Globally and locally attractive solutions for quasi-periodically forced systems

Michele V. Bartuccelli*, **Jonathan H.B. Deane*** and **Guido Gentile**†

*Department of Mathematics and Statistics, University of Surrey, Guildford, GU2 7XH, UK.

E-mails: m.bartuccelli@surrey.ac.uk, j.deane@surrey.ac.uk

†Dipartimento di Matematica, Universit di Roma Tre, Roma, I-00146, Italy.

E-mail: gentile@mat.unirom3.it

Abstract

We consider a class of differential equations, $\ddot{x} + \gamma\dot{x} + g(x) = f(\boldsymbol{\omega}t)$, with $\boldsymbol{\omega} \in \mathbb{R}^d$, describing one-dimensional dissipative systems subject to a periodic or quasi-periodic (Diophantine) forcing. We study existence and properties of the limit cycle described by the trajectory with the same quasi-periodicity as the forcing. For $g(x) = x^{2p+1}$, $p \in \mathbb{N}$, we show that, when the dissipation coefficient is large enough, there is only one limit cycle and that it is a global attractor. In the case of other forces, including $g(x) = x^{2p}$ (with $p = 1$ describing the varactor equation), we find estimates for the basin of attraction of the limit cycle.

1 Introduction

Consider the ordinary differential equation

$$\ddot{x} + \gamma\dot{x} + x^{2p+1} = f(\boldsymbol{\omega}t), \tag{1.1}$$

where $p \in \mathbb{N}$, $\boldsymbol{\omega} \in \mathbb{R}^d$ is the frequency vector, $f(\boldsymbol{\psi})$ is an analytic quasi-periodic function,

$$f(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \mathrm{e}^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}} f_{\boldsymbol{\nu}}, \tag{1.2}$$

with average $\langle f \rangle \equiv f_{\mathbf{0}} \neq 0$, and $\gamma > 0$ is a real parameter (dissipation coefficient). Here and henceforth we denote with $\cdot$ the scalar product in $\mathbb{R}^d$. By the analyticity assumption on f there are two strictly positive constants F and ξ such that one has $|f_{\boldsymbol{\nu}}| \leq F\,\mathrm{e}^{-\xi|\boldsymbol{\nu}|}$ for all $\boldsymbol{\nu} \in \mathbb{Z}^d$.

If $d > 1$ we shall assume a Diophantine condition on the frequency vector $\boldsymbol{\omega}$, that is

$$|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| \geq C_0|\boldsymbol{\nu}|^{-\tau} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}, \tag{1.3}$$

where $|\boldsymbol{\nu}| = |\boldsymbol{\nu}|_1 \equiv |\nu_1| + \ldots + |\nu_d|$, and C_0 and τ are positive constants, with $\tau > d-1$ and C_0 small enough. Note that for $d = 1$ the condition (1.3) is automatically satisfied for all $\boldsymbol{\omega}$.

In this paper we want to show that for γ large enough the system (1.1) admits a global attractor which is a quasi-periodic solution with the same frequency vector $\boldsymbol{\omega}$ as the forcing f. This will be done in two steps: first we prove that for γ large enough there is a quasi-periodic solution $x_0(t)$ with frequency vector $\boldsymbol{\omega}$ (cf. Theorem 1 in Section 2), second we prove that, again for γ large enough, any trajectory is attracted by $x_0(t)$ (cf. Theorem 2 in Section 3).

In particular, this solves for the system (1.1) a problem left as open in [7]. Indeed in [7] we considered a class of ordinary differential equations, including (1.1), and proved existence of a quasi-periodic solution with the same quasi-periodicity as the forcing, but we couldn't conclude that this was the only solution with such a property. The result stated above gives an affirmative answer to this problem for the system (1.1), by showing that the quasi-periodic solution $x_0(t)$ is unique; cf. Theorem 3 in Section 4.

This uniqueness result holds for the more general systems studied in [7], including the resistor-inductor-varactor circuit, or simply varactor equation, described in [7, 3]. In that case the solution $x_0(t)$ is not a global attractor any more, but it turns out to be the only attractor in a neighbourhood of the solution itself.

More precisely the situation is as follows. We can consider systems described by

$$\ddot{x} + \gamma \dot{x} + g(x) = f(\boldsymbol{\omega} t), \tag{1.4}$$

where f is given by (1.2) and g is an analytic function. Studying the behaviour of the system (1.4) for γ large enough suggests to introduce a new parameter $\varepsilon = 1/\gamma$, in terms of which the differential equation (1.4) becomes

$$\varepsilon \ddot{x} + \dot{x} + \varepsilon g(x) = \varepsilon f(\boldsymbol{\omega} t), \tag{1.5}$$

and study what happens for ε small enough.

If we assume that there exists $c_0 \in \mathbb{R}$ such that $g(c_0) = f_{\mathbf{0}}$ and $g'(c_0) := \partial_x g(c_0) \neq 0$, then the system (1.5) admits a quasi-periodic solution $x_0(t)$, analytic in t, with the same frequency vector $\boldsymbol{\omega}$ as the forcing f, and furthermore $x_0(t) = c_0 + O(\varepsilon)$. This was proved in [7], where the solution $x_0(t)$ was explicitly constructed through a suitable summation of the perturbation series

$$x_P(t) = \sum_{k=0}^{\infty} \varepsilon^k x^{(k)}(\boldsymbol{\omega} t), \qquad x^{(k)}(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} \mathrm{e}^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}} x^{(k)}_{\boldsymbol{\nu}}, \tag{1.6}$$

for a function formally solving the equations of motion.

As a drawback of the construction we were not able to prove any uniqueness result about $x_0(t)$. In fact, in principle, there could be other quasi-periodic solutions near $x_0(t)$, possibly with the same frequency vector $\boldsymbol{\omega}$. Neither we could exclude existence of other solutions reducing to c_0 as $\varepsilon \to 0$ or even admitting the same formal expansion (1.6) in powers of ε. In this papers we get rid of these possibilities, and we prove that there exists, in the plane $(x, \dot{x})$, a neighborhood $\mathcal{B}$ of the point $(c_0, 0)$ where $(x_0(t), \dot{x}_0(t))$ is the only stable solution of (1.5). Moreover it turns out to be asymptotically stable, that is it attracts any trajectory starting in $\mathcal{B}$. Therefore, this allows us to formulate a strengthened version of the theorem proved in [7]; cf. Theorem 4 in Section 4.

In general the neighbourhood $\mathcal{B}$ can be very small. In specific cases one can look for improved estimates of $\mathcal{B}$. In particular we can consider the case $g(x) = x^2$, corresponding to the varactor equation studied in [3]. In that case we can give a good description of the basin of attraction of the quasi-periodic solution; cf. Section 5. This provides a complementary result to the analysis performed in [3], where the set of initial data generating unbounded solutions was extensively studied. The analysis can be extended to any function x^{2p}, $p \in \mathbb{N}$. We also construct a positively invariant set containing the attracting periodic orbit which allows us, together with the results of Section 5, to obtain a larger set estimating the basin of attraction (with area growing linearly in γ); cf. Theorem 5 in Section 6.

Finally we show that in the case $g(x) = x^{2p}$, $p \in \mathbb{N}$, there are unbounded solutions which blow up in finite time, and we discuss the implications for the varactor equation in [3]; cf. Theorem 6 in Section 7.

More formal statements of the results will be formulated in the forthcoming sections. Some open problems will be discussed at the end. Here we confine ourselves to noting that, while in the case of periodic forcing standard techniques, like those based on Poincaré sections [9, 12], could be applied, this is not the case for quasi-periodic forcing, where no Poincaré maps can be introduced.

The rest of the paper is organised as follows. Sections 2 and 3 are devoted to the global study of the system (1.1) for γ large enough, whereas in Section 4 we draw the conclusions, and we pass to the study of the system (1.4). In Section 5 we specify the construction envisaged for any $g(x)$ in (1.4) to the case $g(x) = x^{2p}$, with particular care for $p = 1$, and we look for an estimate of the basin of attraction of the attracting periodic solution. Then, in Section 6, we use the techniques of [3] to improve the estimate of the basin of attraction. In Section 7 we show that, again in the case $g(x) = x^{2p}$, there are unbounded solutions which blow up in finite time. Finally in Section 6 we mention some open problems, and possible future directions of research.

2 Existence of the quasi-periodic solution

First we show that for γ large enough there exists a quasi-periodic solution $x_0(t)$.

Theorem 1 *Consider the equation (1.1), with f a non-zero average quasi-periodic function analytic in its argument and with $\boldsymbol{\omega}$ satisfying the Diophantine condition (1.3). There exists $\gamma_0 > 0$ such that for all $\gamma > \gamma_0$ there is a quasi-periodic solution $x_0(t)$ with the same frequency vector as the forcing term. Such a solution extends to a function analytic in $1/\gamma$ in a disc $\mathcal{D}$ of the complex plane tangent to the imaginary axis at the origin and centered on the real axis. Furthermore, $x_0(t) = \alpha + O(1/\gamma)$, where $\alpha = f_{\mathbf{0}}^{1/(2p+1)} \neq 0$.*

Proof. We can apply the results of Section 7 in [7]. If we set $g(x) = x^{2p+1}$, then $g(c_0) = f_{\mathbf{0}}$ yields $c_0 = f_{\mathbf{0}}^{1/(2p+1)}$, so that $g'(c_0) \neq 0$ as by assumption one has $f_{\mathbf{0}} \neq 0$. Both the existence of the analyticity domain $\mathcal{D}$ and the form of the solution itself follow from the analysis in [7]. ■

If γ is large enough, say $\gamma > \overline{\gamma}_0 \geq \gamma_0$, then the solution $x_0(t)$ is of definite sign. In the following we shall assume that this is the case: hence $x_0(t) \neq 0$ for all $t \in \mathbb{R}$.

3 Convergence to the quasi-periodic solution

Given the quasi-periodic solution $x_0(t)$ one can write $x(t) = x_0(t)+\xi(t)$, with $\xi(t)$ satisfying the differential equation

$$\ddot{\xi} + \gamma\dot{\xi} + \xi F(\xi, x_0(t)) = 0, \tag{3.1}$$

where we have defined

$$F(\xi, x) := \frac{1}{\xi}\left((x+\xi)^{2p+1} - x^{2p+1}\right) = \sum_{j=0}^{2p} \binom{2p+1}{j} \xi^{2p+1-j} x^j. \tag{3.2}$$

We can write (3.1) as

$$\begin{cases} \dot{\xi} = y, \\ \dot{y} = -\gamma y - \xi F(\xi, x_0(t)), \end{cases} \tag{3.3}$$

that is $\dot{z} = \Phi(z)$, if we define $z = (\xi, y)$ and $\Phi(z) = (y, -\gamma y - \xi F(\xi, x_0(t))$. We denote by $\varphi(t, z_0)$ the solution of (3.3) with initial datum z_0. Define also $P(\xi, t) := F(\xi, x_0(t))$ and $Q(\xi) := F(\xi, \alpha)$ and set $R(\xi, t) := P(\xi, t)/Q(\xi)$.

Here we prove the following result.

Theorem 2 *Consider the equation (3.1), with $x_0(t)$ the quasi-periodic solution of (1.1) given in Theorem 1. There exists $\gamma_1 > 0$ such that for all $\gamma > \gamma_1$ all trajectories in phase space converge toward the origin as time goes to infinity.*

The proof will pass through several lemmata.

Lemma 1 *Assume $\gamma > \overline{\gamma}_0$ so that $x_0(t)$ exists and $x_0(t) \neq 0$ for all $t \in \mathbb{R}$. There exist two positive constants R_1 and R_2 such that*

$$R_1 < R(\xi, t) < R_2 \tag{3.4}$$

for all $\xi \in \mathbb{R}$ and for all $t \in \mathbb{R}$.

Proof. By (3.2) we can write

$$F(\xi, x) = (2p+1) \int_0^1 \mathrm{d}s \, (x + s\,\xi)^{2p} , \tag{3.5}$$

so that $F(\xi, x) \geq 0$ for all $(\xi, x) \in \mathbb{R}^2$. Moreover $F(0,0) = 0$ and $F(\xi, x) > 0$ for all $\xi \in \mathbb{R}$ if $x \neq 0$, and $\lim_{|\xi|\to\infty} F(\xi, x) = \infty$ for all $x \in \mathbb{R}$. Hence for $\alpha \neq 0$ and $\gamma > \overline{\gamma}_0$, one has both $P(\xi, t) > 0$ and $Q(\xi) > 0$, hence also $R(\xi, t) > 0$ for all $(\xi, t) \in \mathbb{R}^2$. Moreover $\lim_{|\xi|\to\infty} R(\xi, t) = 1$ for all $t \in \mathbb{R}$, so that the assertion follows. ■

Lemma 2 *Consider the equation (3.1), with $x_0(t)$ the quasi-periodic solution of (1.1) given in Theorem 1. There exists $\gamma_2 > 0$ such that for all $\gamma > \gamma_2$ there is a convex set $\mathcal{S}$ containing the origin such that any trajectory starting inside $\mathcal{S}$ tends to the origin as time goes to infinity. One can take $\mathcal{S}$ such that $\partial\mathcal{S}$ crosses the positive and negative y- and ξ-axes at distances $O(\gamma^2)$ and $O(\gamma^{2/(p+1)})$ from the origin respectively.*

Proof. Rescale time through the Liouville transformation

$$\tau = \int_0^t \mathrm{d}t' \, \sqrt{R(\xi(t'), t')}, \tag{3.6}$$

which is well-defined by Lemma 1. Then, if we introduce the coordinate transformation $\psi : (\xi, y) \to (v, y)$ by setting $\xi(t) = v(\tau(t))$ and $y(t) = \sqrt{R(\xi(t), t)}\, w(\tau(t))$, equation (3.3) is transformed into

$$\begin{cases} v' = w, \\ w' = -\dfrac{w}{\sqrt{R}} \left(\gamma + \dfrac{R'}{2\sqrt{R}} \right) - v\, Q(v), \end{cases} \tag{3.7}$$

where primes denote differentiation with respect to τ, $Q(v(\tau)) = Q(\xi(t(\tau))$ and $R = R(v(\tau), t(\tau)) = R(\xi(t(\tau)), t(\tau))$.

The autonomous system

$$\begin{cases} v' = w, \\ w' = -v\, Q(v), \end{cases} \tag{3.8}$$

can be explicitly solved: all trajectories move on the level curves of the function

$$H(v, w) = \frac{1}{2} w^2 + \int_0^v \mathrm{d}v' \, v' Q(v'). \tag{3.9}$$

In (3.7) one has $R'/\sqrt{R} = \dot{R}/R$, with

$$\frac{\dot{R}}{R} = \frac{\dot{P}}{P} - \frac{\dot{Q}}{Q},$$

and it is easy to see (Appendix A) that there are two γ-independent positive constants B_1 and B_2 such that

$$\left| \frac{\dot{R}}{2R} \right| < \frac{1}{\gamma} \left(B_1 + B_2 |w| \right). \tag{3.10}$$

If γ satisfies $\gamma^2 > 2B_1$ we can define $\widetilde{w}$ as

$$\widetilde{w} = \frac{\gamma^2 - B_1}{B_2} > \frac{\gamma^2}{2B_2}, \tag{3.11}$$

so that $\overline{\gamma} := (\gamma + R'/2\sqrt{R})/\sqrt{R} > 0$ for $|w| \leq \widetilde{w}$.

Consider the compact set $\widetilde{\mathcal{P}}$ whose boundary $\partial\widetilde{\mathcal{P}}$ is given by the level curve $H(v,w) = \widetilde{w}^2/2$ of the system (3.8). Such a curve crosses the w-axis at $w = \pm\widetilde{w} = O(\gamma^2)$ and the v-axis at $v = O(\gamma^{2/(p+1)})$. If we take an initial datum $(v(0), w(0)) \in \widetilde{\mathcal{P}}$ then the dissipation coefficient $\overline{\gamma}$ in (3.7), even if it changes with time, always remains strictly positive. Moreover $H' = -\overline{\gamma}w^2 \leq 0$ and $H' = 0$ only for $w = 0$, and for $w = 0$ the vector field in (3.7) vanishes only at $v = 0$, because $Q(v) > 0$ for all v (cf. the proof of Lemma 1). Then we can apply Barbashin-Krasovsky theorem [1, 11], and conclude that the origin is asymptotically stable and that $\widetilde{\mathcal{P}}$ belongs to its basin of attraction.

Let $\mathcal{P}(t)$ be the time-dependent preimage of $\widetilde{\mathcal{P}}$ under the transformation ψ. By Lemma 1 if γ is large enough there is a compact set $\mathcal{S} \subset \mathcal{P}(t)$ for all $t \in \mathbb{R}$, such that the boundary $\partial\mathcal{S}$ crosses the positive and negative y- and ξ-axes at a distances of order γ^2 and $\gamma^{2/(p+1)}$ from the origin, respectively. All trajectories starting from points inside $\mathcal{S}$ are attracted by the origin. ∎

Lemma 3 *Consider the curve $g(\xi,t) = -\gamma^{-1}\xi F(\xi, x_0(t))$ in the plane (ξ, y). There exists $\gamma_3 > 0$ such that for $\gamma > \gamma_3$, outside the set $\mathcal{S}$ defined in Lemma 2, one has*

$$-\frac{1}{2\gamma}\xi^{2p+1} \geq g(\xi,t) \geq -\frac{2}{\gamma}\xi^{2p+1} \tag{3.12}$$

for all $t \in \mathbb{R}$.

Proof. Consider $\xi \geq 0$ (the case $\xi < 0$ can be discussed in the same way). By (3.2) one has

$$\xi F(\xi, x_0(t)) = \xi^{2p+1} + \sum_{j=1}^{2p} \binom{2p+1}{j} \xi^{2p+1-j} x_0^j(t),$$

and, if γ is sufficiently large so that $|x_0(t)| < 2|\alpha|$, then for $\xi \geq 2|\alpha|$ the sum can be bounded by $2^{2p+1}(2|\alpha|)\,\xi^{2p} \equiv C_p\xi^{2p}$. Hence one has

$$\frac{1}{2}\,\xi^{2p+1} \leq \xi F(\xi, x_0(t)) \leq 2\xi^{2p+1}, \tag{3.13}$$

as soon as $\xi \geq 2C_p$ (note that if $\xi \geq 2C_p$ then one has automatically $\xi \geq 2|\alpha|$). Next, we want to show that the latter inequality is satisfied outside $\mathcal{S}$.

Consider the intersection of the graph of $g(\xi,t)$ with $\partial\mathcal{S}$. Let II be the quadrant $\{(\xi,y) \in \mathbb{R}^2 : \xi \geq 0, y < 0\}$; cf. figure 1. The curve $\partial\mathcal{S}\cap$II is below the line

$$y_1(\xi) = y_0\left(1 - \frac{\xi}{\xi_0}\right), \tag{3.14}$$

where $y_0 := -b\gamma^2$, with $b > 0$, is the y-coordinate of the point at which $\partial\mathcal{S}$ crosses the y-axis, and $\xi_0 := a\gamma^{2/(p+1)}$, with $a > 0$, is the ξ-coordinate of the point at which $\partial\mathcal{S}$ crosses the ξ-axis. On the other hand the graph of $g(\xi,t)$ in II is above the curve

$$y_2(\xi) = -\frac{2}{\gamma}\,(\xi + 2|\alpha|)^{2p+1}\,, \tag{3.15}$$

because in (3.2) one has $\xi F(\xi,x) \leq |x+\xi|^{2p+1} + |x|^{2p+1}$.

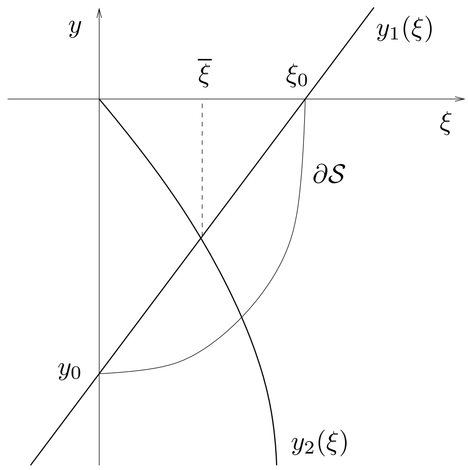

Figure 1: Construction used in the proof of Lemma 3.

As a consequence in II the two curves $\partial\mathcal{S}$ and $g(\xi,t)$ cannot cross each other for $\xi \in (0, a\gamma^\beta/2]$, with $\beta \le 2/(2p+1)$. The latter assertion can be proved by *reductio ad absurdum.* First note that $a\gamma^\beta \le \xi_0$ for γ large enough. Suppose that there exists $\overline{\xi} \le a\gamma^\beta/2$ such that $y_1(\overline{\xi}) = y_2(\overline{\xi})$. Then one has

$$b\frac{\gamma^2}{2} \le b\gamma^2\left(1 - \frac{\overline{\xi}}{\xi_0}\right) = \frac{2}{\gamma}\left(\overline{\xi} + 2|\alpha|\right)^{2p+1} \le \frac{4}{\gamma}\max\{\overline{\xi}, 2|\alpha|\}^{2p+1},$$

that is $b\gamma^3 \le 8\max\{\overline{\xi}, 2|\alpha|\}^{2p+1}$, which is not possible if $\beta \le 2/(2p+1)$ and γ is large enough.

Therefore in II the graph of $g(\xi,t)$ can be outside $\mathcal{S}$ only for $\xi > a\gamma^\beta/2$, which is greater than $2C_p$ for γ large enough. Hence (3.13) is satisfied outside $\mathcal{S}$, so that (3.12) follows. ∎

Lemma 4 *Consider the equation (3.1), with $x_0(t)$ the quasi-periodic solution of (1.1) given in Theorem 1. There exists $\gamma_4 > 0$ such that for all $\gamma > \gamma_4$, if $z \notin \mathcal{S}$,then either $\varphi(t,z)$ enters $\mathcal{S}$ or crosses the y-axis outside $\mathcal{S}$ in a finite positive time.*

Proof. Consider the four quadrants

$$\begin{aligned} \mathrm{I} &= \{(\xi,y) \in \mathbb{R}^2 : \xi > 0, y \ge 0\}, \\ \mathrm{II} &= \{(\xi,y) \in \mathbb{R}^2 : \xi \ge 0, y < 0\}, \\ \mathrm{III} &= \{(\xi,y) \in \mathbb{R}^2 : \xi < 0, y \le 0\}, \\ \mathrm{IV} &= \{(\xi,y) \in \mathbb{R}^2 : \xi \le 0, y > 0\}. \end{aligned} \tag{3.16}$$

In I one has $\dot{\xi} \ge 0, \dot{y} < 0$, in II one has $\dot{\xi} < 0$, in III one has $\dot{\xi} \le 0, \dot{y} > 0$, and in IV one has $\dot{\xi} > 0$. It is easy to see that each trajectory starting in I enters II and each trajectory starting from III enters IV in a finite time (see Appendix B).

Consider now an initial datum z in II but not in to $\mathcal{S}$. Let $\mathcal{C}_1$ be a continuous curve $\xi \to y(\xi)$ in II such that $\dot{y} < 0$ for z in II above $\mathcal{C}_1$; cf. figure 2. Existence of such a curve follows from Lemma 3, which also implies that $\mathcal{C}_1$ is decreasing outside $\mathcal{S}$ (see Appendix C). The curve $\mathcal{C}_1$ divides II into two sets IIa and IIb, with IIa above IIb. Denote by $\mathcal{T}_1$ and $\mathcal{T}_2$ the parts of IIa and IIb, respectively, outside $\mathcal{S}$. Hence for $z \in \mathcal{T}_1$ the trajectory $\varphi(t,z)$ either enters $\mathcal{S}$ or enters $\mathcal{T}_2$. In the latter case it cannot come back to $\mathcal{T}_1$, hence $y(t) \le \overline{y}$, if $(\overline{\xi}, \overline{y}) = \mathcal{C}_1 \cap \partial\mathcal{S}$. This means that if the solution does not enter $\mathcal{S}$ then it has to cross the vertical axis and enter III.

Analogously one discusses the case of initial data z in IV, outside $\mathcal{S}$: their evolution leads either to $\mathcal{S}$ or to I. Hence the lemma is proved. ∎

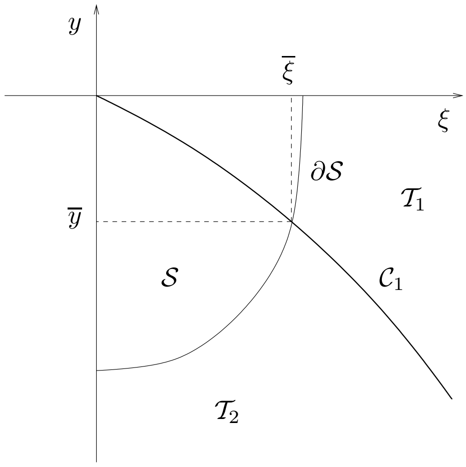

Figure 2: Construction used in the proof of Lemma 4.

Lemma 5 *Consider the equation (3.1), with $x_0(t)$ the quasi-periodic solution of (1.1) given in Theorem 1. There exists $\gamma_5 > 0$ such that for all $\gamma > \gamma_5$ and for all $z \notin \mathcal{S}$ on the vertical axis, either $\varphi(t,z)$ enters $\mathcal{S}$ or the trajectory $\varphi(t,z)$ re-crosses the vertical axis at a point z_1 which is such that $|z| - |z_1| > \delta$ for some positive z-independent constant δ.*

Proof. Fix an initial datum $z \notin \mathcal{S}$ on the vertical axis. This means that at $t = 0$ one has $z = z(0) = (0, y(0))$ outside $\mathcal{S}$. Assume for concreteness $y(0) > 0$ and set $y(0) = 1/\varepsilon^{p+1}$, with $\varepsilon > 0$. As $z(0) \notin \mathcal{S}$ there exists a constant C_2 such that $\gamma^2 \varepsilon^{p+1} \leq C_2$. Consider the change of coordinates

$$X = \varepsilon \xi, \qquad Y = \varepsilon^{p+1} y, \qquad T = \frac{t}{\varepsilon^p}. \tag{3.17}$$

Then the system (3.3) becomes

$$\begin{cases} X' = Y, \\ Y' = -\gamma \varepsilon^p Y - X F(X, \varepsilon x_0(\varepsilon^p T), \end{cases} \tag{3.18}$$

where primes denote differentiation with respect to T. Note that $Y(0) = 1$ and $XF(X, \varepsilon x_0(\varepsilon^p T) = X^{2p+1} + O(\varepsilon\, X^{2p})$. Call S the image of $\mathcal{S}$ under the transformation (3.17); cf. figure 3.

We can rewrite the system (3.18) as

$$\begin{cases} X' = Y, \\ Y' = \Psi(X,Y) \equiv \Psi_1(X,Y) + \Psi_2(X,Y) + \Psi_3(X,Y), \end{cases} \tag{3.19}$$

with

$$\begin{cases} \Psi_1(X,Y) = -XF(X, \varepsilon x_0(0)), \\ \Psi_2(X,Y) = -X\left[\, F(X, \varepsilon x_0(\varepsilon^p T)) - F(X, \varepsilon x_0(0)) \,\right], \\ \Psi_3(X,Y) = -\gamma \varepsilon^p Y. \end{cases} \tag{3.20}$$

If we replace $\Psi(X,Y)$ with $\Psi_1(X,Y)$ in (3.19) the trajectory moves on the level curve $\Gamma = \{(X,Y) \in \mathbb{R}^2 : H(X,Y) = 1/2\}$ for the function

$$H(X,Y) = \frac{1}{2} Y^2 + \int_0^X \mathrm{d}X\, XF(X, \varepsilon x_0(0)) \tag{3.21}$$

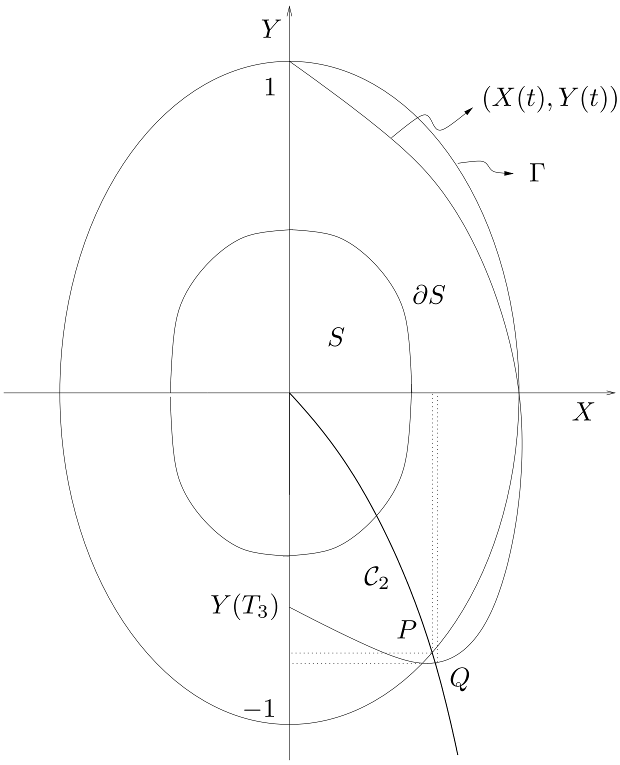

Figure 3: Construction used in the proof of Lemma 5.

and crosses the vertical axis at the point $(0, -1)$, hence at the same distance from the origin as at $t = 0$. By Lemma 3 there exists in II a curve $\mathcal{C}_2$, decreasing outside S, such that

$$-\gamma\varepsilon^p Y \geq \Psi(X, Y) \geq -\gamma\varepsilon^p Y/2,$$

for Y in II below $\mathcal{C}_2$ (see Appendix C). Such a curve can be chosen in such a way that it crosses the level curve Γ in a point $P = (X_P, Y_P)$, with $X_P = 2D_1(\gamma\varepsilon^p)^{1/(2p+1)}$, for some constant D_1 (see Appendix C). Note that the time T_1 necessary to reach such a point is of order 1.

If we take into account the component $\Psi_2(X, Y)$ of the vector field in (3.19), we can move from P at most by a quantity of order ε^{p+1}. Indeed, as long as the motion remains close to that generated by the vector field $\Psi_1(X, Y)$, one has

$$|F(X, \varepsilon x_0(\varepsilon^p T_1)) - F(X, \varepsilon x_0(0))| \leq D_2'\varepsilon\, |x_0(\varepsilon^p T_1) - x_0(0)| \leq D_2''\varepsilon^{p+1}, \tag{3.22}$$

for suitable positive constants D_2' and D_2'', so that the points reached at time T_1 by moving according to the vector fields Ψ_1 and $\Psi_1 + \Psi_2$ cannot be more distant than $D_2\varepsilon^{p+1}$ for some constant D_2. This follows from the fact that the system is quasi-integrable, so that in a time of order 1 the action variable can change at most by a quantity of order of the perturbation as bounded in (3.22); see Appendix D.

Finally the component $\Psi_3(X, Y)$ points inward along the full length of the curve Γ. Define T_2 as the time at which the the trajectory of the full system (3.19) crosses the curve $\mathcal{C}_2$ in a point Q near P. Of course T_2 is near T_1, and so is of order 1, and $X(T_2) \geq X_P/2$ by construction, while $Y(T_2) \geq Y_P - D_2\varepsilon^{p+1}$, with $Y_P > -1$.

From time T_2 onwards, we have

$$\begin{cases} X' = Y, \\ Y' \geq -\gamma\varepsilon^p Y/2, \end{cases} \tag{3.23}$$

as long as the motion remains below $\mathcal{C}_2$. The latter property is easily checked to hold (see Appendix C). Then the trajectory crosses the vertical axis and meanwhile, at least, moves upward in the vertical direction by a quantity $\gamma\varepsilon^p X_P/2 = D_1\gamma\varepsilon^p(\gamma\varepsilon^p)^{1/(2p+1)}$.

Therefore when the trajectory again crosses the vertical axis, this happens at a time T_3 such that $Y(T_3) \geq Y_P - D_2\varepsilon^{p+1} + D_1\gamma\varepsilon^p(\gamma\varepsilon^p)^{1/(2p+1)} > -1 + \Delta Y$, with

$$\Delta Y = D_1\gamma\varepsilon^p(\gamma\varepsilon^p)^{1/(2p+1)} - D_2\varepsilon^{p+1} \geq D_2\varepsilon^{p+1},$$

where the latter inequality holds provided $(\gamma\varepsilon^p)^{1+1/(2p+1)} \geq 2D_2\varepsilon^{p+1}$, that is provided

$$\gamma^{2(p+1)} \geq D_0\varepsilon^{p+1}, \qquad D_0 = (2D_2)^{2p+1}. \tag{3.24}$$

Since $\varepsilon^{p+1}\gamma^2 \leq C_2$, inequality (3.24) is satisfied if $\gamma^{2(p+1)} \geq D_0C_2\gamma^{-2}$, which requires $\gamma^{2(p+2)} \geq D_0C_2$, that is

$$\gamma \geq (D_0C_2)^{1/2(p+2)}, \qquad D_0 = (2D_2)^{2p+1}. \tag{3.25}$$

Under this condition one has $|Y(0)| - |Y(T_3)| = 1 - |Y(T_3)| \geq \Delta Y \geq D_2\varepsilon^{p+1}$, so that, in terms of the original coordinate y, one has $|y(0)| - |y(t_3)| \geq D_2$.

Then, if the trajectory crosses the vertical axis once more in the positive direction (and this necessarily happens if it does not enter $\mathcal{S}$, by Lemma 4), this occurs at a time t_4 such that

$$|y(0)| - |y(t_4)| \geq 2D_2, \tag{3.26}$$

where we recall that the constant D_2 is independent of the initial datum $y(0)$. Simply one can repeat the argument above by taking $(0, y(t_3))$ as initial datum and calling t_4 the time of crossing of the positive ξ-axis. This means that the trajectory either enters $\mathcal{S}$ or, after a complete cycle, moves closer to the origin by a finite positive distance $\delta = 2D_0$. ■

Lemma 6 *Consider the equation (3.1), with $x_0(t)$ the quasi-periodic solution of (1.1) given as in Theorem 1. There exists $\gamma_6 > 0$ such that for all $\gamma > \gamma_6$ for all $z \notin \mathcal{S}$ there is a finite time $t(z)$ such that $\varphi(t(z), z) \in \mathcal{S}$.*

Proof. Consider $z \notin \mathcal{S}$. By Lemma 4 either $\varphi(t, z)$ enters $\mathcal{S}$ or there exists a time t_1 such that $\varphi(t_1, z)$ is on the vertical axis outside $\mathcal{S}$. Hence, without loss of generality, we can consider only initial data $z = (\xi, y)$ outside $\mathcal{S}$ such that $\xi = 0$. Assume $y > 0$ (if $y < 0$ the discussion proceeds in the same way): we can apply Lemma 4 and we find that, as far as the trajectory does not enter $\mathcal{S}$, at each turn it gets closer to $\mathcal{S}$ by a finite quantity. Hence sooner or later it enters $\mathcal{S}$. ■

Theorem 2 follows from the lemmata above: it is enough to take $\gamma_1 = \max\{\overline{\gamma}_0, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}$, so that all lemmata apply.

4 Uniqueness of the quasi-periodic solutions

Let γ be $\gamma > \max\{\gamma_0, \gamma_1\}$. Then there exists a quasi-periodic solution $x_0(t)$ for the system (1.1), and such a solution is a global attractor.

In [7] we explicitly constructed a solution $x_0(t)$ with the properties stated in Theorem 1. Such a solution turns out to be Borel summable for $d = 1$. In general the solution is obtained from the formal series through a suitable summation procedure. Since Theorem 2 implies that if there exists a quasi-periodic solution $x_0(t)$ this has to be unique, we can conclude that for real γ large enough there exists a unique quasi-periodic solution $x_0(t)$ with the same frequency vector as the forcing f. By setting $\varepsilon = 1/\gamma$, in the complex ε-plane, there is a solution $x_1(t)$ which is analytic in a domain containing a disk $\mathcal{D}$ with centre on the real axis and tangent at the origin to the imaginary axis. For real γ such a solution coincides with $x_0(t)$ (as the latter is the only one), hence, by uniqueness of the analytic continuation, the function $x_1(t)$ is the only solution of (1.1) in all the domain $\mathcal{D}$. In particular it is the only one which admits the formal expansion given by perturbation theory.

We can summarise the discussion above through the following statement.

Theorem 3 *Consider the equation (1.1), with f a non-zero average quasi-periodic function analytic in its argument and with $\boldsymbol{\omega}$ satisfying the Diophantine condition (1.3). There exists $\gamma_0 > 0$ such that for all real $\gamma > \gamma_0$ there is a unique quasi-periodic solution $x_0(t)$ with the same frequency vector as the forcing term. Such a solution describes a limit cycle in the plane $(x, \dot{x})$ which is a global attractor.*

Note that the hypotheses made in Theorem 1 are more restrictive than those considered in [7]. In particular we excluded both polynomial nonlinearities and monomial nonlinearities with even degree.

The first request aims to have a strictly positive function $R(\xi, t)$, which was used in order to construct the positively invariant set $\mathcal{S}$. We leave as an open problem to study what happens if the nonlinearity x^{2p+1} in (1.1) is replaced with

$$g(x) = \sum_{j=1}^{2p+1} a_j\, x^j, \qquad a_j \in \mathbb{R}, \qquad a_{2p+1} > 0. \tag{4.1}$$

For $d = 1$ and $\gamma = 0$ it is known that all motions are bounded, also replacing the constants a_j with periodic functions [5, 8, 4]. The same holds for $d > 1$ [10]. One could expect that the presence of friction tends to contract phase space toward some periodic solution (which certainly exists for γ large enough, as proved in [7]), but our results do not allow us to treat, in general, such a case.

If the nonlinearity x^{2p+1} is replaced with an even monomial x^{2p}, with $p \in \mathbb{N}$, then, under the further condition that $f_{\mathbf{0}} > 0$, there is a quasi-periodic solution $x_0(t)$: again this follows from [7]. In such a case $x_0(t)$ is not a global attractor, as there are unbounded solutions; cf. for example [3] for $p = 1$. Still one can prove that the solution found in [7] is unique, in the sense that it is the only attractor in a neighbour of the solution itself, and it is a local attractor. The same result holds, more generally, for any analytic $g(x)$ in (1.4) such that $g(c_0) = f_{\mathbf{0}}$ and $g'(c_0) \neq 0$ for some $c_0 \in \mathbb{R}$. A more formal statement is as follows.

Theorem 4 *Consider the equation (1.4), with f, given by (1.2), and g both analytic in their arguments, and with $\boldsymbol{\omega}$ satisfying the Diophantine condition (1.3). Assume that there exists $c_0 \in \mathbb{R}$ such that $g(c_0) = f_{\mathbf{0}}$ and $g'(c_0) \neq 0$. There exists $\gamma_0 > 0$ such that for $\gamma > \gamma_0$ there is a unique quasi-periodic solution $x_0(t)$ which has the same frequency vector as f, reduces to c_0 in the limit $\gamma \to \infty$, and extends to a function analytic in a disk with center on the positive real axis and boundary tangent to the vertical axis at the origin. Furthermore, there exists $\gamma_1 \geq \gamma_0$ such that for $\gamma > \gamma_1$ there is a neighbourhood $\mathcal{B}$ of the point $(c_0, 0)$, containing the orbit described by $x_0(t)$, with the property that all trajectories starting in $\mathcal{B}$ are attracted to the cycle described by $x_0(t)$ in the plane.*

Proof. The existence of a quasi-periodic solution $x_0(t)$ with the same frequency vector $\boldsymbol{\omega}$ as the forcing was proved in [7]. As a byproduct of the proof, one can write $x_0(t) = c_0 + x_1(\boldsymbol{\omega} t)$, with $x_1(\boldsymbol{\psi})$ analytic in $\boldsymbol{\psi}$ and of order ε, if $\varepsilon = 1/\gamma$ (that is for ε small enough one has $|x_1(\boldsymbol{\psi})| \leq C|\varepsilon|$ for all $\boldsymbol{\psi}$ and for a suitable C). Therefore we can write $x(t) = x_0(t) + \xi(t)$, where $\xi(t)$ satisfies the differential equation

$$\ddot{\xi} + \gamma\dot{\xi} + \xi F(\xi, x_0(t)) = 0, \tag{4.2}$$

with

$$F(\xi, x) = \frac{1}{\xi}\Big(g(x + \xi) - g(x)\Big) = \partial_x g(x) + O(\xi). \tag{4.3}$$

Then we can write (4.2) as a system of first order differential equations,

$$\begin{cases} \dot{\xi} = y, \\ \dot{y} = -\gamma y - \xi F(\xi, x_0(t)), \end{cases} \tag{4.4}$$

and define $R(\xi, t) = F(\xi, x_0(t))/F(\xi, c_0)$. It is easy to see that one has

$$\lim_{\xi \to 0} R(\xi, t) = 1 + O(\varepsilon),$$

so that for ε small enough one has $R_1 < R(\xi, t) < R_2$, for two suitable positive constants R_1 and R_2.

Then we can rescale time and variables by setting

$$\tau = \int_0^t \mathrm{d}t \sqrt{R(\xi(t'), t')}, \qquad \xi(t) = v(\tau(t)), \qquad y(t) = \sqrt{R(\xi(t), t)}\, w(\tau(t)), \tag{4.5}$$

which transforms the system (4.4) into

$$\begin{cases} v' = w, \\ w' = -\dfrac{w}{\sqrt{R}} \left(\gamma + \dfrac{R'}{2\sqrt{R}} \right) - v\, F(v, c_0), \end{cases} \tag{4.6}$$

where primes denote differentiation with respect to τ.

If we neglect the friction term in (4.6) we obtain the autonomous system

$$\begin{cases} v' = w, \\ w' = -vF(v, c_0)) = -\partial_x g(c_0)\, v + O(v^2), \end{cases} \tag{4.7}$$

which admits the constant of motion

$$H(v, w) = \frac{1}{2} w^2 + \frac{1}{2} \partial_x g(c_0)\, v^2 + O(v^3), \qquad \partial_x g(c_0) \neq 0. \tag{4.8}$$

Hence the origin is a stable equilibrium point for (4.7), and the level curves for H are close to ellipses in a neighbourhood $\widetilde{\mathcal{P}}$ of the origin. It is easy to check that in $\widetilde{\mathcal{P}}$ the coefficient of the friction term is strictly positive, for γ large enough, because $R'/2\sqrt{R}$ is (in $\widetilde{\mathcal{P}}$) less than a constant. Hence we can apply Barbashin-Krasovsky's theorem and conclude that the origin is asymptotically stable and $\widetilde{\mathcal{P}}$ belongs to its basin of attraction. If we go back to the original variables (ξ, y) we find that $\widetilde{\mathcal{P}}$ is transformed back to a time-dependent set $\mathcal{P}(t)$. But the dependence on t of $\mathcal{P}(t)$ is very weak (as R is close to 1 for γ large enough), so that there exists a convex set $\mathcal{S} \subset \mathcal{P}(t)$ for all $t \in \mathbb{R}$. Hence any trajectory starting from $\mathcal{S}$ is attracted toward the origin. In terms of the variables (x, y), using once more that the solution $x_0(t)$ is close to c_0 within $O(\varepsilon)$, we can say that, for ε small enough (that is for γ large enough) there exists a neighbourhood $\mathcal{B}$ of the point $(c_0, 0)$ such that it contains the cycle described by the quasi-periodic solution $x_0(t)$ in the plane $(x, \dot{x})$, and any trajectory starting from $\mathcal{B}$ is attracted by such a cycle.

In particular the solution $x_0(t)$ is the only quasi-periodic solution which tends to c_0 as $\varepsilon \to 0$, and for $\varepsilon > 0$ small enough, say $\varepsilon < \varepsilon_0$, it is the only one which admits the formal power expansion (1.6). Such a solution was proved in [7] to be analytic in a domain $\mathcal{D}$ containing the interval $(0, \varepsilon_0)$, hence by the uniqueness of the analytic continuation, we can conclude that $x_0(t)$ is unique in all $\mathcal{D}$. ∎

By looking at the proof of Theorem 4, we see that it proceeds along the same lines of the proof of Lemma 2 in Section 3. By using the definitions of [6], Theorem 4 says that the orbit described by $x_0(t)$ is an attracting set (and an attractor) with fundamental neighbourhood $\mathcal{B}$.

5 The varactor equation, and generalisations

The construction of the set $\mathcal{B}$ in the proof of Theorem 4 can be improved of course in concrete examples. Here we want to consider explicitly the case of even monomials x^{2p}, with particular emphasis for $p = 1$. Hence we discuss explicitly the case $p = 1$, but the discussion can be easily extended to all $p \geq 1$.

So, let us fix $p = 1$. If we define the functions $F(\xi, x)$ and $R(\xi, t)$ as in the proof of Theorem 4, we find

$$\xi\, F(\xi, x) = \xi^2 + 2x\xi, \qquad R(\xi, t) = \frac{\xi + 2x_0(t)}{\xi + 2c_0} = 1 + 2\frac{x_0(t) - c_0}{\xi + 2c_0}, \tag{5.1}$$

where $c_0 = \sqrt{f_{\mathbf{0}}}$, with $f_{\mathbf{0}} > 0$. For γ large enough, that is for ε small enough, one has

$$|x_0(t) - c_0| < C_1\varepsilon, \tag{5.2}$$

for a suitable constant C_1 (see the first lines of the proof of Theorem 4).

The transformation of coordinates described in the proof of Theorem 4 leads to (4.6), with

$$v\, F(v, c_0) = v^2 + 2c_0 v, \tag{5.3}$$

so that, if we set $\overline{\gamma} = (\gamma + R'/2\sqrt{R})/\sqrt{R}$, we can interpret the system as a Hamiltonian system with hamiltonian

$$H(v, w) = \frac{1}{2}w^2 + U(v), \qquad U(v) = \frac{1}{3}v^3 + c_0 v^2, \tag{5.4}$$

in the presence of a friction term with non-constant dissipation coefficient $\overline{\gamma}$. If we neglect the friction term, that is if we put $\overline{\gamma} = 0$, then the system becomes the Hamiltonian system

$$\begin{cases} v' = w, \\ w' = -v^2 - 2c_0 v, \end{cases} \tag{5.5}$$

which admits a stable equilibrium point $P_1 = (0, 0)$, an unstable equilibrium point $P_2 = (-2c_0, 0)$ and the separatrix Γ of equation

$$w = \pm\sqrt{2(H(-2c_0, 0) - U(v))}, \qquad H(-2c_0, 0) = \frac{4c_0^3}{3}, \tag{5.6}$$

which contains a homiclinic orbit to the right of P_2. Consider the open set $\mathcal{D}_1$ containing the point P_1 and with boundary given by (the closure of) the homoclinic orbit in Γ. Any level curve with energy $V(0) < E < V(-2c_0)$ contains a bounded connected component, internal to $\mathcal{D}_1$, which is a closed orbit $\mathcal{C}_E$ for the system (5.5). Call $\mathcal{C}$ the closed curve $\mathcal{C}_E$ such that it intersects the negative v-axis at $v = -2c_0 + C_2\varepsilon^\beta$, for suitable constants C_2 and β to be fixed, and $\mathcal{D}$ the bounded open set with boundary $\mathcal{C}$.

From (5.1), one has

$$|R(\xi, t) - 1| < \frac{C_1}{C_2}\varepsilon^{1-\beta}, \tag{5.7}$$

so that $R(\xi, t) \to 1$ as $\varepsilon \to 0$, provided $\beta < 1$; for instance one can take $\beta = 1/2$. Then, if γ is large enough, for any point $(v, w) \in \overline{\mathcal{D}}$ one has

$$|R'(\xi, t)| \leq \left|\frac{2\xi'(c_0 - x_0)}{(\xi + 2c_0)^2}\right| + \left|\frac{2x_0'}{\xi + 2c_0}\right| \leq C_3\varepsilon^{1-2\beta} + C_4\varepsilon^{1-\beta}, \tag{5.8}$$

for suitable constants C_3 and C_4, so that, by using also that $\xi = v$ and

$$|\xi + 2c_0| \geq C_2\varepsilon^\beta \tag{5.9}$$

for all $(v, w) \in \overline{\mathcal{D}}$, one obtains

$$|\gamma - \gamma'| \leq \left|\frac{R'}{2R}\right| \leq C_5\varepsilon^{1-2\beta} \leq C_5\varepsilon^{2(1-\beta)}\gamma < \gamma \tag{5.10}$$

for a suitable constant C_5, hence $\gamma' > 0$ for γ large enough. Therefore we can conclude that $\overline{\mathcal{D}}$ is positively invariant, and we can apply once more Barbashin-Krasovsky's theorem to conclude that the equilibrium point P_1 is asymptotically stable and the set $\mathcal{D}$ is contained inside its basin of attraction.

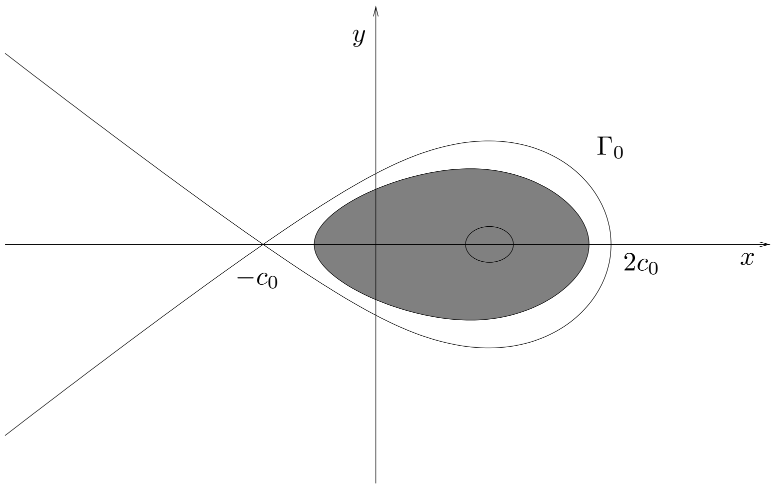

Figure 4: Estimate of the basin of attraction (grey set) of the limit cycle (solid line inside the grey set) for large γ. The curve Γ_0 is the separatrix of the system described by the differential equations $\ddot{x} + x^2 = c_0^2$.

If we go back to the original coordinates, we see that, for γ large enough, there is a quasi-periodic solution $x_0(t)$ which moves very close to the point $(c_0, 0)$ (within a distance of order $\varepsilon = 1/\gamma$); cf. figure 4. Its basin of attraction contains a large set whose boundary is at a distance of order ε^β from the separatrix Γ_0 of the system, that is from the curve of equation $y = \pm\sqrt{2(2c_0^3 - x^3 + 3c_0^2 x)/3}$.

6 Improved estimates for the basins of attraction

We start by constructing an invariant set $\mathcal{A}$, valid for all $p \in \mathbb{N}$, which has the property that its area is $O(\gamma)$. The construction is surely not optimal but is included because it is, amongst many sets constructed, one which grows with γ in the vertical direction. Constructing invariant sets containing the attracting orbit with γ considered to be large is not difficult, but finding ones that grow with γ was found to be less trivial. The construction follows very closely the ideas exploited in [3].

Rewrite (1.4), with $g(x) = x^{2p}$ as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = f(\boldsymbol{\omega} t) - \gamma y - x^{2p}, \end{cases} \tag{6.1}$$

so that the vector field generated by the differential equation is defined by $\boldsymbol{\phi}(t) = (y, f(\boldsymbol{\omega} t) - \gamma y - x^{2p})$. Let $f^{2p} \leq f(\boldsymbol{\omega} t) \leq F^{2p}$. Note that in this and the following section, f is just a constant and not the function $t \to f(\boldsymbol{\omega} t)$. We adopt this notation here to conform to that used in the analysis in [3]. For the same reason we denote in boldface vectors in $\mathbb{R}^2$, so that $\boldsymbol{a} \cdot \boldsymbol{b}$ denotes the scalar product in $\mathbb{R}^2$.

The two vector fields, $\boldsymbol{\phi}_F = (y, F^{2p} - \gamma y - x^{2p})$ and $\boldsymbol{\phi}_f = (y, f^{2p} - \gamma y - x^{2p})$, have no explicit time-dependence and also have the property that for all t, $\boldsymbol{\phi}(t) = \mu \boldsymbol{\phi}_F + (1-\mu)\boldsymbol{\phi}_f$ where μ is a (time-dependent) scalar.

Following [3], we let the boundary of invariant set $\mathcal{A}$ be a hexagon GHIJKL whose edges are straight lines, except for HI and KL. GH and JK are horizontal and LG and IJ are vertical — see figure 5. The co-ordinates of points H and K are $(0, y_H)$ and $(0, y_K)$ respectively. The dotted curves in figure 5 are $P_F : y = (F^{2p} - x^{2p})/\gamma$ (upper) and $P_f : y = (f^{2p} - x^{2p})/\gamma$ (lower). As shown in [3], only in the region between these curves is the sign of $\dot{y}$ ambiguous; above P_F, $\dot{y} < 0$ and below P_f, $\dot{y} > 0$. Provided that $y_H \geq F^{2p}/\gamma$, which will turn out to be automatically satisfied, $\boldsymbol{\phi}(t)$ will be into $\mathcal{A}$ along GH, and point J being below P_f guarantees that $\boldsymbol{\phi}(t)$ will always be into KJ, both results holding for all t. Furthermore, the sign of $\dot{x} = y$ guarantees that $\boldsymbol{\phi}(t)$ will be into LG and IJ, again for all t. It therefore remains to prove that $\boldsymbol{\phi}(t)$ is into the curved sides HI and KL for all t.

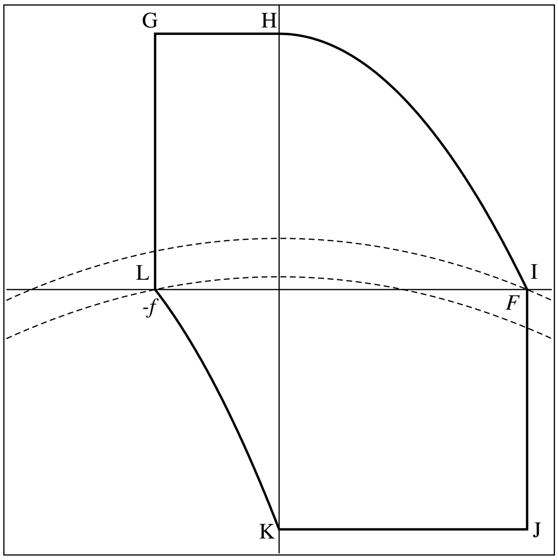

Figure 5: The absorbing set GHIJKL, $\mathcal{A}$. Also shown as dotted lines are the curves $P_F : y = (F^{2p} - x^{2p})/\gamma$ and $P_f : y = (f^{2p} - x^{2p})/\gamma$.

Let us define HI by $y = \lambda_1 \left(F^{2p} - x^{2p}\right)$ with $\lambda_1 \in \mathbb{R}$ positive; then the inward-pointing normal, $\boldsymbol{n}_1 = (-2p\lambda_1 x^{2p-1}, -1)$. In order to prove that $\boldsymbol{\phi}(t)$ is into HI for all time, we need only show that $\boldsymbol{n}_1 \cdot \boldsymbol{\phi}_F \geq 0$ for $x \in [0, F]$. We have $\boldsymbol{n}_1 \cdot \boldsymbol{\phi}_F = \left[F^{2p} - x^{2p}\right] \left[\lambda_1(\gamma - 2p\lambda_1 x^{2p-1}) - 1\right]$ and, since $F^{2p} - x^{2p} \geq 0$, and the second bracket in the scalar product reaches its minimum value over $[0, F]$ at $x = F$, we require

$$2pF^{2p-1}\lambda_1^2 - \gamma\lambda_1 + 1 \leq 0, \tag{6.2}$$

in order for the scalar product to be non-negative. Solving this quadratic and choosing the larger solution gives

$$\lambda_1 = \frac{\gamma}{4pF^{2p-1}} \left[1 + \sqrt{1 - \frac{8pF^{2p-1}}{\gamma^2}}\right] \tag{6.3}$$

provided $\lambda_1 \in \mathbb{R}$. This is true for $\gamma^2 \geq 8pF^{2p-1}$. We now show that this condition on γ also forces $y_H = \lambda_1 F^{2p} \geq F^{2p}/\gamma$, or $\lambda_1 \geq 1/\gamma$, to be satisfied. Letting $q = 4pF^{2p-1}$, the condition for λ_1 to be real is $\gamma^2/q \geq 2$; using this in the definition of λ_1 gives $(\gamma^2/q)\left(1 + \sqrt{1 - 2q/\gamma^2}\right) \geq 2\left(1 + \sqrt{1 - 2q/\gamma^2}\right) \geq 1$, which is clearly true. Hence, boundary HI has been constructed in such a way that $\boldsymbol{\phi}(t)$ is into it for all time, provided that H is above P_F: this is true if $\lambda_1 \geq 1/\gamma$. Note that $\lambda_1 \sim \gamma/(2pF^{2p-1})$ as $\gamma \to \infty$, and so $y_H = \lambda_1 F^{2p} = O(\gamma)$.

We now define KL by $y = \lambda_2 \left(f^{2p} - (x + 2f)^{2p}\right)$ where $\lambda_2 \in \mathbb{R} > 0$. This has inward normal $\boldsymbol{n}_2 = \left(2p\lambda_2(x + 2f)^{2p-1}, 1\right)$ and the inequality we need to consider is now

$$\boldsymbol{\phi}_f \cdot \boldsymbol{n}_2 = \lambda_2 \left[(x + 2f)^{2p} - f^{2p}\right] \left[\gamma - 2p\lambda_2(x + 2f)^{2p-1}\right] + f^{2p} - x^{2p} \geq 0 \tag{6.4}$$

for $x \in [-f, x_K]$ where x_K, satisfying $F \geq x_K > -f$, is to be defined. The first term consists of a product of two terms, the first of which is positive for $x \in (-f, \infty)$ and the second of which is a monotonically decreasing function of x for $x \in \mathbb{R} \setminus \{-2f\}$; it is zero at $x = x_0 = (\gamma/2p\lambda_2)^{1/(2p-1)} - 2f$. Choosing $x_K = x_0 = 0$ ensures that the product term in (6.4) is non-negative and gives

$$\lambda_2 = \frac{\gamma}{2p(2f)^{2p-1}}. \tag{6.5}$$

The last term in inequality (6.3) is non-negative for $x \in [-f, f]$ and so the scalar product is non-negative for λ_2 as given above, provided that $x_K = 0$. We also require point J to be below p_f. The y-co-ordinate of J $= y_K = \lambda_2\left(f^{2p} - (2f)^{2p}\right)$ and so this condition becomes

$$\gamma^2 \geq \frac{p\left(F^{2p} - f^{2p}\right)}{f\left(1 - 4^{-p}\right)}. \tag{6.6}$$

We can now state our result.

Theorem 5 *Define $\mathcal{A}$ as the hexagonal set whose vertices are G, H, I, J, K and L, where*

$$\begin{gathered} G = (-f, \lambda_1 F^{2p}), \quad H = (0, \lambda_1 F^{2p}), \quad I = (F, 0), \\ J = (F, -\lambda_2 f^{2p}(4^p - 1)), \quad K = (0, -\lambda_2 f^{2p}(4^p - 1)), \quad L = (-f, 0), \end{gathered} \tag{6.7}$$

and where λ_1 and λ_2 are given by equations (6.3) and (6.4), respectively. Let the edges LG, GH, IJ and JK of $\mathcal{A}$ be straight lines and let HI be given by $y = \lambda_1\left(F^{2p} - x^{2p}\right)$ and KL, by $y = \lambda_2\left(f^{2p} - (x + 2f)^{2p}\right)$. Then, provided that

$$\gamma^2 \geq \max\left(8pF^{2p-1}, \frac{p\left(F^{2p} - f^{2p}\right)}{f\left(1 - 4^{-p}\right)}\right), \tag{6.8}$$

set $\mathcal{A}$ is an invariant set containing the limit cycle described by the solution $x_0(t)$ in the plane.

Figure 6 compares the set GHIJKL with the actual basin of attraction for $p = 1, 2$ and 3.

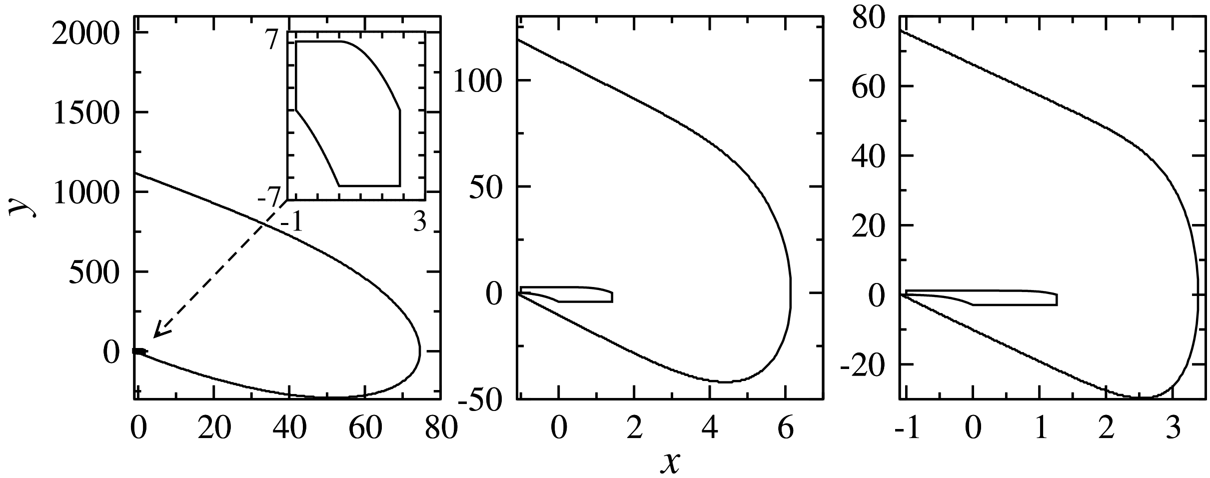

Figure 6: The set GHIJKL compared to the set $\mathcal{B}^c$ (cf. [3]) with $1 \leq f(t) = (5 + 3\sin t)/2 \leq 4$, $\gamma = 9$ and $p = 1$ (left), $p = 2$ (middle) and $p = 3$ (right). The set $\mathcal{B}$ is defined such that all trajectories starting outside $\mathcal{B}$ are (numerically) found to remain outside, hence it provides an estimate for the basin of attraction of the periodic solution, $x_0(t)$.

The existence of the set $\mathcal{A}$, with the properties stated in Theorem 5 allows us to improve the estimate $\mathcal{D}$ of the basin of attraction found in Section 5 for the varactor equation studied in [3], that is for the system (7.1), where $f(\omega t) = \alpha + \beta \sin t$, with $\alpha > 0$ and $|\beta| < \alpha$. Define $\mathcal{D}_0 = \mathcal{D} \cup \mathcal{A}$. First note that both $\mathcal{D}$ and $\mathcal{A}$ are strictly contained in $\mathcal{D}_0$, as follows easily from the remark that $\mathcal{D}_0$ is inside the separatrix Γ_0 (cf. the end of Section 5), and at a distance $O(1/\sqrt{\gamma})$ from it. The inclusion $\mathcal{D} \subset \mathcal{D}_0$ is obvious as $\mathcal{A}$ grows linearly at γ in the vertical direction. The inclusion $\mathcal{A} \not\subseteq \mathcal{D}_0$ follows from the fact that the separatrix intersects the x-axis in $x = -c_0$ and $x = 2c_0$, with $c_0 = \sqrt{\alpha}$; hence $x_L = -\sqrt{\alpha - \beta} > -c_0$ and $x_I = \sqrt{\alpha + \beta} \leq \sqrt{2\alpha} < 2c_0$. Then the set $\mathcal{D}_1 := \mathcal{A} \setminus (\mathcal{A} \cap \mathcal{D})$ is non-empty. Furthermore a trajectory starting in $\mathcal{D}_1$ can go out of $\mathcal{D}_1$ only by entering $\mathcal{D}_0$ (by the invariance of $\mathcal{A}$). This means that all trajectories starting in $\mathcal{D}_0$ are attracted by the limit cycle described by $x_0(t)$.

7 Blow up in finite time

We prove finite-time blow up for the system (6.1) by first finding an invariant set $\mathcal{J}$ (Lemma 7) and then constructing an invariant subset, $\mathcal{S}(-X_0)$, of $\mathcal{J}$, the latter construction being carried out in Lemma 8. Within $\mathcal{S}(-X_0)$, a differential inequality must hold and we show in Theorem 6 that all solutions of this differential inequality must blow up in finite time.

Lemma 7 *Define* $h(x) = 2px^{2p-1}\left(F^{2p} - x^{2p}\right) - \gamma^2\left(F^{2p} - f^{2p}\right)$ *and the set*

$$\mathcal{D} = \{(x, y)|x \le -\xi, (F^{2p} - x^{2p})/\gamma \le y \le 0\}, \tag{7.1}$$

where $-\xi$ *is the root of* $h(x)$ *with* $-\xi < -F$. *Then* ξ *as defined exists uniquely and* $\mathcal{D}$ *is an invariant set.*

Proof. The set $\mathcal{D}$ is clearly absorbing along its horizontal, $y = 0$, $x \le -\xi$, and vertical, $x = -\xi$, $(F^{2p} - \xi^{2p})/\gamma \le y \le 0$ boundaries. We therefore only need to prove that it is also absorbing along the curved boundary, $y = (F^{2p} - x^{2p})/\gamma$ for $x \le -\xi$. The appropriate normal here is $\boldsymbol{n} = \left(2px^{2p-1}/\gamma, 1\right)$ and so we require $\boldsymbol{n} \cdot \boldsymbol{\phi}_f \ge 0$, which is equivalent to proving that $h(x) \ge 0$, for $x \le -\xi$.

To this end, first note some elementary properties of $h(x)$: (i) $h(-F) = \gamma^2\left(F^{2p} - f^{2p}\right) < 0$; (ii) $h(x)$ has exactly three stationary points for $x \in \mathbb{R}$: $h'(x) = 0$ for $x = 0$ and $x = x_\pm = \pm F[(2p-1)/(4p-1)]^{1/2p}$; (iii) $h''(x_-) > 0$; and (iv) $h(x) \to \infty$ as $x \to -\infty$. From these, it becomes clear that $-F < x_- < 0$ and $h(x_-)$ is a minimum; and, in the light of (i) and (iv), $h(x)$ has exactly one real root, $-\xi, \in (-\infty, -F)$. Hence, $h(x) \ge 0$ for $x \le -\xi$ and the invariance of set $\mathcal{D}$ is proved. ■

We now define, for $x \le -X_0$, a curve $G(x, y) : y = -b\left(-X_0 - x\right)^\rho$ with $-X_0 \le -\xi < 0$, $\rho = 3/2$, and $b > 0$ to be found. In order that $\mathcal{S}(-X_0) \subset \mathcal{J}$, $-X_0 \le -\xi$ and $b > 0$. In the proof of Theorem 6, it will be required that $\mathcal{S}(-X_0)$ has infinite area for all $p \in \mathbb{N}$, and hence $\rho < 2$ — otherwise, the curves $(F^{2p} - \xi^{2p})/\gamma$ and G could intersect at some finite $x < -\xi$ when $p = 1$. In order that solutions blow up in *finite* time, $\rho > 1$ will also be required: hence, we choose $\rho = 3/2$.

The proof of the invariance of $\mathcal{S}$ now follows.

Lemma 8 *The set*

$$\mathcal{S}(-X_0) = \{(x, y)|x \le -\xi,\ U \ge y \ge (F^{2p} - x^{2p})/\gamma\}, \tag{7.2}$$

where

$$U = \begin{cases} -b\left(-X_0 - x\right)^{3/2} & x \le -X_0 \\ 0 & \textit{otherwise} \end{cases} \tag{7.3}$$

with $-X_0 \le -\xi$ *and* b *sufficiently small, is an invariant subset of* $\mathcal{J}$, *and contains points* (x, y) *with* $y \to -\infty$.

Proof. The fact that the vector field is into all the boundaries of $\mathcal{S}$ except $y = -b\left(-X_0 - x\right)^{3/2}$ has been proved in Lemma 7; therefore, we only need to consider the boundary $y = -b\left(-X_0 - x\right)^{3/2}$. The appropriate normal is that which points into $\mathcal{S}$, this being $\boldsymbol{n} = (3b/2)\sqrt{-X_0 - x}, -1)$. The correct choice for the vector field here is $\boldsymbol{\phi}_F = (y, F^{2p} - \gamma y - x^{2p})$, which gives

$$\boldsymbol{n} \cdot \boldsymbol{\phi}_F = -b(-X_0 - x)^{3/2}\left[(3b/2)\sqrt{-X_0 - x} + \gamma\right] + x^{2p} - F^{2p} \ge 0,\ \ x \le -X_0. \tag{7.4}$$

Since $-X_0 - x \ge 0$, we can substitute $v^2 = -X_0 - x$, giving

$$L(v) = (X_0 + v^2)^{2p} - F^{2p} - bv^3(3bv/2 + \gamma) \ge 0,\ \ v \ge 0. \tag{7.5}$$

Let $l(x) = x^p$; then the mean value theorem states that, for $b > a$, $l(b) - l(a) = (b - a)pc^{p-1}$, where $c \in (a, b)$. Applying this to the first two terms in equation (7.5), we have

$$\begin{aligned} L(v) &\ge \left[(X_0 + v^2)^2 - F^2\right] pF^{2p-2} - bv^3(3bv/2 + \gamma) \\ &= v^4(pF^{2p-2} - 3b^2/2) - \gamma bv^3 + 2X_0 pF^{2p-2}v^2 + (X_0^2 - F^2)pF^{2p-2} \\ &\ge v^2\left[v^2(pF^{2p-2} - 3b^2/2) - \gamma bv + 2X_0 pF^{2p-2}\right] = v^2 M(v), \end{aligned} \tag{7.6}$$

where we have used $X_0 > F$ to obtain the last inequality. Hence, we need to show that for $b > 0$ sufficiently small, $M(v) \geq 0$ for $v \geq 0$. First, write $M(v) = a_2 v^2 - a_1 bv + a_0$, $a_1, a_0 > 0$, and let b be sufficiently small that $a_2 > 0$. Let $\tilde{M}(v) = M(v)/a_2 = v^2 - 2c_1 bv + c_0$, with $c_1, c_0 > 0$. Then $\tilde{M}(v) = (v - bc_1)^2 + c_0 - b^2 c_1^2$, and it is plain that b can be chosen to be small enough that $\tilde{M}(v)$, and so $L(v)$, are non-negative for $v \geq 0$.

We will also require $\mathcal{S}(-X_0)$ to extend to infinite negative y values; it has this property provided that the curves $y = -b\,(-X_0 - x)^{3/2}$ and $y = (F^{2p} - x^{2p})/\gamma$ nowhere intersect for $y \leq 0$. This, too, is clearly true if b is small enough. ■

It is possible that some of the conditions applied in the above proof could be relaxed, but a 'better' invariant set $\mathcal{S}$ is not required in the proof of the following theorem.

Theorem 6 *For all $\boldsymbol{x}_0 = (x_0, y_0) \in \mathcal{J}$, $\exists - X_0 \leq -\xi$ which is such that $\boldsymbol{x}_0 \in \mathcal{S}(-X_0)$. All solutions starting from such an $\boldsymbol{x}_0$ blow up in finite time.*

Proof. Let $\boldsymbol{x}_0 = (x(0), y(0)) = (x_0, y_0) \in \mathcal{J}$. Then there always exists an $-X_0 \leq -\xi$ such that $\boldsymbol{x}_0$ is in a subset $\mathcal{S}(-X_0)$ of $\mathcal{J}$: choose any $-X_0 > x_0$. Also, since $\dot{x} \leq 0$ in $\mathcal{J}$, $x \leq x_0$ and so $-X_0 - x \geq -X_0 - x_0 > 0$. Additionally, since $\boldsymbol{x}_0 \in \mathcal{S}(-X_0)$, the differential inequality $-b(-X_0 - x)^{3/2} \geq y = \dot{x} \geq (F^{2p} - x^{2p})/\gamma$ applies for all time $t \geq 0$, by the invariance of $\mathcal{S}(-X_0)$.

In fact, only the upper bound is important here, and with the substitution $u = -X_0 - x > 0$ and $u_0 = -X_0 - x_0 > 0$, this becomes $\dot{u} \geq bu^{3/2}$. Integrating gives

$$\int_{u_0}^{u} \mathrm{d}\psi\, \psi^{-3/2} \geq b \int_0^t \mathrm{d}s, \tag{7.7}$$

giving $u_0^{-1/2} - u^{-1/2} \geq bt/2$, which, after re-arranging, gives

$$u \geq \left[u_0^{-1/2} - bt/2\right]^{-2} \qquad \text{for } t \in [0, t_\infty), \tag{7.8}$$

where $bt_\infty = 2u_0^{-1/2}$. The above inequality shows that $x(t)$ tends to $-\infty$ within a finite time $t \leq t_\infty$. By the invariance of $\mathcal{S}(-X_0)$, $x(t)$ cannot tend to $-\infty$ without also $y(t) \to -\infty$, and so finite time blow up is proven. ■

Note that for $p = 1$ there is a set $\mathcal{B}$ (cf. figure 6) such that all solutions starting from $\mathcal{B}$ reach $\mathcal{J}$ in a finite time: this was proved in [3]. Therefore for $p = 1$ Theorem 6 shows that all trajectories starting from $\mathcal{B}$ blow up in finite time.

8 Conclusions, extensions and open problems

We conclude this paper with a list of open problems (some already mentioned in the previous sections).

The first one concerns possible extensions of the proof of Theorem 3 to the case of more general polynomials of the form (4.1). A natural question is under which conditions there is still a global attractor, in these cases, when the dissipation coefficient is large enough.

A characterisation of the set $\mathcal{B}$ can be given in some concrete cases, such as that of the varactor equation considered in Section 5. Its diameter is of order 1. We have seen in Section 6 that we can improve the estimate by obtaining a set whose size increases linearly in γ in the vertical direction, but in such a way that it is still expected to be strictly included inside the actual basin of attraction. It would be worthwhile to attempt constructions of sets contained inside the basins of attraction that are as large as possible.

We also leave as an open problem for the varactor equation the proof that any bounded solution is attracted by $x_0(t)$. On the basis of numerical simulations, we conjecture that this is the case.

Another interesting problem is whether one can weaken the hypotheses on the function g, both for determining the existence of a quasi-periodic solution with the same frequency vector as the forcing and, in that case, for proving its uniqueness and attractivity.

Finally, extensions to higher dimensional cases would be desirable.

A Proof of (3.10)

One has $\dot{P} = \partial_\xi P\,\dot{\xi} + \partial_t P$ and $\dot{Q} = \partial_\xi Q\,\dot{\xi}$, so that

$$\frac{\dot{P}}{P} - \frac{Q}{Q} = \left(\frac{\partial_\xi P}{P} - \frac{\partial_\xi Q}{Q}\right)\dot{\xi} + \frac{\partial_t P}{P} = \frac{\partial_\xi P - \partial_\xi Q}{P} + \frac{\partial_\xi Q}{PQ}\,(Q-P) + \frac{\partial_t P}{P}. \tag{A.1}$$

One can write

$$\begin{aligned}
\partial_t P &= \sum_{j=1}^{2p} \binom{2p+1}{j} j\xi^{2p+1-j} x_0^{j-1}(t)\,\dot{x}_0(t), &\text{(A.2)}\\
Q - P &= \sum_{j=1}^{2p} \binom{2p+1}{j} \xi^{2p+1-j}\,(\alpha - x_0(t))^j\,, \\
\partial_\xi P - \partial_\xi Q &= \sum_{j=1}^{2p} \binom{2p+1}{j} (2p+1-j)\,\xi^{2p-j}\,(x_0(t)-\alpha)^j\,,
\end{aligned}$$

where $\dot{x}_0(t) = O(1/\gamma)$ and $x_0(t) - \alpha = O(1/\gamma)$.

Finally $|x_0(t)| \le 2|\alpha|$ for all $t \in \mathbb{R}$ if γ is large enough, and both ξ^{2p+1-j}/P and ξ^{2p+1-j}/Q tend to zero as $\xi \to \infty$ for $j \ge 1$. Hence (3.10) follows, with the constants B_1 and B_2 depending on p but not on γ.

B Initial data in I and III

Take an initial datum $z = (\xi, y)$ in I. If $y = 0$ then $\dot{\xi} = 0$ and $\dot{y} = -\xi F(\xi, x_0(t)) < 0$, so that the trajectory enters II. If $y > 0$ then $\dot{\xi} > 0$ and $\dot{y} < 0$.

Moreover $\partial_\xi(\xi F(\xi, x)) = \partial_\xi (x+\xi)^{2p+1} = (2p+1)\,(x+\xi)^{2p} \ge 0$ for all $x \in \mathbb{R}$, so that, by using the fact that $\xi(t) \ge \xi(0)$ as long as $(\xi(t), y(t))$ remains in I, one has in I

$$\xi F(\xi, x_0(t)) \ge \inf_{t\in\mathbb{R}} \xi F(\xi, x_0(t)) \ge \xi(0) \inf_{t\in\mathbb{R}} F(\xi(0), x_0(t)) \ge c > 0, \tag{B.1}$$

where we used that $F(\xi, x)$ is strictly greater than a positive constant for $x \neq 0$ (see the proof of Lemma 1). Therefore we obtain

$$\dot{y} \le -\gamma y - c, \tag{B.2}$$

which implies that $y(t)$ reaches the ξ-axis in a finite time.

Analogously one discusses the case of initial data z in III.

C On the curves $\mathcal{C}_1$ and $\mathcal{C}_2$

Call $\mathcal{T}$ the subset of II outside $\mathcal{S}$.

Define $\mathcal{C}_1$ as a continuous curve in II such that in $\mathcal{T}$ it is given by the graph of the function $\xi \to -\xi^{2p+1}/4\gamma$. In (3.3) one can write $\dot{y} = \gamma(-y + g(\xi,t))$, with $g(\xi,t)$ defined in Lemma 3. By Lemma 3, in $\mathcal{T}$ one has $g(\xi,t) \le -\xi^{2p+1}/2\gamma$, so that at all points in $\mathcal{T}$ above $\mathcal{C}_1$ one has

$$-y + g(\xi,t)) = |y| + g(\xi,t) \le \frac{1}{4\gamma}\,\xi^{2p+1} - \frac{1}{2\gamma}\,\xi^{2p+1} \le -\frac{1}{4\gamma}\,\xi^{2p+1},$$

hence $\dot{y} < 0$.

Define $\mathcal{C}_2$ as a continuous curve in II such that in $\mathcal{T}$ it is given by the graph of the function $\xi \to -4\xi^{2p+1}/\gamma$. By Lemma 3, one has $g(\xi,t) \ge -2\xi^{2p+1}/\gamma$, so that in all points of $\mathcal{T}$ below $\mathcal{C}_2$ one has $y \le -4\xi^{2p+1}/\gamma \le 2g(\xi,t)$, hence

$$-\gamma y \ge \gamma(-y + g(\xi,t)) \ge -\gamma y/2,$$

so that $-\gamma y \ge \dot{y} \ge -\gamma y/2$. In terms of the rescaled variables (X,Y) this yields $Y' \equiv \Psi(X,Y)$, with

$$-\gamma\varepsilon^p Y \ge \Psi(X,Y) \ge -\gamma\varepsilon^p Y/2, \tag{C.1}$$

as asserted after (3.21).

The point P is given by the intersection of the curve $Y_1(X) = -4X^{2p+1}/\gamma\varepsilon^p$ withe the level curve Γ. Hence

$$\frac{1}{2} = \frac{1}{2}\left(\frac{4X_P^{2p+1}}{\gamma\varepsilon^p}\right)^2 + X\,F(X,\varepsilon x_0(0)) = \frac{2^{4p+1}}{\gamma^2\varepsilon^{2p}}X_P^{4p+2} + X_P^{2p+1} + O(\varepsilon\,X^{2p}),$$

hence $X_P = O((\gamma\varepsilon^p)^{1/(2p+1)})$.

Now consider the solution of (3.19) with initial datum $Z(T_2) = (X(T_2), Y(T_2))$. We want to check that the solution remains below $\mathcal{C}_2$ until it crosses the Y-axis. The solution of

$$\begin{cases} X' = Y, \\ Y' = \Psi(X,Y), \end{cases} \tag{C.2}$$

with $\Psi(X,Y)$ satisfying the bounds (C.1), moves below the line with slope $-\gamma\varepsilon^p$ passing through $Z(T_2)$, that is below the line of equation

$$Y_1(X) = Y_0 - \gamma\varepsilon^p X \tag{C.3}$$

with Y_0 determined by the request that for $X = \overline{X} \equiv X(T_2)$ one has

$$Y_0 - \gamma\varepsilon^p\overline{X} = -\frac{4}{\gamma\varepsilon^p}\overline{X}^{2p+1}, \tag{C.4}$$

where the graph of $-4X^{2p+1}/\gamma\varepsilon^p$ describes the curve $\mathcal{C}_2$ in the coordinates (X,Y). By using that $\overline{X}$ is close to X_P one realises that Y_0 in (C.4) has to be negative. In turn this implies that the line of equation (C.3) is below the curve $\mathcal{C}_2$, so that also the assertion after (3.23) is proved.

D Variations in finite times for quasi-integrable systems

The system obtained from (3.19) by replacing $\Psi(X,Y)$ with $\Psi_1(X,Y)$ is an integrable Hamiltonian system, with Hamiltonian (3.21). For $\varepsilon = 0$ the Hamiltonian reduces to

$$H_0(X,Y) = \frac{1}{2}Y^2 + \frac{1}{2p+2}X^{2p+2}, \tag{D.1}$$

which can be written in terms of the action-angle variables (I,φ) as

$$H_0(X,Y) = \mathcal{H}_0(I) = c_p I^{(2n+2)/(n+2)}, \tag{D.2}$$

where c_p is a suitable p-dependent positive constant. By taking into account the other terms of the vector field, we obtain

$$H(X,Y) = \mathcal{H}(I) = c_p I^{(2n+2)/(n+2)} + O(I^{(2n+1)/(n+2)}). \tag{D.3}$$

The equations obtained by adding to $\Psi_1(X,Y)$ the vector field $\Psi_2(X,Y)$ are still Hamiltonian, and are described by the non-autonomous Hamiltonian

$$\mathcal{H}(I) + \mathcal{H}_1(I,\varphi,t), \tag{D.4}$$

with $\mathcal{H}_0$ given as in (D.3) and $\mathcal{H}_1$ of order ε^{p+1} as long as the action variables remain of order 1.

The corresponding equations of motion are

$$\begin{cases} \dot{I} = -\partial_\varphi \mathcal{H}_1(I,\varphi), \\ \dot{\varphi} = \omega_0(I) + \partial_I \mathcal{H}_1(I,\varphi), \end{cases}$$

with $\omega_0(I) = \partial_I \mathcal{H}_0(I)$. Then one immediately realises that in a time of order 1 the action variables remain close to their initial values. In turn this implies that also the angle variables are changed by order ε^{p+1} with respect their unperturbed values. In terms of the original coordinates (X,Y) this means that the solution remains within a distance $O(\varepsilon^{p+1})$ with respect the unperturbed value.

References

[1] E.A. Barbašin, N.N. Krasovskiĭ, *On stability of motion in the large* (Russian), Doklady Akad. Nauk SSSR (N.S.) **86** (1952), 453-456.

[2] M.V. Bartuccelli, J.H.B. Deane, G. Gentile, S. Gourley, *Global attraction to the origin in a parametrically driven nonlinear oscillator*, Appl. Math. Comput. **153** (2004), no. 1, 1–11.

[3] M.V. Bartuccelli, J.H.B. Deane, G. Gentile, L. Marsh, *Invariant sets for the varactor equation*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., to appear.

[4] Sh.-N. Chow, M. van Noort, Y. Yi, *Quasiperiodic dynamics in Hamiltonian $1\frac{1}{2}$ degree of freedom systems far from integrability*, J. Differential Equations **212** (2005), no. 2, 366–393.

[5] R. Dieckerhoff, E. Zehnder, *Boundedness of solutions via the twist-theorem*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), no. 1, 79–95.

[6] J.-P. Eckmann, D. Ruelle, *Ergodic theory of chaos and strange attractors*, Rev. Modern Phys. **57** (1985), no. 3, part 1, 617–656.

[7] G. Gentile, M.V. Bartuccelli, J.H.B. Deane, *Summation of divergent series and Borel summability for strongly dissipative equations with periodic or quasi-periodic forcing terms*, J. Math. Phys. **46** (2005), no. 6, 062704, 21 pp.

[8] S. Laederich, M. Levi, *Invariant curves and time-dependent potentials*, Ergodic Theory Dynam. Systems **11** (1991), no. 2, 365–378.

[9] M. Levi, *Nonchaotic behaviour in the Josephson junction*, Phys. Rev. A (3) **37** (1988), no. 3, 927–931.

[10] M. Levi, E. Zehnder, *Boundedness of solutions for quasiperiodic potentials*, SIAM J. Math. Anal. **26** (1995), no. 5, 1233–1256.

[11] N.N. Krasovskiĭ, *Stability of motion. Applications of Lyapunov's second method to differential systems and equations with delay*, Stanford University Press, Stanford, Calif., 1963.

[12] R. Martins, *The effect of inversely unstable solutions on the attractor of the forced pendulum equation with friction*, J. Differential Equations **212** (2005), no. 2, 351–365.