive operator on $\mathrm{L}^2(\Lambda)$, which we denote by $H_{\Lambda,\mathrm{D}}(A,0)$.

(ii) The two operators

$$H_{\Lambda,\mathrm{X}}(A, V^{(\omega)}) := H_{\Lambda,\mathrm{X}}(A,0) + V^{(\omega)}, \qquad \mathrm{X} = \mathrm{D} \ \text{or} \ \mathrm{X} = \mathrm{N}, \tag{2.5}$$

are self-adjoint and bounded below on $\mathrm{L}^2(\Lambda)$ as form sums for all ω in some subset $\Omega_{\mathsf{F}} \in \mathscr{A}$ of Ω with full probability, in symbols, $\mathbb{P}(\Omega_{\mathsf{F}}) = 1$.

(iii) The mapping $H_{\Lambda,X}(A,V) : \Omega_{\mathsf{F}} \ni \omega \mapsto H_{\Lambda,\mathrm{X}}(A, V^{(\omega)})$ is measurable. We call it the finite-volume magnetic Schrödinger operator with random potential *V and Dirichlet or Neumann boundary condition if $\mathrm{X} = \mathrm{D}$ or $\mathrm{X} = \mathrm{N}$, respectively.*

(iv) The spectrum of $H_{\Lambda,\mathrm{X}}(A, V^{(\omega)})$ is purely discrete for all $\omega \in \Omega_{\mathsf{F}}$.

(v) The (random) finite-volume density-of-states measure*, defined by the trace*

$$\nu_{\Lambda,\mathrm{X}}^{(\omega)}(I) := \mathrm{Tr}\Big[\chi_I\big(H_{\Lambda,\mathrm{X}}(A, V^{(\omega)})\big)\Big], \tag{2.6}$$

is a positive Borel measure on the real line $\mathbb{R}$ for all $\omega \in \Omega_{\mathsf{F}}$. Here $\chi_I\big(H_{\Lambda,\mathrm{X}}(A, V^{(\omega)})\big)$ is the spectral projection operator of $H_{\Lambda,\mathrm{X}}(A, V^{(\omega)})$ associated with the energy regime $I \in \mathscr{B}(\mathbb{R})$. Moreover, the (unbounded left-continuous) distribution function

$$N_{\Lambda,\mathrm{X}}^{(\omega)}(E) := \nu_{\Lambda,\mathrm{X}}^{(\omega)}\big(\,]-\infty, E[\,\big) = \mathrm{Tr}\Big[\Theta\big(E - H_{\Lambda,\mathrm{X}}(A, V^{(\omega)})\big)\Big] < \infty \tag{2.7}$$

of $\nu_{\Lambda,\mathrm{X}}^{(\omega)}$, called the finite-volume integrated density of states*, is finite for all energies $E \in \mathbb{R}$.*

Proof. The proofs of assertions (i), (ii) and (iv) are contained in Appendix A because (B) and (F) imply (A.1) and (A.2). Assertion (iii) is a consequence of considerations in [35], see also Sect. V.1 in [10], and of a straightforward generalization to non-zero vector potentials. Assertion (v) follows from (ii) and (iv). □

Remark 2.4. Counting multiplicity, $\nu^{(\omega)}_{\Lambda,\mathrm{X}}(I)$ is just the number of eigenvalues of the operator $H_{\Lambda,\mathrm{X}}(A, V^{(\omega)})$ in the Borel set $I \subseteq \mathbb{R}$. Since this number is almost-surely finite if I is bounded, the mapping $\nu_{\Lambda,\mathrm{X}} : \omega \mapsto \nu^{(\omega)}_{\Lambda,\mathrm{X}}$ is a random Borel measure.

The precise definition of the infinite-volume magnetic Schrödinger operator on $\mathrm{L}^2(\mathbb{R}^d)$ and a compilation of its basic properties are given in

Proposition 2.2. *Assume that the vector potential A and the random potential V enjoy the properties* (C) *and* (S). *Then*

(i) *the operator $\mathscr{C}_0^\infty(\mathbb{R}^d) \ni \psi \mapsto \frac{1}{2}\sum_{j=1}^d (\mathrm{i}\partial_j + A_j)^2\,\psi + V^{(\omega)}\psi$ is essentially self-adjoint for all ω in some subset $\Omega_\mathrm{S} \in \mathscr{A}$ of Ω with full probability, $\mathbb{P}(\Omega_\mathrm{S}) = 1$. Its self-adjoint closure on $\mathrm{L}^2(\mathbb{R}^d)$ is denoted by $H(A, V^{(\omega)})$.*

(ii) *The mapping $H(A, V) : \Omega_\mathrm{S} \ni \omega \mapsto H(A, V^{(\omega)})$ is measurable. We call it the* infinite-volume magnetic Schrödinger operator with random potential V.

Proof. For assertion (i) see [24, Prop. 2.3], which generalizes [10, Prop. V.3.2] to the case of continuously differentiable vector potentials $A \neq 0$. Note that the assumption of a vanishing divergence, $\sum_{j=1}^d \partial_j A_j = 0$, in [24, Prop. 2.3] is not needed in the argument. Assertion (ii) is a straightforward generalization of [10, Prop. V.3.1] to continuously differentiable $A \neq 0$, see also [34, Prop. 2 on p. 288]. □

Remark 2.5. For alternative or weaker criteria instead of (S) guaranteeing the almost-sure self-adjointness of $H(0, V)$, see [49, Thm. 5.8] or [34, Thm. 1 on p. 299].

If A has the property (C), the infinite-volume magnetic Schrödinger operator without scalar potential, $H(A, 0)$, is unitarily invariant under so-called *magnetic translations* [67]. The latter form a family of unitary operators $\{T_x\}_{x\in\mathbb{R}^d}$ on $\mathrm{L}^2(\mathbb{R}^d)$ defined by

$$(T_x\psi)\,(y) := \mathrm{e}^{\mathrm{i}\Phi_x(y)}\psi(y - x), \qquad \psi \in \mathrm{L}^2(\mathbb{R}^d), \tag{2.8}$$

where $\Phi_x(y) := \int_{\mathscr{K}(x,y)} \mathrm{d}r \cdot (A(r) - A(r - x))$ is an integral along some smooth curve $\mathscr{K}$ with initial point $x \in \mathbb{R}^d$ and terminal point $y \in \mathbb{R}^d$. Since A and its x-translate $A(\cdot - x)$ give rise to the same magnetic field and $\mathbb{R}^d$ is simply connected, the integral $\Phi_x(y)$ is actually independent of $\mathscr{K}$.

Remark 2.6. (i) For the vector potential in the symmetric gauge (see Remark 2.3(ii)) one has $\Phi_x(y) = \sum_{j,k=1}^d x_j B_{jk}(y_k - x_k)/2$.

(ii) For a discussion in the case of more general configuration spaces and magnetic fields, see for example [44].

(iii) In the situation of Prop. 2.2 and if the random potential V has property (E), we have

$$T_x\, H(A, V^{(\omega)})\, T_x^\dagger = H(A, V^{(\mathscr{T}_x\omega)}) \tag{2.9}$$

for all $\omega \in \Omega_\mathrm{S}$ and all $x \in \mathbb{Z}^d$ or $x \in \mathbb{R}^d$, depending on whether V is $\mathbb{Z}^d$- or $\mathbb{R}^d$-ergodic. Hence, following standard arguments, $H(A, V)$ is an *ergodic operator* and its spectral components are non-random, see [62, Thm. 2.1]. Moreover, the discrete spectrum of $H(A, V^{(\omega)})$ is empty for $\mathbb{P}$-almost all $\omega \in \Omega$, see [34, 10, 62].

2.4. The integrated density of states. The quantity of main interest in the present paper is the integrated density of states and its corresponding measure, called the density-of-states measure. The next theorem, which we recall from [29], deals with its definition and its representation as an infinite-volume limit of the suitably scaled finite-volume counterparts (2.7).

Proposition 2.3. *Let $\chi_{\Lambda(0)}$ denote the multiplication operator associated with the indicator function of the unit cube $\Lambda(0)$. Assume that the potentials A and V have the properties* (C)*,* (S)*,* (I) *and* (E)*. Then the* (infinite-volume) integrated density of states

$$N(E) := \mathbb{E}\Big\{ \operatorname{Tr}\big[\chi_{\Lambda(0)}\, \Theta\big(E - H(A,V)\big)\, \chi_{\Lambda(0)}\big]\Big\} < \infty \tag{2.10}$$

is well defined for all energies $E \in \mathbb{R}$ in terms of the (spatially localized) spectral family of the infinite-volume operator $H(A,V)$. It is the (unbounded left-continuous) distribution function of some positive Borel measure ν on the real line $\mathbb{R}$. Moreover, let $\Lambda \subset \mathbb{R}^d$ stand for bounded open cubes centered at the origin. Then there is a set $\Omega_0 \in \mathscr{A}$ of full probability, $\mathbb{P}(\Omega_0) = 1$, such that the limit relation

$$N(E) = \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{N^{(\omega)}_{\Lambda,\mathrm{X}}(E)}{|\Lambda|} \tag{2.11}$$

holds for both boundary conditions $\mathrm{X} = \mathrm{D}$ *and* $\mathrm{X} = \mathrm{N}$*, all $\omega \in \Omega_0$ and all $E \in \mathbb{R}$ except for the (at most countably many) discontinuity points of N.*

Proof. See [29]. □

Remark 2.7. (i) A proof of the *existence* of the integrated density of states N under slightly different hypotheses was outlined in [43]. It uses functional-analytic arguments first presented in [36] for the case $A = 0$. A different approach to the existence of the density-of-states measure ν for $A \neq 0$, using Feynman-Kac(-Itô) functional-integral representations of Schrödinger semigroups [58, 9], can be found in [62, 8]. The latter approach dates back to [47, 46] for the case $A = 0$. To our knowledge, it works straightforwardly in the case $A \neq 0$ for $\mathrm{X} = \mathrm{D}$ only. For $A \neq 0$ the *independence* of the infinite-volume limit in (2.11) of the boundary condition X (previously claimed without proof in [43]) follows from [45] if the random potential V is bounded and from [19] if V is bounded from below. So the main new point about Prop. 2.3 is that it also applies to a wide class of V unbounded from below. Even for $A = 0$, Prop. 2.3 is partially new in that the corresponding result [49, Thm. 5.20] only shows vague convergence of the underlying measures, see the next remark.

(ii) An immediate corollary of Prop. 2.3 is the vague convergence [6, Def. 30.1] of the *spatial eigenvalue concentrations* $|\Lambda|^{-1}\, \nu^{(\omega)}_{\Lambda,\mathrm{X}}$ in the infinite-volume limit $\Lambda \uparrow \mathbb{R}^d$ to the non-random positive Borel measure ν uniquely corresponding to the integrated density of states (2.10) in the sense that $N(E) = \nu(]-\infty, E[)$ for all $E \in \mathbb{R}$, that is,

$$\nu = \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{\nu^{(\omega)}_{\Lambda,\mathrm{X}}}{|\Lambda|} \qquad \text{(vaguely)} \tag{2.12}$$

for both $\mathrm{X} = \mathrm{D}$ and $\mathrm{X} = \mathrm{N}$ and $\mathbb{P}$-almost all $\omega \in \Omega$.

One may relate properties of the density-of-states measure ν to simple spectral properties of the infinite-volume magnetic Schrödinger operator. Examples are the support of ν and the location of the almost-sure spectrum of $H(A, V^{(\omega)})$ or the absence of a point component in the Lebesgue decomposition of ν and the absence of "immobile eigenvalues" of $H(A, V^{(\omega)})$. This is the content of

Corollary 2.1. *Under the assumptions of Prop. 2.3 and letting $I \in \mathscr{B}(\mathbb{R})$, the following equivalence holds: $\nu(I) = 0$ if and only if $\chi_I\big(H(A, V^{(\omega)})\big) = 0$ for $\mathbb{P}$-almost all $\omega \in \Omega$. This immediately implies:*

(i) $\operatorname{supp} \nu = \operatorname{spec} H(A, V^{(\omega)})$ *for $\mathbb{P}$-almost all $\omega \in \Omega$. [Here* $\operatorname{spec} H(A, V^{(\omega)})$ *denotes the spectrum of $H(A, V^{(\omega)})$ and* $\operatorname{supp} \nu := \{E \in \mathbb{R} : \nu(]E - \varepsilon, E + \varepsilon[) > 0 \text{ for all } \varepsilon > 0\}$ *is the topological support of ν.]*

(ii) $0 = \nu(\{E\}) \big[= \lim_{\varepsilon \downarrow 0} [N(E + \varepsilon) - N(E)] \big]$ *if and only if $E \in \mathbb{R}$ is not an eigenvalue of $H(A, V^{(\omega)})$ for $\mathbb{P}$-almost all $\omega \in \Omega$.*

Proof. See [29]. □

The equivalence (ii) of the above corollary is a continuum analogue of [15, Prop. 1.1], see also [49, Thm. 3.3]. In the one-dimensional case [48] and the multi-dimensional lattice case [18], the equivalence has been exploited to show for $A = 0$ the (global) continuity of the integrated density of states N under practically no further assumptions on the random potential beyond those ensuring the existence of N. The proof of such a statement in the multi-dimensional continuum case is considered an important open problem [60]. For $A \neq 0$ one certainly needs additional assumptions as [20] illustrates, see Remark 4.3(ii) below. Under the additional assumptions of Corollary 3.1 below, we will show that the integrated density of states is not only continuous, but even absolutely continuous in the case of a constant magnetic field of arbitrary strength.

3. Existence of the Density of States for Certain Random Potentials

In this section we provide conditions under which the integrated density of states N (or, equivalently, its measure ν) is absolutely continuous with respect to the Lebesgue measure. As a by-product, we get rather explicit upper bounds on the resulting Lebesgue density $\mathrm{d}N(E)/\mathrm{d}E = \nu(\mathrm{d}E)/\mathrm{d}E$, called the density of states. Results of this genre date back to [65] and go nowadays under the name Wegner estimates.

3.1. A Wegner estimate. The main aim of this subsection is to extend the Wegner estimate in [23] to the case with magnetic fields. For this purpose we recall from there

Definition 3.1. *A random potential $V : \Omega \times \mathbb{R}^d \to \mathbb{R}$ admits a* (U, λ, u, ϱ)-decomposition *if there exists a random potential $U : \Omega \times \mathbb{R}^d \to \mathbb{R}$, a random variable $\lambda : \Omega \to \mathbb{R}$ and a real-valued $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ such that*

(i) $V^{(\omega)} = U^{(\omega)} + \lambda^{(\omega)} u$ *for $\mathbb{P}$-almost all $\omega \in \Omega$,*

(ii) the conditional probability distribution of λ relative to the sub-sigma-algebra generated by the family of random variables $\{U(x)\}_{x \in \mathbb{R}^d}$ has a jointly measurable density $\varrho : \Omega \times \mathbb{R} \to [0, \infty[$ with respect to the Lebesgue measure on $\mathbb{R}$.

The condition $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ was missed out in [23, Def. 2]. We now state the following generalization of [23, Thm. 2] which in its turn relies on a result in [11].

Theorem 3.1. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded open cube. Let $\Lambda = \left(\overline{\bigcup_{j=1}^{J} \Lambda_j} \right)^{\mathrm{int}}$ be decomposed into the interior of the closure of finitely many, $J \in \mathbb{N}$, pairwise disjoint bounded open cubes $\Lambda_j \subset \mathbb{R}^d$. Let the potentials A and V be supplied with the properties* (B) *and* (F)*, respectively. Assume that for each $j \in \{1, \ldots, J\}$ the random potential V admits a $(U_j, \lambda_j, u_j, \varrho_j)$-decomposition subject to the following three conditions: there exist five strictly positive constants $v_1, v_2, \beta, R, Z > 0$ such that for all $j \in \{1, \ldots, J\}$,*

(i) $v_1 \chi_{\Lambda_j}(x) \le u_j(x)$ *and* $u_j(x)\chi_{\Lambda_j}(x) \le v_2$ *for Lebesgue-almost all* $x \in \mathbb{R}^d$,

(ii) $\operatorname*{ess\,sup}_{\xi \in \mathbb{R}} \left(\varrho_j^{(\omega)}(\xi)\, \max\{\mathrm{e}^{-\beta v_1 \xi}, \mathrm{e}^{-\beta v_2 \xi}\} \right) \le R$ *for $\mathbb{P}$-almost all $\omega \in \Omega$,*

(iii) $\mathbb{E}\left\{ \mathrm{Tr}\left[\mathrm{e}^{-\beta H_{\Lambda_j,\mathrm{N}}(A,U_j)} \right] \right\} \le \left|\Lambda_j\right| Z.$

Then the averaged number of eigenvalues of the finite-volume operator $H_{\Lambda,\mathrm{X}}(A, V)$ in any non-empty energy regime $I \in \mathscr{B}(\mathbb{R})$ of finite Lebesgue measure $|I|$ is bounded from above according to

$$\mathbb{E}\left[\nu_{\Lambda,X}(I) \right] \le |\Lambda|\, |I|\, \frac{RZ}{v_1}\, \mathrm{e}^{\beta \sup I} \tag{3.1}$$

for both boundary conditions X. [*Here* $\sup I$ *denotes the least upper bound of* $I \subset \mathbb{R}$.]

Remark 3.1. The (Chebyshev-Markov) inequality $\chi_{[1,\infty[}(|\xi|) \le |\xi|$ implies

$$\mathbb{P}\left\{ I \cap \operatorname{spec} H_{\Lambda,\mathrm{X}}(A, V) \ne \emptyset \right\} = \mathbb{E}\left[\chi_{[1,\infty[}\big(\nu_{\Lambda,\mathrm{X}}(I)\big) \right] \le \mathbb{E}\left[\nu_{\Lambda,\mathrm{X}}(I) \right]. \tag{3.2}$$

Therefore the Wegner estimate (3.1) in particular bounds the probability of finding at least one eigenvalue of $H_{\Lambda,\mathrm{X}}(A, V)$ in a given energy regime $I \in \mathscr{B}(\mathbb{R})$. Such bounds are a key ingredient of proofs of Anderson localization for multi-dimensional random Schrödinger operators, see [10, 49, 11, 24, 61] and references therein.

Proof (of Theorem 3.1). Since we follow exactly the strategy of the proof of [23, Thm. 2], we only remark that the two main steps in this proof remain valid in the presence of a vector potential A. The first step, used in inequality (27) of [23], concerns the lowering of the eigenvalues of the operator $H_{\Lambda,\mathrm{X}}(A, V^{(\omega)})$ by so-called Dirichlet-Neumann bracketing in case $X = D$ and by the (subsequent) insertion of interfaces in Λ with the requirement of Neumann boundary conditions. For $A \ne 0$, supplied with property (B), the validity of these two techniques is established in Appendix A. The second step is an application of a spectral-averaging estimate of [11], which is re-phrased as Lemma 3.1 below. Since there the operator L is only required to be self-adjoint and does not enter the r.h.s. of (3.3), it makes no difference if L is taken as $H_{\Lambda,\mathrm{X}}(0, U_j)$ (as is done in [23]) or as $H_{\Lambda,\mathrm{X}}(A, U_j)$ for each $j \in \{1, \ldots, J\}$. □

An essential tool in the preceding proof is the (simple extension of the) abstract one-parameter *spectral-averaging estimate* of [11]; in this context see also [13].

Lemma 3.1. *Let K, L and M be three self-adjoint operators acting on a Hilbert space $\mathscr{H}$ with K and M bounded such that $\kappa := \inf_{K\varphi \ne 0} \langle \varphi\,, M\varphi \rangle / \langle \varphi\,, K^2 \varphi \rangle > 0$ is strictly positive. Moreover, let $g \in \mathrm{L}^\infty(\mathbb{R})$. Then the inequality*

$$\int_{\mathbb{R}} \mathrm{d}\xi\ |g(\xi)|\, \langle \psi\,, K\,\chi_I(L + \xi M)\, K\,\psi \rangle \le |I|\, \frac{\|g\|_\infty}{\kappa}\, \langle \psi, \psi \rangle \tag{3.3}$$

holds for all $\psi \in \mathscr{H}$ and all $I \in \mathscr{B}(\mathbb{R})$.

Proof. Since the assumption $\kappa > 0$ implies the operator inequality $\kappa K^2 \leq M$, the lemma is proven as Cor. 4.2 in [11] for any positive bounded function g with compact support. It extends to positive bounded functions with arbitrary supports by a monotone-convergence argument. □

3.2. Upper bounds on the density of states. If the fraction RZ/v_1 on the r.h.s of the Wegner estimate (3.1) is independent of Λ for sufficiently large $|\Lambda|$, this estimate enables one to prove the absolute continuity of the infinite-volume density-of-states measure with a magnetic field.

Corollary 3.1. *Let A and V have the properties* (C)*,* (S)*,* (I) *and* (E)*. Suppose furthermore:*

(i) there exists a sequence (Λ) of bounded open cubes $\Lambda \subset \mathbb{R}^d$ with $\Lambda \uparrow \mathbb{R}^d$ such that infinitely many of them admit a decomposition $\Lambda = \left(\overline{\bigcup_{j=1}^{J} \Lambda_j}\right)^{\text{int}}$ into a finite number J (depending on Λ) of pairwise disjoint open cubes $\Lambda_1, \ldots, \Lambda_J$.

(ii) V obeys the assumptions of Theorem 3.1 for every such decomposition with constants β, v_1, R, $Z > 0$, all of them not depending on Λ.

Then the density-of-states measure ν is absolutely continuous with respect to the Lebesgue measure. Moreover, its Lebesgue density w, called the density of states*, is locally bounded according to*

$$w(E) := \frac{\nu(\mathrm{d}E)}{\mathrm{d}E} = \frac{\mathrm{d}N(E)}{\mathrm{d}E} \leq \frac{RZ}{v_1}\, \mathrm{e}^{\beta E} =: W(E) \tag{3.4}$$

for Lebesgue-almost all energies $E \in \mathbb{R}$.

Proof. Let $I \subset \mathbb{R}$ be bounded and open. Then (2.12) together with [6, Satz 30.2] implies that $\nu(I) \leq \liminf_{\Lambda\uparrow\mathbb{R}^d} |\Lambda|^{-1} \nu^{(\omega)}_{\Lambda,\mathrm{X}}(I)$ for $\mathbb{P}$-almost all $\omega \in \Omega$. Therefore, by the non-randomness of the density-of-states measure ν and Fatou's lemma we have

$$\nu(I) \leq \liminf_{\Lambda\uparrow\mathbb{R}^d} \frac{\mathbb{E}\left[\nu_{\Lambda,\mathrm{X}}(I)\right]}{|\Lambda|} \leq |I|\, \frac{RZ}{v_1}\, \mathrm{e}^{\beta \sup I}. \tag{3.5}$$

Here we used (3.1) and the assumption that the constants involved there do not depend on Λ. Now the Radón-Nikodým theorem yields the claimed absolute continuity of ν. □

4. Examples Illustrating the Results of Section 3

Assumption (iii) of Theorem 3.1 may be checked in various ways. For example, by the diamagnetic inequality (A.24) of Appendix A for Neumann partition functions one sees that a possible choice of Z in (3.1) is

$$Z_1 := \max_{1\leq j\leq J} \left\{ |\Lambda_j|^{-1} \mathbb{E}\left[\mathrm{Tr}\left(\mathrm{e}^{-\beta H_{\Lambda_j,\mathrm{N}}(0,U_j)}\right)\right]\right\}. \tag{4.1}$$

This yields an upper bound on $\mathbb{E}\left[\nu_{\Lambda,X}(I)\right]$ in (3.1) which is *independent* of the magnetic field and, in particular, coincides with the one in [23, Thm. 2]. Rather weak conditions on the random potential U_j assuring the finiteness of the expectation value in (4.1) can be found in [21].

Another choice of Z results from applying the following averaged Golden-Thompson inequality.

Lemma 4.1. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded open cube and assume that A and V enjoy properties* (B) *and* (F). *Then the* averaged partition function *of $H_{\Lambda,\mathrm{X}}(A,V)$ is bounded for all $\beta > 0$ according to*

$$\mathbb{E}\left\{\mathrm{Tr}\left[\mathrm{e}^{-\beta\, H_{\Lambda,\mathrm{X}}(A,V)}\right]\right\} \leq \mathrm{Tr}\left[\mathrm{e}^{-\beta\, H_{\Lambda,\mathrm{X}}(A,0)}\right] \operatorname*{ess\,sup}_{x\in\Lambda}\left\{\mathbb{E}\left[\mathrm{e}^{-\beta\, V(x)}\right]\right\}, \tag{4.2}$$

provided that the essential supremum on the r.h.s. is finite.

Proof. We proceed as in the proof of [36, Thm. 3.4(ii)] and define $V_n^{(\omega)}(x) := \max\{-n, V^{(\omega)}(x)\}$ for $n \in \mathbb{N}$ and $\omega \in \Omega_{\mathsf{F}}$. The Golden-Thompson inequality [53] yields

$$\mathrm{Tr}\left[\mathrm{e}^{-\beta\, H_{\Lambda,\mathrm{X}}(A,V_n^{(\omega)})}\right] \leq \mathrm{Tr}\left[\mathrm{e}^{-\beta H_{\Lambda,\mathrm{X}}(A,0)}\,\mathrm{e}^{-\beta\, V_n^{(\omega)}}\right]. \tag{4.3}$$

We then evaluate the trace on the r.h.s. in an orthonormal eigenbasis of $H_{\Lambda,\mathrm{X}}(A,0)$. Using Fubini's theorem, the probabilistic expectation of the quantum-mechanical expectation of $\exp(-\beta V_n)$ with respect to a normalized eigenfunction of $H_{\Lambda,\mathrm{X}}(A,0)$ is estimated by $\operatorname{ess\,sup}_{x\in\Lambda}\left\{\mathbb{E}\left[\exp(-\beta\, V_n(x))\right]\right\}$, which is smaller than the second factor on the r.h.s. of (4.2) since $V \leq V_n$. The proof is completed by noting that the l.h.s. of (4.3) converges for $n \to \infty$ to the trace on the l.h.s. of (4.2) by monotone convergence of forms [51, Thm. S.16], similar to the proof of [36, Prop. 2.1(e)]. □

Using (4.2) one gets

$$Z_2 := \max_{1\leq j\leq J}\left\{|\Lambda_j|^{-1}\,\mathrm{Tr}\left[\mathrm{e}^{-\beta H_{\Lambda_j,\mathrm{N}}(A,0)}\right] \operatorname*{ess\,sup}_{x\in\Lambda_j}\mathbb{E}\left[\mathrm{e}^{-\beta U_j(x)}\right]\right\} \tag{4.4}$$

as another choice for Z in (3.1). By (A.24) one may further estimate the magnetic Neumann partition function in (4.4) according to

$$\mathrm{Tr}\left[\mathrm{e}^{-\beta\, H_{\Lambda,\mathrm{N}}(A,0)}\right] \leq \mathrm{Tr}\left[\mathrm{e}^{-\beta\, H_{\Lambda,\mathrm{N}}(0,0)}\right] \leq |\Lambda|\left(|\Lambda|^{-1/d} + (2\pi\beta)^{-1/2}\right)^d. \tag{4.5}$$

The second inequality follows from the explicitly known [53, p. 266] spectrum of $H_{\Lambda,\mathrm{N}}(0,0)$. Applying (4.5) to (4.4) one weakens Z_2 to a rather explicit choice of Z in (3.1) given by

$$Z_3 := \max_{1\leq j\leq J}\left\{\left(|\Lambda_j|^{-1/d} + (2\pi\beta)^{-1/2}\right)^d \operatorname*{ess\,sup}_{x\in\Lambda_j}\mathbb{E}\left[\mathrm{e}^{-\beta U_j(x)}\right]\right\}. \tag{4.6}$$

4.1. Alloy-type random potentials. The existence of a (U, λ, u, ϱ)-decomposition of V as required in Theorem 3.1 is immediate for alloy-type random potentials whose coupling strengths are distributed according to a Borel probability measure on the real line with a bounded Lebesgue density. To illustrate the essentials of Theorem 3.1 we first consider the case of positive potentials.

Corollary 4.1. *Let A and V have the properties* (B) *and* (A). *Assume that $u_0 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ and that the probability distribution of λ_0 has a Lebesgue density $g \in \mathrm{L}^\infty(\mathbb{R})$ with support in the positive half-line $[0, \infty[$. Furthermore, suppose that there exist two strictly positive constants $v_1, v_2 > 0$ such that*

$$v_1 \chi_{\Lambda(0)}(x) \leq u_0(x) \quad \textit{and} \quad u_0(x)\chi_{\Lambda(0)}(x) \leq v_2 \tag{4.7}$$

for Lebesgue-almost all $x \in \mathbb{R}^d$. Then for each bounded open cube of the form

$$\Lambda = \Big(\overline{\bigcup_{j \in \Lambda \cap \mathbb{Z}^d} \Lambda(j)} \Big)^{\mathrm{int}}, \tag{4.8}$$

one has

$$\mathbb{E}\left[\nu_{\Lambda,\mathrm{X}}(I) \right] \leq |\Lambda| \, |I| \, W_{\mathsf{A}}(\sup I) \tag{4.9}$$

for both $\mathrm{X} = \mathrm{D}$ and $\mathrm{X} = \mathrm{N}$ and all $I \in \mathscr{B}(\mathbb{R})$. Here W_{A} is the function

$$\mathbb{R} \ni E \mapsto W_{\mathsf{A}}(E) := \big(1 + (2\pi\beta)^{-1/2}\big)^d \, \frac{\|g\|_\infty}{v_1} \, \mathrm{e}^{\beta E} \tag{4.10}$$

with $\beta \in]0, \infty[$ serving as a variational parameter.

Proof. For each $j \in \Lambda \cap \mathbb{Z}^d$, the choice $u_j(x) := u_0(x - j)$ and $U_j^{(\omega)}(x) := V^{(\omega)}(x) - \lambda_j^{(\omega)} u_j(x)$ yields a (U_j, λ_j, u_j, g)-decomposition of V in the sense of Definition 3.1. It remains to verify the three assumptions of Theorem 3.1. Assumption (i) is guaranteed by (4.7). Assumption (ii) is fulfilled with $R = \|g\|_\infty$. To verify assumption (iii), we make use of (4.6) and observe that $U_j^{(\omega)} \geq 0$. □

Remark 4.1. (i) The estimates in the proof of Corollary 4.1, when specializing the fraction RZ/v_1 of Theorem 3.1 to W_{A}, were unnecessarily rough for the sake of simplicity. In specific examples the upper bound W_{A} may be improved. Moreover, more general alloy-type random potentials are also covered by Theorem 3.1. In particular, the random potential may be unbounded from below, see the next corollary. Furthermore, one may allow for correlated coupling strengths $\{\lambda_j\}$ as long as the relevant conditional probabilities have bounded Lebesgue densities.

(ii) Apart from the existence of a bounded Lebesgue density for the coupling strength λ_0 one further restrictive assumption of Corollary 4.1 is the fact that the single-site potential u_0 must possess a definite sign. The latter may be slightly weakened such that one may treat certain u_0 taking on values of both signs by choosing a more complicated decomposition different from the natural one used in the proof of Corollary 4.1. This basically corresponds to the linear-transformation technique introduced in [63] which turns certain given alloy-type random potentials into ones with positive single-site potentials and correlated coupling strengths, see the previous Remark 4.1(i). In any case, the fact that u_0 must possess a sufficiently large support is believed to be important for the absolute continuity of the integrated density of states in the presence of a magnetic field, see Remark 4.3(ii).

(iii) We only know of [12, 4, 5, 64] where Wegner estimates for magnetic Schrödinger operators with alloy-type random potentials have been derived.[1] The Wegner estimate of [4] is proven for energies in pre-supposed gaps of the spectrum of $H(A, 0)$. The other three works consider the case of two space dimensions and a perpendicular constant magnetic field, see Subsect. 4.3, especially Remark 4.3(iii) and 4.3(iv).

We close this subsection by considering the example of an unbounded below alloy-type random potential with exponentially decaying probability density for its (independent) coupling strengths. This example is marginal in the sense that any such density has to fall off at minus infinity at least as fast as exponentially in order to ensure the applicability of Theorem 3.1.

Corollary 4.2. *Let A and V have the properties* (B) *and* (A). *Assume a Laplace distribution for λ_0, that is*

$$\mathbb{P}\{\lambda_0 \in I\} = \frac{1}{2\alpha}\int_I \mathrm{d}\xi\ \mathrm{e}^{-|\xi|/\alpha}, \qquad I \in \mathscr{B}(\mathbb{R}), \tag{4.11}$$

with some $\alpha > 0$. Furthermore, suppose that $u_0 \in \mathrm{L}^\infty(\mathbb{R}^d)$ and that (4.7) *holds with some $v_1, v_2 > 0$ and let*

$$K_\beta := -\operatorname*{ess\,inf}_{x\in\Lambda(0)} \sum_{j\in\mathbb{Z}^d} \ln\left\{1 - [\beta\alpha u_0(x-j)]^2\right\} < \infty \tag{4.12}$$

be finite for some $\beta \in]0, (\alpha\|u_0\|_\infty)^{-1}[$. Finally, let Λ be of the form (4.8). *Then* (4.9) *holds where W_{A} may be taken as the function*

$$E \mapsto W_{\mathsf{A}}(E) := \left(1 + (2\pi\beta)^{-1/2}\right)^d \frac{1-(\beta\alpha v_1)^2}{2\alpha v_1}\mathrm{e}^{\beta E + K_\beta} \tag{4.13}$$

with $\beta \in \left\{\beta' \in]0, (\alpha\|u_0\|_\infty)^{-1}[:\ K_{\beta'} < \infty\right\}$ serving as a variational parameter.

Proof. The proof is analogous to that of Corollary 4.1. To verify the assumptions of Theorem 3.1 we note that assumption (i) is guaranteed by (4.7). Assumption (ii) is fulfilled with $R = (2\alpha)^{-1}$ if $\beta \in]0, (\alpha v_2)^{-1}]$. As for assumption (iii), we make use of (4.6) and explicitly compute the involved expectation if $\beta \in]0, (\alpha\|u_0\|_\infty)^{-1}[$. □

4.2. Gaussian random potentials. As another application of Theorem 3.1 we note that the Wegner estimate derived previously [23, Thm. 1] for certain Gaussian random potentials and the case without magnetic field remains valid in the present setting. The reason for this is the fact that *every* Wegner estimate stemming from [23, Thm. 2] is also one in the presence of a magnetic field thanks to the diamagnetic inequality.

Corollary 4.3. *Let A and V have the properties* (B) *and* (G). *Moreover, assume that there exist a finite signed Borel measure μ on $\mathbb{R}^d$, which is normalized in the sense that $\int_{\mathbb{R}^d}\mu(\mathrm{d}^d x)\int_{\mathbb{R}^d}\mu(\mathrm{d}^d y)\, C(x-y) = C(0)$, an open subset $\Gamma \subset \mathbb{R}^d$ with volume $|\Gamma| > 0$ and a constant $\gamma > 0$ such that the covariance function C of V obeys*

$$\gamma\,\chi_\Gamma(x) \le (C(0))^{-1}\int_{\mathbb{R}^d}\mu(\mathrm{d}^d y)\, C(x-y) =: (C(0))^{-1/2}\, u(x) \tag{4.14}$$

[1] See, however, note added in proof.

for all $x \in \mathbb{R}^d$. Then for each $\ell > 0$, for which there exists a bounded open cube $\Lambda^{(\ell)} \subseteq \Gamma$ with edges of length ℓ parallel to the co-ordinate axes, and each bounded open cube $\Lambda \subset \mathbb{R}^d$ satisfying the matching condition $|\Lambda|^{1/d}/\ell \in \mathbb{N}$, one has

$$\mathbb{E}\left[\nu_{\Lambda,\mathrm{X}}(I)\right] \leq |\Lambda|\,|I|\,W_{\mathsf{G}}(\sup I) \tag{4.15}$$

for both $\mathrm{X} = \mathrm{D}$ and $\mathrm{X} = \mathrm{N}$ and all $I \in \mathscr{B}(\mathbb{R})$. Here W_{G} is the function

$$E \mapsto W_{\mathsf{G}}(E) := \left(2\ell^{-1} + (2\pi\beta)^{-1/2}\right)^d \frac{\exp\left\{\beta E + \beta^2 C_\ell/2\right\}}{\sqrt{2\pi C(0)}\, b_\ell} \tag{4.16}$$

where we introduced the constants $C_\ell := C(0)\left(1 + B_\ell^2 - b_\ell^2\right)$, $B_\ell := (C(0))^{-1/2} \sup_{x\in\Lambda^{(\ell)}} u(x)$ and $b_\ell := (C(0))^{-1/2} \inf_{x\in\Lambda^{(\ell)}} u(x) \geq \gamma$. Finally, $\beta \in]0,\infty[$ serves, besides ℓ, as a second variational parameter.

Proof. The key input is the fact that every Gaussian random potential V admits a (U, λ, u, ϱ)-decomposition in the sense of Definition 3.1. More precisely, $\lambda^{(\omega)} := (C(0))^{-1/2} \int_{\mathbb{R}^d} \mu(\mathrm{d}^d x) V^{(\omega)}(x)$ is a standard Gaussian random variable with Lebesgue density $\varrho(\xi) := (2\pi)^{-1/2} \exp\left(-\xi^2/2\right)$. This random variable and the Gaussian random field $U^{(\omega)}(x) := V^{(\omega)}(x) - \lambda^{(\omega)} u(x)$, where u is defined in (4.14), are stochastically independent. For details see the proof of [23, Thm. 1]. To obtain the specific form W_{G}, which is independent of the magnetic field, we used (4.6). □

Remark 4.2. (i) Without loss of generality, every measure μ yielding (4.14) may be normalized in the sense of the assumption in the above corollary. The measure μ allows one to apply Corollary 4.3 to Gaussian random potentials with certain covariance functions taking on also negative values. Examples are given in [23, 30].

(ii) If $C(x) \geq 0$ for all $x \in \mathbb{R}^d$, we may choose μ equal to Dirac's point measure at the origin. Due to the continuity of C and since $C(0) > 0$, condition (4.14) is then fulfilled with some sufficiently small cube Γ containing the origin and $\gamma = \inf_{x\in\Gamma} C(x)/C(0)$. Under stronger conditions on the vector potential A the Wegner estimate for this case has been stated in [24, Prop. 2.14] where it serves as one input for a proof of Anderson localization by certain Gaussian random potentials, see Remark 3.1.

(iii) Choosing $\ell = |E|^{-1/4}$ and $\beta = (2C_\ell)^{-1}\left(\sqrt{E^2 + 2d\,C_\ell} - E\right)$ we obtain the following leading low- and high-energy behaviour:

$$\lim_{E\to-\infty} \frac{\ln W_{\mathsf{G}}(E)}{E^2} = -\frac{1}{2C(0)}, \qquad \lim_{E\to\infty} \frac{W_{\mathsf{G}}(E)}{E^{d/2}} = \frac{(e/(\pi d))^{d/2}}{\sqrt{2\pi}\, u(0)}. \tag{4.17}$$

Since W_{G} provides an upper bound on the density of states (see Corollary 3.1), its low-energy behaviour is optimal in the sense that it coincides with that of the derivative of the known low-energy behaviour of the integrated density of states [43, 62, 8]. This is not true for the high-energy behaviour. It is known [43, 62] that the high-energy growth of the integrated density of states is neither affected by the random potential nor by the magnetic field and proportional to $E^{d/2}$ for $E \to \infty$ in analogy to Weyl's celebrated asymptotics for the free particle [66]. Note that the constant on the r.h.s. of the second equation in (4.17) is smaller than the one given by [23, Eq. (14)].

4.3. Two space dimensions: random Landau Hamiltonians. In this subsection we consider the special case of two space dimensions and a perpendicular constant magnetic field of strength $B := B_{12} > 0$. Accordingly, the vector potential in the symmetric gauge is given by

$$A(x) = \frac{B}{2}\begin{pmatrix}-x_2\\ x_1\end{pmatrix}, \qquad x = \begin{pmatrix}x_1\\ x_2\end{pmatrix} \in \mathbb{R}^2. \tag{4.18}$$

This case has received considerable attention during the last three decades [2, 37] in the physics of low-dimensional electronic structures.

The magnetic Schrödinger operator on $\mathrm{L}^2(\mathbb{R}^2)$ modelling the non-relativistic motion of a particle with unit charge on the Euclidean plane $\mathbb{R}^2$ under the influence of this magnetic field is the *Landau Hamiltonian*. Its spectral resolution dates back to Fock [25] and Landau [38] and is given by the strong-limit relation

$$H(A,0) = \frac{B}{2}\sum_{l=0}^{\infty}(2l+1)\,P_l. \tag{4.19}$$

The energy eigenvalue $(l+1/2)B$ is called the l^{th} *Landau level* and the corresponding orthogonal eigenprojection P_l is an integral operator with continuous complex-valued kernel

$$P_l(x,y) := \frac{B}{2\pi}\exp\left[\mathrm{i}\frac{B}{2}(x_2y_1 - x_1y_2) - \frac{B}{4}|x-y|^2\right]\mathrm{L}_l\left(\frac{B}{2}|x-y|^2\right), \tag{4.20}$$

given in terms of the l^{th} Laguerre polynomial $\xi \mapsto \mathrm{L}_l(\xi) := \frac{1}{l!}\mathrm{e}^{\xi}\frac{\mathrm{d}^l}{\mathrm{d}\xi^l}\left(\xi^l\mathrm{e}^{-\xi}\right)$, $\xi \geq 0$, [27, Sect. 8.97]. The diagonal $P_l(x,x) = B/(2\pi)$ is naturally interpreted as the degeneracy per area of the l^{th} Landau level.

Using definition (2.10) with $V = 0$, the integrated density of states of the Landau Hamiltonian (4.19) turns out to be the well-known "staircase" function

$$N(E) = \frac{B}{2\pi}\sum_{l=0}^{\infty}\Theta\Big(E - \Big(l+\frac{1}{2}\Big)B\Big), \qquad V = 0, \tag{4.21}$$

which is obviously not absolutely continuous with respect to the Lebesgue measure. For the derivation of (4.21) one may apply [51, Thm. VI.23] because the operator $P_l\,\chi_{\Lambda(0)}$ is Hilbert-Schmidt, more precisely $\mathrm{Tr}[\chi_{\Lambda(0)}P_l\chi_{\Lambda(0)}] = B/(2\pi) < \infty$. Alternatively one may compute [45, App. B] the infinite-area limit $\lim_{\Lambda\uparrow\mathbb{R}^2}|\Lambda|^{-1}\,\mathrm{Tr}[\Theta(E - H_{\Lambda,\mathrm{X}}(A,0))]$ for some boundary condition X. The result coincides with (4.21) by Prop. 2.3. Informally, the density of states associated with (4.21) is a series of Dirac delta functions supported at the Landau levels. The corresponding infinities are indicated by vertical lines in Fig. 4.1 and together form what might be called a "Dirac half-comb". By adding a random potential V to (4.19), the delta peaks are expected to be smeared out. In fact, under the assumptions of Corollary 3.1 they are smeared out completely in the sense that the density of states w of the arising *random Landau Hamiltonian* $H(A,V) = H(A,0) + V$ is shown there to be locally bounded.

For example, in the presence of a Gaussian random potential with the Gaussian covariance function $C(x) = C(0)\exp\big\{-|x|^2/(2\tau^2)\big\} > 0$, $\tau > 0$, Fig. 4.1 contains the

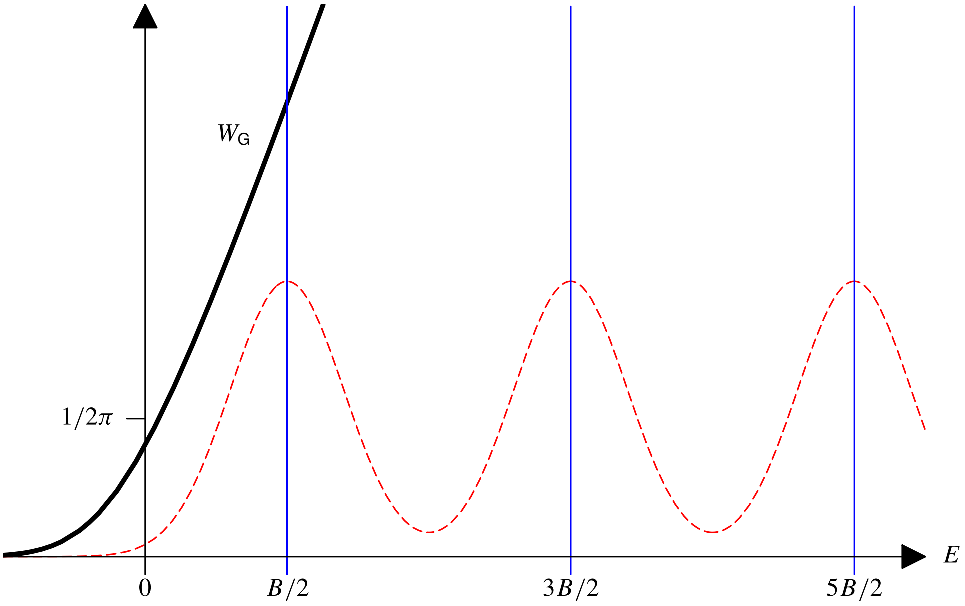

Fig. 4.1. Plot of the upper bound $W_G(E)$ on $w(E)$ as a function of the energy E. Here w is the density of states of the Landau Hamiltonian with a Gaussian random potential with Gaussian covariance function. The dashed line is a plot of the graph of an approximation to w. The exact w is unknown. Vertical lines indicate the delta peaks which reflect the non-existence of the density of states without random potential V. The step function $\Theta(E)/2\pi$ (not shown) is the free density of states characterized by $B = 0$ and $V = 0$. (See text).

graph of the upper bound W_G on w given in (4.16) after (numerically) minimizing with respect to β, ℓ and a certain one-parameter subclass of possible decompositions of V. Here we picked a (small) disorder parameter, $C(0) = (B/5)^2$, and a (large) correlation length, $\tau = 100B^{-1/2}$. We recall that the function W_G is independent of B due to our application of the diamagnetic inequality, but nevertheless provides an upper bound on w for all $B \geq 0$. Therefore $W_G(E)$ is a rather rough estimate of $w(E)$ already for energies $E < B/2$ and, in particular, starts increasing significantly at too low energies. Nevertheless, the upper bound shows that the density of states w has no infinities for arbitrarily weak disorder, that is, for arbitrarily small $C(0) > 0$. In fact, in the above situation we believe the graph of w to look similar to the dashed line in Fig. 4.1.

We conclude this subsection with several remarks:

Remark 4.3. (i) Unfortunately, our upper bound W in (3.4) is never sharp enough to reflect the expected "magneto-oscillations" of w. Instead, by construction W is always increasing.

(ii) The assumptions of Corollary 3.1 guarantee in particular that there occurs no point component in the Lebesgue decomposition of the density-of-states measure ν. Using Corollary 2.1, this implies that any given energy $E \in \mathbb{R}$, in particular any Landau-level energy, is $\mathbb{P}$-almost surely no eigenvalue under these assumptions. This stands in contrast to a certain situation with random point impurities, in which case the authors of [20] show that finitely many Landau-level energies remain infinitely degenerate eigenvalues if B is sufficiently large.

(iii) Exploiting the existence of spectral gaps of $H(A, 0)$, a Wegner estimate for Landau Hamiltonians with alloy-type random potentials is derived in [12, 4, 5] which proves that ν is absolutely continuous when restricted to intervals *between* the Landau-level energies. For this result to hold the authors were able to weaken the assumption (4.7) on the size of the support of the single-site potential which our Corollary 4.1 requires. On the other hand, absolute continuity of ν at *all* energies is proven in [12] only for bounded random potentials under the present assumptions on the support.

(iv) In [64] a Wegner estimate for alloy-type random potentials is derived without assuming a definite sign of the single-site potential. However, this estimate holds only between the Landau-level energies for sufficiently strong magnetic field and does not enable one to deduce the (local) existence of the density of states, because it has the "wrong" volume dependence.

(v) In [30] the integrated density of states associated with the *restricted* random Landau Hamiltonian $P_l H(A, V) P_l$ of a single but arbitrary Landau level is shown to be absolutely continuous for Gaussian random potentials satisfying the assumptions of Corollary 4.3 (for $d = 2$).

A. On Finite-Volume Schrödinger Operators with Magnetic Fields

For convenience of the reader (and the authors), this appendix defines non-random magnetic Schrödinger operators with Neumann boundary conditions and compiles some of their basic properties. In passing, the more familiar basic properties of the corresponding operators with Dirichlet boundary conditions are briefly recalled, see for example [42, 9]. In particular, we prove a diamagnetic inequality for Neumann Schrödinger operators and Dirichlet-Neumann bracketing for a wide class of vector potentials including singular ones. Altogether, this appendix may be understood to extend some of the results in the key papers [31, 32, 3, 57] to the case of Neumann boundary conditions.

Throughout this appendix, $\Lambda \subseteq \mathbb{R}^d$ denotes a non-empty open, not necessarily proper subset of d-dimensional Euclidean space $\mathbb{R}^d$ with $d \in \mathbb{N}$. Moreover, $a : \mathbb{R}^d \to \mathbb{R}^d$ stands for a vector potential and $v : \mathbb{R}^d \to \mathbb{R}$ for a scalar potential with $v_\pm := \big(|v| \pm v\big)/2$ denoting its positive respectively negative part. We will assume throughout that

$$|a|^2 ,\ v_+ \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d). \tag{A.1}$$

The negative part v_- is assumed to be a *form perturbation* either of $H_{\Lambda,\mathrm{N}}(a, 0)$ or even of $H_{\Lambda,\mathrm{N}}(0, 0)$. By this we mean that v_- is form-bounded [52, Def. p. 168] with form bound strictly smaller than one either relative to $H_{\Lambda,\mathrm{N}}(a, 0)$ or even to $H_{\Lambda,\mathrm{N}}(0, 0)$. Both operators will be defined in Lemma A.1 below. The operator $H_{\Lambda,\mathrm{N}}(0, 0)$ is the usual Neumann Laplacian, up to a factor of $-1/2$.

Remark A.1. By the diamagnetic inequality, see Prop. A.2 below, we will see that v_- is a form perturbation of $H_{\Lambda,\mathrm{N}}(a, 0)$ if it is one of $H_{\Lambda,\mathrm{N}}(0, 0)$. If Λ is a bounded open cube, an easy-to-check sufficient criterion for v_- to be even infinitesimally form-bounded [52, Def. p. 168] relative to $H_{\Lambda,\mathrm{N}}(0, 0)$ can be taken from [36, Lemma 2.1] and reads

$$v_- \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d) \tag{A.2}$$

with $p = 1$ if $d = 1$, some $p > 1$ if $d = 2$ and some $p \geq d/2$ if $d \geq 3$.

A.1. Definition of magnetic Neumann Schrödinger operators. In a first step, we consider the case $v = 0$ and $|a|^2 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ or, equivalently, $a \in \left(\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)\right)^d$, that is, $a_j \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ for all $j \in \{1, \dots, d\}$. We define the sesquilinear form

$$h^{a,0}_{\Lambda,\mathrm{N}}(\varphi, \psi) := \frac{1}{2} \sum_{j=1}^{d} \left\langle (\mathrm{i}\nabla + a)_j\, \varphi,\ (\mathrm{i}\nabla + a)_j\, \psi \right\rangle \tag{A.3}$$

for all φ and ψ in its form domain

$$W^{1,2}_a(\Lambda) := \left\{ \phi \in \mathrm{L}^2(\Lambda) :\ (\mathrm{i}\nabla + a)\,\phi \in \left(\mathrm{L}^2(\Lambda)\right)^d \right\}, \tag{A.4}$$

which might be called a *magnetic Sobolev space*, see [39, Sect. 7.20] in case $\Lambda = \mathbb{R}^d$. Here and in the following, $\nabla - \mathrm{i}a$ denotes the gauge-covariant gradient in the sense of distributions on $\mathscr{C}^\infty_0(\Lambda)$. In particular, this means

$$\begin{aligned} W^{1,2}_a(\Lambda) = \bigcap_{j=1}^{d} \Big\{ \phi \in \mathrm{L}^2(\Lambda)\ :\ &\text{there is } \phi_j \in \mathrm{L}^2(\Lambda) \text{ such that} \\ &\langle\, \phi\,,\, \mathrm{i}\partial_j\eta + a_j\eta \,\rangle = \langle\, \phi_j\,,\, \eta \,\rangle \quad \text{for all } \eta \in \mathscr{C}^\infty_0(\Lambda) \Big\}. \end{aligned} \tag{A.5}$$

Remark A.2. We emphasize that the condition $\psi \in W^{1,2}_a(\Lambda)$ allows for the case that neither $\nabla\psi$ nor $a\psi$ belongs to $\left(\mathrm{L}^2(\Lambda)\right)^d$. In general, $\psi \in W^{1,2}_a(\Lambda)$ only implies $\nabla\psi \in \left(\mathrm{L}^1_{\mathrm{loc}}(\Lambda)\right)^d$ and $|\,\psi\,| \in W^{1,2}(\Lambda) := \left\{\phi \in \mathrm{L}^2(\Lambda) :\ \nabla\phi \in \left(\mathrm{L}^2(\Lambda)\right)^d\right\}$, the usual first-order Sobolev space of L^2-type. The latter statement is a consequence of the diamagnetic inequality, see Remark A.5(iv) below and [59]. If even $|a|^2 \in \mathrm{L}^\infty(\mathbb{R}^d)$, the magnetic Sobolev space coincides with the usual one, $W^{1,2}_a(\Lambda) = W^{1,2}(\Lambda)$, up to equivalence of norms.

Basic facts about $h^{a,0}_{\Lambda,\mathrm{N}}$ are summarized in

Lemma A.1. *The form $h^{a,0}_{\Lambda,\mathrm{N}}$ is densely defined on $\mathrm{L}^2(\Lambda)$, symmetric, positive and closed. It therefore uniquely defines a self-adjoint positive operator $H_{\Lambda,\mathrm{N}}(a,0)$ on $\mathrm{L}^2(\Lambda)$ which, up to a factor of $-1/2$, is called* magnetic Neumann Laplacian.

Proof. Since $\mathscr{C}^\infty_0(\Lambda) \subset W^{1,2}_a(\Lambda) \subset \mathrm{L}^2(\Lambda)$ and $\mathscr{C}^\infty_0(\Lambda)$ is dense in $\mathrm{L}^2(\Lambda)$, the form $h^{a,0}_{\Lambda,\mathrm{N}}$ is densely defined. Its symmetry and positivity are obvious from the definition. To prove that $h^{a,0}_{\Lambda,\mathrm{N}}$ is also closed we have to show that the space $W^{1,2}_a(\Lambda)$ is complete with respect to the (metric induced by the form-) norm

$$\sqrt{\langle \phi, \phi \rangle + h^{a,0}_{\Lambda,\mathrm{N}}(\phi, \phi)}. \tag{A.6}$$

To this end, we proceed along the lines of Sects. 7.20 and 7.3 in [39] and let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence in $W^{1,2}_a(\Lambda)$ which is Cauchy with respect to the norm (A.6). By completeness of $\mathrm{L}^2(\Lambda)$, there exist functions $\phi, \psi_j \in \mathrm{L}^2(\Lambda)$, $j \in \{1, \dots, d\}$, such that $\phi_n \to \phi$ and $(\mathrm{i}\nabla + a)_j\phi_n \to \psi_j$ strongly in $\mathrm{L}^2(\Lambda)$ as $n \to \infty$. Since $(\mathrm{i}\nabla + a)_j\phi_n \to (\mathrm{i}\nabla + a)_j\phi$ in the sense of distributions on $\mathscr{C}^\infty_0(\Lambda)$ as $n \to \infty$, we have $(\mathrm{i}\nabla + a)_j\phi = \psi_j$ and hence $\phi \in W^{1,2}_a(\Lambda)$. The existence and uniqueness of $H_{\Lambda,\mathrm{N}}(a,0)$ follow now from the one-to-one correspondence between densely defined, symmetric, bounded below, closed forms and self-adjoint, bounded below operators, see [51, Thm. VIII.15]. $\square$

Remark A.3. (i) We recall that the operator $H_{\Lambda,\mathrm{N}}(a,0)$ has the subspace

$$\mathscr{D}\left(H_{\Lambda,\mathrm{N}}(a,0)\right) := \Big\{\psi \in W_a^{1,2}(\Lambda) \;:\; \text{there is } \widetilde{\psi} \in \mathrm{L}^2(\Lambda) \text{ such that} \quad h_{\Lambda,\mathrm{N}}^{a,0}(\varphi,\psi) = \langle \varphi\,,\,\widetilde{\psi}\,\rangle \quad \text{for all } \varphi \in W_a^{1,2}(\Lambda)\Big\} \tag{A.7}$$

of its underlying form domain as its operator domain and acts according to $H_{\Lambda,\mathrm{N}}(a,0)\,\psi = \widetilde{\psi}$.

(ii) Let $D_j(a)$ denote the closure of the symmetric operator $\mathscr{C}_0^\infty(\Lambda) \ni \psi \mapsto (\mathrm{i}\nabla + a)_j\,\psi \in \mathrm{L}^2(\Lambda)$. Being the closure of a symmetric operator, $D_j(a)$ is symmetric. The domain of its adjoint $D_j^\dagger(a)$ is given by

$$\mathscr{D}\big(D_j^\dagger(a)\big) := \left\{\psi \in \mathrm{L}^2(\Lambda) :\ (\mathrm{i}\nabla + a)_j\,\psi \in \mathrm{L}^2(\Lambda)\right\}, \tag{A.8}$$

because the adjoint of $\mathscr{C}_0^\infty(\Lambda) \ni \psi \mapsto (\mathrm{i}\nabla + a)_j\,\psi$ coincides with that of its closure. While for a proper subset $\Lambda \neq \mathbb{R}^d$ the operator $D_j(a)$ is not self-adjoint, it is so for $\Lambda = \mathbb{R}^d$ [57, Lemma 2.5]. In the latter case it may physically be interpreted, up to a sign, as the j^{th} component of the velocity (operator). By construction the magnetic Neumann Laplacian is a form sum of d operators in accordance with

$$H_{\Lambda,\mathrm{N}}(a,0) = \frac{1}{2}\sum_{j=1}^{d} D_j(a)\,D_j^\dagger(a)\,, \tag{A.9}$$

where the self-adjoint positive operator $D_j(a)\,D_j^\dagger(a)$ comes from the closed form $\langle D_j^\dagger(a)\,\varphi,\ D_j^\dagger(a)\,\psi\rangle$ with form domain (A.8). Note that (A.8) is just the j^{th} set of the intersection on the r.h.s. of (A.5). See also Thm. X.25 in [52].

(iii) Restricting the form $h_{\Lambda,\mathrm{N}}^{a,0}$ to the domain $\mathscr{C}_0^\infty(\Lambda) \subset W_a^{1,2}(\Lambda)$, one obtains a form which is closable in $W_a^{1,2}(\Lambda)$ with respect to the norm (A.6), see [57, 42, 9]. Its closure $h_{\Lambda,\mathrm{D}}^{a,0}$ is uniquely associated with another self-adjoint positive operator $H_{\Lambda,\mathrm{D}}(a,0)$ on $\mathrm{L}^2(\Lambda)$ which, up to a factor of $-1/2$, is called *magnetic Dirichlet Laplacian*. For general $a \in \left(\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)\right)^d$ the space $\mathscr{C}_0^\infty(\Lambda)$ is not contained in $\mathscr{D}\left(H_{\Lambda,\mathrm{N}}(a,0)\right)$, see (A.7). As a consequence, $H_{\Lambda,\mathrm{N}}(a,0)$ in general cannot be restricted to $\mathscr{C}_0^\infty(\Lambda)$. This stands in contrast to the case $a = 0$ where $H_{\Lambda,\mathrm{D}}(0,0)$ is the *Friedrichs extension* of the restriction of $H_{\Lambda,\mathrm{N}}(0,0)$ to $\mathscr{C}_0^\infty(\Lambda)$. As the Dirichlet counterpart of (A.9) we only have the inequality $H_{\Lambda,\mathrm{D}}(a,0) \leq \frac{1}{2}\sum_{j=1}^{d} D_j^\dagger(a)\,D_j(a)$ which is meant in the sense of forms [53, Def. on p. 269]. The operators $H_{\mathbb{R}^d,\mathrm{N}}(a,0)$ and $H_{\mathbb{R}^d,\mathrm{D}}(a,0)$ are equal, see [57].

(iv) In the *free* case, which is characterized by $a = 0$ and $v = 0$, the just defined operators $H_{\Lambda,\mathrm{D}}(0,0)$ and $H_{\Lambda,\mathrm{N}}(0,0)$ coincide, up to a factor of $-1/2$, with the usual Dirichlet- and Neumann-Laplacian [53, p. 263], respectively.

In a second and final step, we let $v_+ \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and assume v_- to be a form perturbation of $H_{\Lambda,\mathrm{N}}(a,0)$. As a consequence, the sesquilinear form

$$h_{\Lambda,\mathrm{N}}^{a,v}(\varphi,\psi) := h_{\Lambda,\mathrm{N}}^{a,0}(\varphi,\psi) + \left\langle v_+^{1/2}\varphi, v_+^{1/2}\psi\right\rangle - \left\langle v_-^{1/2}\varphi, v_-^{1/2}\psi\right\rangle \tag{A.10}$$

is well defined for all φ and ψ in its form domain $\mathscr{Q}\big(h^{a,v}_{\Lambda,\mathrm{N}}\big) := W^{1,2}_a(\Lambda) \cap \mathscr{Q}(v_+)$, where

$$\mathscr{Q}(v_+) := \Big\{\phi \in \mathrm{L}^2(\Lambda) : \, v_+^{1/2}\phi \in \mathrm{L}^2(\Lambda)\Big\}. \tag{A.11}$$

Basic facts about $h^{a,v}_{\Lambda,\mathrm{N}}$ are summarized in

Lemma A.2. *The form $h^{a,v}_{\Lambda,\mathrm{N}}$ is densely defined on $\mathrm{L}^2(\Lambda)$, symmetric, bounded below and closed. It therefore uniquely defines a self-adjoint, bounded below operator $H_{\Lambda,\mathrm{N}}(a, v)$ on $\mathrm{L}^2(\Lambda)$ which is called* magnetic Neumann Schrödinger operator.

Proof. The domain $W^{1,2}_a(\Lambda) \cap \mathscr{Q}(v_+)$ of $h^{a,v_+}_{\Lambda,\mathrm{N}}$ is dense in $\mathrm{L}^2(\Lambda)$, because both $W^{1,2}_a(\Lambda)$ and $\mathscr{Q}(v_+)$ contain $\mathscr{C}^\infty_0(\Lambda)$. Hence $H_{\Lambda,\mathrm{N}}(a, v_+)$ is well defined as a form sum of $H_{\Lambda,\mathrm{N}}(a, 0)$ and v_+. Moreover, $h^{a,v_+}_{\Lambda,\mathrm{N}}$ is symmetric, positive and closed, because it is the sum of two of such forms. Since $H_{\Lambda,\mathrm{N}}(a, 0) \leq H_{\Lambda,\mathrm{N}}(a, v_+)$, the negative part v_- of v is also a form perturbation of $H_{\Lambda,\mathrm{N}}(a, v_+)$. The proof of the lemma is then completed by the KLMN-theorem [52, Thm. X.17]. □

Remark A.4. Since the form domain of $h^{a,0}_{\Lambda,\mathrm{D}}$ is contained in $W^{1,2}_a(\Lambda)$, the negative part v_- of v is also a form perturbation of $H_{\Lambda,\mathrm{D}}(a, 0) \leq H_{\Lambda,\mathrm{D}}(a, v_+)$. Hence one may apply the KLMN-theorem to define, similarly to $H_{\Lambda,\mathrm{N}}(a, v)$, what is called the *magnetic Dirichlet Schrödinger operator* and denoted as $H_{\Lambda,\mathrm{D}}(a, v)$.

An immediate consequence of the definition of $H_{\Lambda,\mathrm{X}}(a, v)$ is the fact that so-called *decoupling* and *Dirichlet-Neumann bracketing* continues to hold for $a \neq 0$ as in the case $a = 0$, see Props. 3 and 4 in Sect. XIII.15 of [53], and [14, 45] for smooth $a \neq 0$.

Proposition A.1. *Let $|a|^2$, $v_+ \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and v_- be a form perturbation of $H_{\Lambda,\mathrm{N}}(a, 0)$. Moreover, let $\Lambda_1, \Lambda_2 \subset \mathbb{R}^d$ be a disjoint pair of non-empty open sets.*

(i) Then the orthogonal decomposition

$$H_{\Lambda_1\cup\Lambda_2,\mathrm{X}}(a, v) = H_{\Lambda_1,\mathrm{X}}(a, v) \oplus H_{\Lambda_2,\mathrm{X}}(a, v) \tag{A.12}$$

holds for both $\mathrm{X} = \mathrm{D}$ *and* $\mathrm{X} = \mathrm{N}$ *on* $\mathrm{L}^2(\Lambda_1 \cup \Lambda_2) = \mathrm{L}^2(\Lambda_1) \oplus \mathrm{L}^2(\Lambda_2)$.

(ii) Let $\Lambda := \overline{\Lambda_1 \cup \Lambda_2}^{\,\mathrm{int}}$ be defined as the interior of the closure of the union of Λ_1 and Λ_2, and suppose that the interface $\Lambda \setminus (\Lambda_1 \cup \Lambda_2)$ is of d-dimensional Lebesgue measure zero. Then the inequalities

$$H_{\Lambda_1\cup\Lambda_2,\mathrm{N}}(a, v) \leq H_{\Lambda,\mathrm{N}}(a, v) \leq H_{\Lambda,\mathrm{D}}(a, v) \leq H_{\Lambda_1\cup\Lambda_2,\mathrm{D}}(a, v) \tag{A.13}$$

hold in the sense of forms.

Proof. The proofs of Props. 3 and 4 in Sect. XIII.15 of [53] for the free case carry over to the case $a \neq 0$ and $v \neq 0$. In particular, the inclusion relations between the various form domains for $a = 0$ and $v = 0$ hold analogously for the form domains in the case $a \neq 0$ and $v \neq 0$. □

A.2. Diamagnetic inequality. A useful tool in the study of Schrödinger operators with magnetic fields is

Proposition A.2. *Let* $\Lambda \subseteq \mathbb{R}^d$ *be open,* $|a|^2$, $v_+ \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ *and* v_- *be a form perturbation of* $H_{\Lambda,\mathrm{N}}(0,0)$*. Then* v_- *is a form perturbation of* $H_{\Lambda,\mathrm{N}}(a,0)$ *with form bound not exceeding the one for* $a = 0$ *and the inequality*

$$\left| \mathrm{e}^{-t\, H_{\Lambda,\mathrm{X}}(a,v)} \psi \right| \leq \mathrm{e}^{-t\, H_{\Lambda,\mathrm{X}}(0,v)} \, |\psi| \tag{A.14}$$

holds for all $\psi \in \mathrm{L}^2(\Lambda)$*, all* $t \geq 0$ *and both* $\mathrm{X} = \mathrm{D}$ *and* $\mathrm{X} = \mathrm{N}$.

Remark A.5. (i) For the Dirichlet version $\mathrm{X} = \mathrm{D}$ of the *diamagnetic inequality* (A.14) to hold, it would be sufficient that v_- is a form perturbation of $H_{\Lambda,\mathrm{D}}(0,0)$.

(ii) Inequality (A.14) for $\Lambda = \mathbb{R}^d$ dates back to [31, 56, 28, 32, 3, 59, 57]. It is also known to hold for $\Lambda \neq \mathbb{R}^d$ and $\mathrm{X} = \mathrm{D}$, even under the weaker assumptions $|a|^2$, $v_+ \in \mathrm{L}^1_{\mathrm{loc}}(\Lambda)$, see [50, 42]. These assumptions still guarantee that the operators $H_{\Lambda,\mathrm{D}}(a,v)$ and $H_{\Lambda,\mathrm{N}}(a,v)$ are definable as self-adjoint operators via forms. However, for arbitrary open $\Lambda \neq \mathbb{R}^d$ the proof of (A.14) for $\mathrm{X} = \mathrm{N}$ would be more complicated than the one which we will give under the stronger assumptions of Prop. A.2. The reason is that a gauge function more fancy than that in Lemma A.3 would be needed in order to avoid integration of a_j across the boundary of Λ. For a "simply shaped" Λ, like a cube, such complications do not arise which implies that our proof would go through for cubes under the weaker assumptions.

(iii) If $a = 0$ inequality (A.14) is equivalent to the assertion that $H_{\Lambda,\mathrm{X}}(0,v)$ is the (negative of the) generator of a positivity-preserving one-parameter operator semigroup on $\mathrm{L}^2(\Lambda)$, see [52, pp. 186]. For general $a \in \left(\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)\right)^d$ inequality (A.14) asserts that the semigroup generated by $H_{\Lambda,\mathrm{X}}(0,v)$ dominates the one generated by $H_{\Lambda,\mathrm{X}}(a,v)$.

(iv) It follows from [28, 59] that (A.14) is equivalent to the following pair of statements:

(a) $\psi \in \mathscr{D}\big(H_{\Lambda,\mathrm{X}}(a,v)\big)$ implies $|\psi| \in \mathscr{Q}\big(h^{0,v}_{\Lambda,\mathrm{X}}\big)$,

(b) $h^{0,v}_{\Lambda,\mathrm{X}}(\varphi, |\psi|) \leq \mathrm{Re}\,\langle \varphi \,\mathrm{sgn}\,\psi\,,\ H_{\Lambda,\mathrm{X}}(a,v)\,\psi\rangle$
for all $\varphi \in \mathscr{Q}\big(h^{0,v}_{\Lambda,\mathrm{X}}\big)$ with $\varphi \geq 0$ and all $\psi \in \mathscr{D}\big(H_{\Lambda,\mathrm{X}}(a,v)\big)$,

where the *signum function* associated with ψ is defined by $(\mathrm{sgn}\,\psi)(x) := \psi(x)/|\psi(x)| \in \mathbb{C}$ if $\psi(x) \neq 0$ and zero otherwise. If $a = 0$ these statements boil down to a Beurling-Deny criterion [17, Thm. 1.3.2] for $H_{\Lambda,\mathrm{X}}(0,v)$ which guarantees that it generates a positivity-preserving semigroup. Inequality (b) with $\mathrm{X} = \mathrm{N}$ and $v = 0$ basically corresponds to the germinal *distributional inequality of Kato*, which he proved [31] for $a \in \left(\mathscr{C}^1(\mathbb{R}^d)\right)^d$. In case $\Lambda \neq \mathbb{R}^d$ and $\mathrm{X} = \mathrm{N}$, we are not aware of a reference proving (A.14) or (a) and (b) for singular a.

Our proof of the diamagnetic inequality (A.14) for $\mathrm{X} = \mathrm{N}$ will mimic the proof in [57], where the case $\Lambda = \mathbb{R}^d$ is considered, see also Sect. 1.3 in [16]. It relies on the fact that for one dimension the vector potential can be removed by a gauge transformation. More precisely, for each $j \in \{1, \ldots, d\}$ the operator $D^\dagger_j(a)$ is unitarily equivalent to $D^\dagger_j(0)$.

Lemma A.3. *Let* $|a|^2 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ *and define a* (gauge) *function* $\lambda_j : \mathbb{R}^d \to \mathbb{R}$ *through*

$$\lambda_j\,(x) := \int_0^{x_j} \mathrm{d}y_j\; a_j\,\bigl(x_1, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_d\bigr)\,. \tag{A.15}$$

For open $\Lambda \subseteq \mathbb{R}^d$ *it induces a densely defined and self-adjoint multiplication operator* λ_j *on* $\mathrm{L}^2\,(\Lambda)$. *The corresponding unitary operator* $\mathrm{e}^{-\mathrm{i}\lambda_j}$ *maps* $\mathscr{D}\bigl(D_j^\dagger(a)\bigr)$ *onto* $\mathscr{D}\bigl(D_j^\dagger(0)\bigr)$, *recall* (A.8), *and one has*

$$D_j^\dagger(a)\,\psi = \mathrm{e}^{\mathrm{i}\lambda_j}\,D_j^\dagger(0)\,\mathrm{e}^{-\mathrm{i}\lambda_j}\,\psi \tag{A.16}$$

for all $\psi \in \mathscr{D}\bigl(D_j^\dagger(a)\bigr)$.

Proof. Fubini's theorem and the Cauchy-Schwarz inequality show that $\lambda_j \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$. Therefore, the induced multiplication operator on its maximal domain $\mathscr{D}(\lambda_j) := \bigl\{\psi \in \mathrm{L}^2\,(\Lambda) : \lambda_j\psi \in \mathrm{L}^2\,(\Lambda)\,\bigr\} \supset \mathscr{C}_0^\infty\,(\Lambda)$ is densely defined and self-adjoint. Moreover, since $\psi \in \mathscr{D}\bigl(D_j^\dagger(a)\bigr)$ implies $\nabla_j\psi \in \mathrm{L}^1_{\mathrm{loc}}\,(\Lambda)$, we are allowed to use the product and chain rule for distributional derivatives [26, pp. 150] which yield $\nabla_j\,\bigl(\mathrm{e}^{-\mathrm{i}\lambda_j}\,\psi\bigr) = \mathrm{e}^{-\mathrm{i}\lambda_j}\,\nabla_j\psi - \mathrm{e}^{-\mathrm{i}\lambda_j}\,\mathrm{i}a_j\psi$. □

Proof (of Prop. A.2). For X = D see [50, 42, 9]. The proof for X = N consists of three steps.

In the first step, we assume $v \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ to be bounded from below. In this case $H_{\Lambda,\mathrm{N}}(a, v)$ is a form sum of $d + 1$ operators each of which is bounded from below, recall Remark A.3(ii) and Lemma A.2. Hence we may employ the strong Lie-Trotter product formula generalized to form sums of several operators [33] and write

$$\mathrm{e}^{-tH_{\Lambda,\mathrm{N}}(a,v)} = \operatorname*{s-lim}_{n\to\infty}\Bigl(\mathrm{e}^{-tD_1(a)D_1^\dagger(a)/2n}\cdots\mathrm{e}^{-tD_d(a)D_d^\dagger(a)/2n}\mathrm{e}^{-tv/n}\Bigr)^n\,. \tag{A.17}$$

Gauge equivalence (A.16) now shows that

$$\mathrm{e}^{-tD_j(a)D_j^\dagger(a)/2n} = \mathrm{e}^{\mathrm{i}\lambda_j}\,\mathrm{e}^{-tD_j(0)D_j^\dagger(0)/2n}\,\mathrm{e}^{-\mathrm{i}\lambda_j} \tag{A.18}$$

for all $j \in \{1, \dots, d\}$ and all $t \geq 0$. By the distributional inequality $\bigl|\nabla_j\,|\psi|\,\bigr| \leq \bigl|\nabla_j\psi\bigr|$, valid for all $\psi \in \mathscr{D}\bigl(D_j^\dagger(0)\bigr)$ [39, Thm. 6.17], the operator $D_j(0)\,D_j^\dagger(0)$ obeys a Beurling-Deny criterion [17, Thm. 1.3.2] and hence is the generator of a positivity-preserving semigroup. It follows that

$$\Bigl|\mathrm{e}^{-tD_j(a)D_j^\dagger(a)/2n}\psi\Bigr| \leq \mathrm{e}^{-tD_j(0)D_j^\dagger(0)/2n}\,|\psi| \tag{A.19}$$

for all $\psi \in \mathrm{L}^2\,(\Lambda)$ and all $t \geq 0$. This together with (A.17) implies the assertion (A.14) (with X = N) for scalar potentials $v \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ which are bounded from below.

In the second step, we prove that if v_- is a form perturbation of $H_{\Lambda,\mathrm{X}}(0, 0)$ then it is also one of $H_{\Lambda,\mathrm{X}}(a, 0)$ with form bound not exceeding the one for $a = 0$ (see [3] or [58, Thm. 15.10] for the case $\Lambda = \mathbb{R}^d$). This follows from (A.23) below with $v = 0$ and $\alpha = 1/2$ together with the fact that the form bound of v_- relative to $H_{\Lambda,\mathrm{X}}(a, 0)$ can be expressed as

$$\lim_{E\to\infty}\Bigl\|\bigl(H_{\Lambda,\mathrm{X}}(a,0)+E\bigr)^{-1/2}\,v_-\,\bigl(H_{\Lambda,\mathrm{X}}(a,0)+E\bigr)^{-1/2}\Bigr\|\,, \tag{A.20}$$

see [16, Prop. 1.3(ii)]. Here $\|\cdot\|$ denotes the (uniform) norm of bounded operators on $L^2(\Lambda)$.

In the third step, we extend the validity of (A.14) (with X = N) to scalar potentials v with $v_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and v_- being a form perturbation of $H_{\Lambda,\mathrm{N}}(0,0)$. To this end, we approximate v by v_n defined through $v_n(x) := \max\{-n, v(x)\}$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$. Monotone convergence for forms [51, Thm. S.16] yields the convergence of $H_{\Lambda,\mathrm{N}}(a, v_n)$ to $H_{\Lambda,\mathrm{N}}(a, v)$ in the strong resolvent sense as $n \to \infty$. It follows that

$$\operatorname*{s-lim}_{n\to\infty} \mathrm{e}^{-tH_{\Lambda,\mathrm{N}}(a,v_n)} = \mathrm{e}^{-tH_{\Lambda,\mathrm{N}}(a,v)} \tag{A.21}$$

for all $t \geq 0$. Since (A.14) (with X = N) holds for each v_n by the first step, the proof is complete. □

A.3. Some consequences. We list some immediate consequences of the diamagnetic inequality. For this purpose, we assume the situation of Prop. A.2.

(i) Powers of the resolvent of the self-adjoint operator $H_{\Lambda,\mathrm{X}}(a, v)$ may be expressed in terms of its semigroup by using the functional calculus. This gives the integral representation

$$\big(H_{\Lambda,\mathrm{X}}(a,v) - z\big)^{-\alpha} = \frac{1}{(\alpha-1)!}\int_0^\infty \mathrm{d}t\, t^{\alpha-1}\, \mathrm{e}^{tz}\, \mathrm{e}^{-tH_{\Lambda,\mathrm{X}}(a,v)}, \tag{A.22}$$

which is valid for all $\alpha > 0$, all $z \in \mathbb{C}$ with $\operatorname{Re} z < \inf\operatorname{spec} H_{\Lambda,\mathrm{X}}(a,v)$ and both X = D and X = N. Here $\alpha \mapsto (\alpha-1)!$ denotes Euler's gamma function [27]. Inequality (A.14) then implies the *diamagnetic inequality for powers of the resolvent*

$$\big|\,\big(H_{\Lambda,\mathrm{X}}(a,v) - z\big)^{-\alpha}\,\psi\big| \leq \big(H_{\Lambda,\mathrm{X}}(0,v) - \operatorname{Re} z\big)^{-\alpha}\,|\psi|\,, \tag{A.23}$$

valid for all $\psi \in L^2(\Lambda)$ and all $z \in \mathbb{C}$ with $\operatorname{Re} z < \inf\operatorname{spec} H_{\Lambda,\mathrm{X}}(0,v)$. We recall [55] that the ground-state energy goes up when the magnetic field is turned on, in symbols, $\inf\operatorname{spec} H_{\Lambda,\mathrm{X}}(0,v) \leq \inf\operatorname{spec} H_{\Lambda,\mathrm{X}}(a,v)$. This follows from Remark A.5(iv)(b) or inequality (A.24) below if its r.h.s. is finite.

(ii) If $H_{\Lambda,\mathrm{X}}(0,v)$ has purely discrete spectrum or, equivalently [53, Thm. XIII.64], has compact resolvent, the Dodds-Fremlin-Pitt theorem [3, Thm. 2.2] together with (A.23) implies that $H_{\Lambda,\mathrm{X}}(a,v)$ has also compact resolvent and hence purely discrete spectrum. In turn, $H_{\Lambda,\mathrm{X}}(0,v)$ has purely discrete spectrum if the free operator $H_{\Lambda,\mathrm{X}}(0,0)$ has and if v is a form perturbation of $H_{\Lambda,\mathrm{X}}(0,0)$ [53, Thm. XIII.68]. While $H_{\Lambda,\mathrm{D}}(0,0)$ has purely discrete spectrum for arbitrary bounded open $\Lambda \subset \mathbb{R}^d$, $H_{\Lambda,\mathrm{N}}(0,0)$ only has if Λ possesses an additional property, for example the *segment property*, see [53, pp. 255]. For example, if Λ is a bounded open *cube* the spectra of $H_{\Lambda,\mathrm{D}}(a,-v_-)$ and $H_{\Lambda,\mathrm{N}}(a,-v_-)$ are both purely discrete. Moreover, by the min-max principle the addition of the positive multiplication operator v_+ to $H_{\Lambda,\mathrm{X}}(a,-v_-)$ cannot create essential spectrum. As a consequence, $H_{\Lambda,\mathrm{X}}(a,v)$ has purely discrete spectrum for both X = D and X = N if Λ is a bounded open cube.

(iii) The diamagnetic inequality (A.14) together with Lemma 15.11 in [58] implies the *diamagnetic inequality for partition functions*

$$\operatorname{Tr}\big[\mathrm{e}^{-tH_{\Lambda,\mathrm{X}}(a,v)}\big] \leq \operatorname{Tr}\big[\mathrm{e}^{-tH_{\Lambda,\mathrm{X}}(0,v)}\big] \tag{A.24}$$

for all $t > 0$ and both $\mathrm{X} = \mathrm{D}$ and $\mathrm{X} = \mathrm{N}$, provided that the r.h.s. is finite. The latter is the case if Λ is a bounded open cube, for example. This follows from Dirichlet-Neumann bracketing (see (A.13) with $a = 0$), the facts that $v_+ \geq 0$ and v_- is a form perturbation of $H_{\Lambda,\mathrm{N}}(0, 0)$, and the finiteness of the free Neumann partition function (see [36, Prop. 2.1(c)] or (4.5)).

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Note added in proof. After submission of the present paper we learned of the interesting paper *The L^p-theory of the spectral shift function, the Wegner estimate, and the integrated density of states for some random operators*, Commun. Math. Phys. **218**, 113–130 (2001), by J. M. Combes, P. D. Hislop and S. Nakamura. Among other things, their approach yields Wegner estimates for rather general magnetic fields and certain bounded random potentials. While these estimates do not imply absolute continuity of the integrated density of states, they yield Hölder continuity of arbitrary order strictly smaller than one. The recent preprint *The integrated density of states for some random operators with nonsign definite potentials*, mp_arc 01-139 (2001), by P. D. Hislop and F. Klopp extends part of this result to single-site potentials taking values of both signs.

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