On the Truncation of Systems with Non-Summable Interactions *

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Abstract

In this note we consider long range q-states Potts models on $\mathbf{Z}^d$, $d \geq 2$. For various families of non-summable ferromagnetic pair potentials $\phi(x) \geq 0$, we show that there exists, for all inverse temperature $\beta > 0$, an integer N such that the truncated model, in which all interactions between spins at distance larger than N are suppressed, has at least q distinct infinite-volume Gibbs states. This holds, in particular, for all potentials whose asymptotic behaviour is of the type $\phi(x) \sim \|x\|^{-\alpha}$, $0 \leq \alpha \leq d$. These results are obtained using simple percolation arguments.

Keywords: long range Potts model, percolation, phase transition, truncation, non-summable interaction.

1 Introduction

The existence of phase transitions, in lattice systems of equilibrium statistical mechanics, is a mathematically well posed problem in the canonical framework of infinite-volume Gibbs states with *summable* interactions [14]. For such interactions, the Dobrushin Uniqueness Theorem guarantees uniqueness of the Gibbs state at high temperature, and non-uniqueness at low temperature is usually obtained as a consequence of high sensitivity to boundary conditions. The two-dimensional Ising model, for example, can be prepared in distinct thermodynamic states by taking the thermodynamic limit along sequences of boxes with different boundary conditions. The combination of these high and low temperature behaviours leads to a well defined finite critical temperature $T_c > 0$. In general, the low temperature phases, $0 < T < T_c$, are described by local deviations from the ground state configurations of the hamiltonian, at which the measure concentrates when $T \to 0$.

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For *non-summable* interactions, infinite Gibbs states and the thermodynamic limit are not defined. Physical quantities like the free energy or pressure density don't exist, due to the fact that in the limit of large volumes, the energy of the system grows faster than its size. Even the use of boundary conditions, in finite volumes, poses problems. The pathological behaviour of non-summable systems was well described by Dyson [10]: "*When [the potential is non-summable] there is an infinite energy-gap between the ground states and all other states, so that the system is completely ordered at all finite temperatures, and there can be no question of a phase transition*". In other words, since the cost for flipping any given spin is infinite, the temperature, even very high, can never obtain to create local deviations from the ground states, as the ones described above in the case of summable potentials: at any temperature the system is "frozen" in one of its ground states, and no critical temperature can be defined. Therefore one can infer that infinitely large lattice systems with non-summable potentials have, independently of the method used to describe them, trivial thermodynamic behaviour. Nevertheless, various ways of rescaling the thermodynamic potentials have been considered in the litterature, in view of obtaining non-trivial pseudo-densities in the thermodynamic limit. See for example [17] for a different scaling of the free energy of gravitational and electrostatic particle systems, or [8], [25], [7], where mean field versions of ferromagnetic spin models with non-summable interactions have been studied numerically.

In the present note, we study non-summable systems by using a simpler approach which is the following. Let $\mu_{\phi,\Lambda}$ denote the Gibbs distribution of the system with ferromagnetic pair interaction ϕ in finite volume Λ. Since nothing can be said, in the sense of weak convergence, about the existence of the thermodynamic limit

$$\lim_{\Lambda} \mu_{\phi,\Lambda}\,,$$

we first *truncate* the potential ϕ by suppressing interactions between points at distance larger than N:

$$\phi_N(x) := \begin{cases} \phi(x) & \text{if } \|x\| \leq N\,, \\ 0 & \text{if } \|x\| > N\,, \end{cases} \tag{1}$$

and then study the double limiting procedure

$$\lim_{N\to\infty} \lim_{\Lambda} \mu_{\phi_N,\Lambda}\,.$$

Since ϕ_N has finite range, the infinite-volume truncated model with measure $\mu_{\phi_N} = \lim_\Lambda \mu_{\phi_N,\Lambda}$ is well defined. When ϕ is non-summable, the phenomenon which we observe is the following: for *any temperature, the measure μ_{ϕ_N} becomes sensitive to boundary conditions once N is large enough (but finite)*. That is, if ϕ has a finite set of ground state configurations indexed by s, then the measures $\mu^s_{\phi_N} = \lim_\Lambda \mu^s_{\phi_N,\Lambda}$ obtained by taking the thermodynamic limit along a sequence of boxes with different boundary condition s, differ once N is sufficiently large: $\mu^s_{\phi_N} \neq \mu^{s'}_{\phi_N}$ when $s \neq s'$. Moreover, in the limit $N \to \infty$, each $\mu^s_{\phi_N}$ converges weakly to δ_s, the Dirac measure concentrated on the ground state configuration s. This concentration phenomenon in the limit $N \to \infty$ *at any fixed temperature* $T > 0$ is thus similar to the one discussed above for summable interactions in the limit $T \to 0$, and is in agreement with Dyson's heuristic description of non-summable interactions. Notice that in our approach, the interaction between any pair of spins is restored in the limit $N \to \infty$, and no mean field rescaling is ever used.

In Section 2 we show that the scenario presented above indeed occurs for the ferromagnetic Potts model ($d \geq 2$), for various families of non-summable potentials. Our results include, for instance, all potentials with slow algebraic decay (see Theorem 1):

$$\phi(x) \sim \frac{1}{\|x\|^\alpha} \quad \text{for some} \quad 0 \leq \alpha \leq d\,, \tag{2}$$

which are the usual non-summable potentials considered in physics. Nevertheless, our aim is to treat general interactions, which need not be asymptotically regular as in (2), but can have an irregular structure. We also give two results for *sparse* interactions, which are of independent interest. In Section 3 we give the inequality which allows to study this problem via independent long range percolation and then reformulate and prove all our results in this setting. As will be seen, the proofs are simple geometric arguments. In Section 4 we conclude with some general remarks.

This work originated with the study of the problem of truncation in independent long range percolation [11] (see [22], [24], [3] for other cases treated in the litterature). Therefore, the results presented in Section 3 give more particular cases where this problem can be solved.

2 Long Range Potts Ferromagnet

We consider the lattice $\mathbf{Z}^d$, $d \geq 2$, with the norm $\|x\| = \max_{k=1,\dots,d} |x_k|$. Interactions are defined via a ferromagnetic potential, which is any function $\phi : \mathbf{Z}^d \backslash \{0\} \to [0, +\infty)$ such that $\sup_{x \neq 0} \phi(x) < +\infty$, with the symmetry

$$\phi(x) = \phi(y) \quad \text{when} \quad \|x\| = \|y\|\,. \tag{3}$$

Let $N \in \mathbf{N}$. To each potential ϕ can be associated a truncated potential ϕ_N, defined as in (1). In the q-state Potts model, $q \geq 2$ is any fixed integer, and at each site $x \in \mathbf{Z}^d$ lives a spin $\sigma_x \in \{1, 2, \dots, q\}$. When $q = 2$ it thus reduces to the Ising model. Spin configurations are elements of $\Omega = \{1, 2, \dots, q\}^{\mathbf{Z}^d}$. Consider a finite box $\Lambda_L = [-L, +L]^d \cap \mathbf{Z}^d$, $L \geq 1$. For $\sigma \in \Omega_{\Lambda_L} = \{1, 2, \dots, q\}^{\Lambda_L}$, the truncated Potts Hamiltonian with boundary condition $\eta \in \Omega$ is given by

$$H^\eta_{N,\Lambda_L}(\sigma) = -\sum_{\substack{\{x,y\} \subset \Lambda_L \\ x \neq y}} \phi_N(x-y)\delta(\sigma_x, \sigma_y) - \sum_{x \in \Lambda_L, y \in \Lambda_L^c} \phi_N(x-y)\delta(\sigma_x, \eta_y)\,,$$

where $\delta(a, b) = 1$ if $a = b$, 0 otherwise. We will mainly be interested in considering the pure s boundary condition, in which $\eta_j = s$ for all $j \in \mathbf{Z}^d$. We have, with some abuse of notation,

$$H^s_{N,\Lambda_L}(s) = \min_{\sigma \in \Omega_{\Lambda_L}} H^s_{N,\Lambda_L}(\sigma)\,.$$

Therefore, we also call the pure configurations s ground state configurations. On Ω_{Λ_L}, the truncated Gibbs measure at inverse temperature $\beta > 0$ with pure s boundary condition is defined by:

$$\mu^{\beta,s}_{\phi_N,\Lambda_L}(\sigma) := \frac{1}{Z^{\beta,s}_{\phi_N,\Lambda_L}} \exp\big(- \beta H^s_{N,\Lambda_L}(\sigma)\big)\,,$$

where $Z^{\beta,s}_{\phi_N,\Lambda_L}$ is a normalizing factor. Let $\mathcal{F}$ be the σ-algebra on Ω generated by cylinder events. We consider the infinite-volume Gibbs measures $\mu^{\beta,s}_{\phi_N}$ on $(\Omega, \mathcal{F})$, obtained by taking limits along an increasing sequence of boxes [1] (this limit is to be understood in the sense of subsequences):

$$\mu^{\beta,s}_{\phi_N}(A) := \lim_{L\to\infty} \mu^{\beta,s}_{\phi_N,\Lambda_L}(A) \quad \forall A \in \mathcal{F}\,.$$

A phase transition occurs in the truncated model if $\mu^{\beta,s}_{\phi_N} \neq \mu^{\beta,s'}_{\phi_N}$ for $s' \neq s$.

Let $B_N := \Lambda_N \setminus \{0\}$. When ϕ is summable, i.e. when

$$\sum_{x\neq 0} \phi(x) := \lim_{N\to\infty} \sum_{x\in B_N} \phi(x) \tag{4}$$

exists, the untruncated Gibbs measures (with $N = \infty$) $\mu^{\beta,s}_{\phi}$ are well defined, and the problem of knowing if $\mu^{\beta,s}_{\phi} \neq \mu^{\beta,s'}_{\phi}$ for some $s' \neq s$ depends strongly on the temperature. When ϕ is not summable these measures are not defined, and we study $\mu^{\beta,s}_{\phi_N}$ at large N. We remind that for fixed N, in the limit of very low temperature, $\beta \to \infty$, the typical configurations of $\mu^{\beta,s}_{\phi_N}$ concentrate on the ground state configuration s ([23]). When the temperature is fixed and N becomes large, we observe essentially the same phenomenon. In view of the argument of Dyson cited in the Introduction, it is reasonable to believe that at any fixed $\beta > 0$, each of the measures $\mu^{\beta,s}_{\phi_N}$ concentrates, when $N \to \infty$, on a single configuration, which is the ground state s. This is the statement of the following conjecture. Let δ_s denote the Dirac mass on $(\Omega, \mathcal{F})$ concentrated on the ground state configuration s, and write $\mu^{\beta,s}_{\phi_N} \Rightarrow \delta_s$ when $\mu^{\beta,s}_{\phi_N}$ converges weakly to δ_s in the limit $N \to \infty$.

Conjecture 1. ($d \geq 2$) *If $\phi \geq 0$ satisfies* (3) *and is non-summable, i.e.*

$$\sum_{x\neq 0} \phi(x) = +\infty\,, \tag{5}$$

then $\mu^{\beta,s}_{\phi_N} \Rightarrow \delta_s$ for all $\beta > 0$ and for all $s \in \{1, 2, \ldots, q\}$.

[1] Here we extend $\mu^{\beta,s}_{\phi_N,\Lambda_L}$ to a measure on $(\Omega, \mathcal{F})$ in the standard way:

$$\mu^{\beta,s}_{\phi_N,\Lambda_L}(A) := \sum_{\sigma\in\Omega_{\Lambda_L}} \mu^{\beta,s}_{\phi_N,\Lambda_L}(\sigma) 1_A(\sigma \cdot s) \quad \forall A \in \mathcal{F}\,,$$

where the configuration $\sigma \cdot s \in \Omega$ coincides with σ on Λ_L and with s on Λ_L^c.

Observe that $\mu^{\beta,s}_{\phi_N} \Rightarrow \delta_s$ implies $\mu^{\beta,s}_{\phi_N}(\sigma_0 = s) \to 1$ in the limit $N \to \infty$, i.e. a phase transition occurs in the truncated model for large enough N. Since the system is ferromagnetic, the sequence $\left(\mu^{\beta,s}_{\phi_N}(\sigma_0 = s)\right)_{N\geq 1}$ is non-decreasing, but the fact that it converges to 1 is not trivial. Another way to formulate the conjecture is: $\beta_c(\phi_N) \to 0$ when $N \to \infty$, where $\beta_c(\phi_N)$ is the critical inverse temperature of the model with potential ϕ_N, i.e.

$$\beta_c(\phi_N) := \inf\left\{\beta > 0 : \mu^{\beta,s}_{\phi_N} \neq \mu^{\beta,s'}_{\phi_N} \text{ for } s \neq s'\right\}.$$

The conjecture is difficult to prove in such generality, since we don't assume any kind of regularity on ϕ. For example, the potential

$$\phi(x) = \begin{cases} \epsilon > 0 & \text{if } \|x\| = k! \text{ for some } k \in \mathbf{N}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

which will enter in the family of interactions considered in Theorem 2, satisfies the hypothesis of the conjecture. Usual perturbation techniques, such as Pirogov-Sinai Theory [23], are of no use for studying the truncated version of this kind of potential, since the domain of validity for the temperature shrinks to zero when the range of interaction, here N, grows.

Remark 1. Observe that since the Gibbs state of any one-dimensional model with finite range interactions is always unique (see for example Theorem (8.39) in [14]), Conjecture 1 is *false* in dimension 1. This shows that a symmetry of the kind (3) is necessary.

For $d \geq 2$, our first result shows that the conjecture is valid under some assumption on the speed of divergence of the series (5).

Theorem 1. ($d \geq 2$) *If $\phi \geq 0$ satisfies* (3) *and* (5) *diverges faster than logarithmically, i.e.*

$$\limsup_{N\to\infty} \frac{1}{\log|B_N|} \sum_{x\in B_N} \phi(x) = +\infty\,, \tag{6}$$

then $\mu^{\beta,s}_{\phi_N} \Rightarrow \delta_s$ for all $\beta > 0$ and for all $s \in \{1, 2, \ldots, q\}$.

Remark 2. As can be seen easily, condition (6) is satisfied by all potentials which have slow algebraic decay, as in (2):

$$\liminf_{\|x\|\to\infty} \|x\|^\alpha \phi(x) > 0 \qquad \text{for some } 0 \leq \alpha < d\,.$$

We shall see later, in Remark 3, that for such potentials the range $0 \leq \alpha < d$ can be extended to $0 \leq \alpha \leq d$, using the multiscale analysis of [3].

Sparse Interactions. We also give two results for potentials which don't have the symmetry (3). Namely, we consider interactions only along directions parallel to the coordinate axis e_i, $i = 1, \ldots, d$, where $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, ..., $e_d = (0, 0, \ldots, 1)$. That is, we are given a sequence $(\phi_n)_{n \geq 1}$, $\phi_n \geq 0$, and [2]

$$\phi(x) = \begin{cases} \phi_{\|x\|} & \text{if } x \text{ is parallel to some } \ e_i \,, i = 1, 2, \ldots, d \,, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

In $d = 2$, for example, there are only vertical and horizontal couplings. For potentials of the form (7), assumption (5) of Conjecture 1 becomes:

$$\sum_{n \geq 1} \phi_n = +\infty \,. \tag{8}$$

The first result is for sequences $(\phi_n)_{n \geq 1}$ which don't converge to zero:

Theorem 2. *($d \geq 2$). If*

$$\limsup_{n \to \infty} \phi_n > 0 \,, \tag{9}$$

then $\mu^{\beta,s}_{\phi_N} \Rightarrow \delta_s$ for all $\beta > 0$ and for all $s \in \{1, 2, \ldots, q\}$.

Notice that (9) implies (8), but with no information on the speed of divergence. Our second result is where we prove our conjecture under the general condition (8), with no assumption on the speed of divergence, but only in dimensions three or more:

Theorem 3. *($d \geq 3$). If*

$$\sum_{n \geq 1} \phi_n = +\infty \,, \tag{10}$$

then $\mu^{\beta,s}_{\phi_N} \Rightarrow \delta_s$ for all $\beta > 0$ and for all $s \in \{1, 2, \ldots, q\}$.

3 Independent Long Range Percolation

To show that the truncated q-states Potts model exhibits a phase transition, it is sufficient to show that

$$\mu^{\beta,s}_{\phi_N}(\sigma_0 = s) > \frac{1}{q} \,. \tag{11}$$

Our purpose is to obtain the following stronger limiting behaviour:

$$\lim_{N \to \infty} \mu^{\beta,s}_{\phi_N}(\sigma_0 = s) = 1 \,, \tag{12}$$

which is equivalent to $\mu^{\beta,s}_{\phi_N} \Rightarrow \delta_s$, as can be verified easily. To obtain (12) we shall reformulate our problem in the framework of long range independent percolation.

[2]The reader should pay attention to the following: we use n to index elements of the sequence $(\phi_n)_{n \geq 1}$, whereas N is used as the parameter of truncation for ϕ_N.

Consider the graph $(\mathbf{Z}^d, \mathcal{E}^d)$, $d \geq 1$, where $\mathcal{E}^d$ is the set of all unoriented edges $e = \{x, y\} \subset \mathbf{Z}^d \times \mathbf{Z}^d$, $x \neq y$. Edge configurations are elements $\omega \in \{0,1\}^{\mathcal{E}^d}$. For a given function $p : \mathbf{Z}^d \backslash \{0\} \to [0,1]$ with

$$p(x) = p(y) \quad \text{when} \quad \|x\| = \|y\| \,, \tag{13}$$

called edge probability, we consider the long range percolation process in which each edge $e = \{x, y\}$ is open ($\omega(e) = 1$) with probability $p(x-y)$, and closed ($\omega(e) = 0$) with probability $1 - p(x-y)$, independently of other edges. This process is described by the product measure on the σ-field on $\{0,1\}^{\mathcal{E}^d}$ generated by cylinders, given by

$$P = \prod_{e \in \mathcal{E}^d} \mu_e \,, \tag{14}$$

where $\mu_e(\omega(e) = 1) = p(x-y)$ is a Bernoulli measure on $\{0,1\}$, independent of the state of other edges. Observe that P is well-defined even when p is non-summable. Define the truncated edge probability

$$p_N(x) := \begin{cases} p(x) & \text{if} \quad \|x\| \leq N \,, \\ 0 & \text{if} \quad \|x\| > N \,, \end{cases}$$

and denote by P_N the truncated product measure defined as in (14) with p_N instead of p. We shall be interested in the percolation probability $P_N(0 \leftrightarrow \infty)$, which is the probability of the event in which there exists, in the truncated model, a path of open edges connecting the origin to infinity. When $P_N(0 \leftrightarrow \infty) > 0$, we say the truncated system percolates.

As can be seen in the following proposition, the percolation probability is a relevant quantity for showing all our results for the dependent Potts model, once the edge probability is well chosen in function of the potential ϕ.

Proposition 1. *Define $p(x)$ by*

$$p(x) := \frac{1 - e^{-2\beta\phi(x)}}{1 + (q-1)e^{-2\beta\phi(x)}} \,. \tag{15}$$

Then the magnetisation of the truncated long range q-states Potts model with potential $\phi_N(x)$ at temperature β and the probability of percolation of the origin in the independent truncated long range percolation process with probabilities $p_N(x)$ are related by the following inequality:

$$\mu^{\beta,s}_{\phi_N}(\sigma_0 = s) \geq \frac{1}{q} + \frac{q-1}{q} P_N(0 \leftrightarrow \infty) \,. \tag{16}$$

Using (11), (16) shows that percolation in the truncated independent model implies phase transition in the truncated Potts model. This holds once the potential ϕ and the probability p are related by (15), which is well suited for our purposes since (15) implies that ϕ and p are bound to have the same asymptotic behaviour: as can be seen, there exist two positive functions $C_\pm = C_\pm(\beta, q)$ such that

$$C_- \phi(x) \leq p(x) \leq C_+ \phi(x) \quad \forall x \neq 0 \,.$$

In particular, non-summability of ϕ implies non-summability of p. Although it does not appear exactly in this form in the litterature, (16) is standard, and can be obtained via the random cluster representation of the measure $\mu^{\beta,s}_{\phi_N}$, and its domination properties with respect to Bernoulli product measures. We refer to [1] or [15] for details.

By Proposition 1, long range independent percolation can be used to show all the results stated previously for the Potts model. We therefore reformulate and prove the equivalent of the results of Section 2 in the context of independent percolation, in which our conjecture is that when (13) holds and

$$\sum_{x\neq 0} p(x) = +\infty\,, \tag{17}$$

then $\lim_{N\to\infty} P_N(0 \leftrightarrow \infty) = 1$. A simple use of the Borel-Cantelli Lemma shows that (17) implies $P(0 \leftrightarrow \infty) = 1$. Therefore the conjecture is that for non-summable edge probabilities, $\lim_{N\to\infty} P_N(0 \leftrightarrow \infty) = P(0 \leftrightarrow \infty)$. Our first result for percolation is

Theorem 4. $(d \geq 2)$ *There exists* $c = c(d) > 0$ *such that if* (13) *holds and if*

$$\limsup_{N\to\infty} \frac{1}{\log|B_N|} \sum_{x\in B_N} p(x) \geq c\,, \tag{18}$$

then $\lim_N P_N(0 \leftrightarrow \infty) = 1$.

Remark 3. As can be seen easily, (18) is satisfied when

$$\lambda := \liminf_{\|x\|\to\infty} \|x\|^\alpha p(x) > 0\,, \qquad \text{for some} \qquad 0 \leq \alpha < d\,. \tag{19}$$

In fact Theorem 4 can be obtained, under condition (19), using the multiscale analysis of Berger [3], for all $0 \leq \alpha \leq d$. Our Theorem allows p to have gaps, but does not cover the case $\alpha = d$, unless λ is assumed sufficiently large.

Proof of Theorem 4: The proof is a simple (almost trivial) blocking argument. Fix N large. For each $s \in S := \{(s_1, \dots, s_d) : s_i = \pm 1\}$, define the quadrant $Q^N_s(0) := \{y \in \mathbf{Z}^d : 0 < s_i y_i \leq N,\ i = 1, 2, \dots, d\}$. For any $x \in \mathbf{Z}^d$, let $Q^N_s(x) := x + Q^N_s(0)$. A site x is good if there exists, for all $s \in S$, a site $y \in Q^N_s(x)$ such that the edge $\{x, y\}$ is open (see Figure 1). We have

$$\begin{aligned}
P_N(x \text{ is good}) &= \prod_{s\in S}\Big[1 - \prod_{y\in Q^N_s(x)} (1 - p(x-y))\Big] \\
&= \Big[1 - \prod_{y\in Q^N_s(0)} (1 - p(y))\Big]^{|S|} \quad \forall s \in S \\
&\geq \Big[1 - \exp\Big(-\sum_{y\in Q^N_s(0)} p(y)\Big)\Big]^{|S|} \quad \forall s \in S \\
&\geq \Big[1 - \exp\Big(-c_1 \sum_{y\in B_N} p(y)\Big)\Big]^{|S|}\,,
\end{aligned} \tag{20}$$

where we used the inequality $\log(1-t) \leq -t$ valid for all $t < 1$, and $c_1 > 0$ is a constant that depends only on the dimension. Next, consider a partition of $\mathbf{Z}^d$ into disjoint blocks of linear size $3N$, obtained by translates of the block $C^N(0) := [0, 3N)^d \cap \mathbf{Z}^d$. That is, each block of this partition is of the form $C^N(z) = 3zN + C^N(0)$, for some renormalized vertex $z \in \mathbf{Z}^d$. We say a block $C^N(z)$ is good if each $x \in C^N(z)$ is good. Now for two points $x \neq x'$, the events $\{x \text{ is good}\}$ and

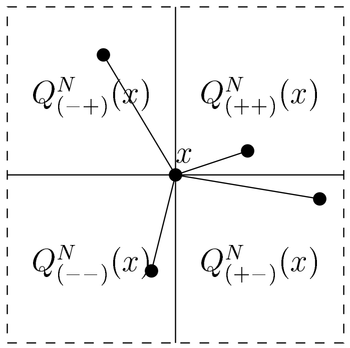

Figure 1: Illustration of a good site x in the two-dimensional case.

$\{x' \text{ is good}\}$ are not necessarily independent, but since they are increasing, FKG inequality gives

$$P_N(C^N(z) \text{ is good}) \geq \prod_{x \in C^N(z)} P_N(x \text{ is good}).$$

For small enough $\epsilon > 0$ we have $1 - \epsilon \geq e^{-2\epsilon}$. Using $|C^N(z)| \leq c_2 |B_N|$ and (20) we thus get, when N is large enough,

$$\begin{aligned} P_N(C^N(z) \text{ is good}) &\geq \exp\Big[-2c_2|S||B_N| \exp\Big(-c_1 \sum_{y \in B_N} p(y)\Big)\Big] \\ &= \exp\Big[-2c_2|S| \exp\Big(\log|B_N| - c_1 \sum_{y \in B_N} p(y)\Big)\Big]. \end{aligned}$$

If (18) holds with a well chosen constant $c > 0$ we get

$$\limsup_{N \to \infty} P_N(C^N(z) \text{ is good}) = 1,$$

i.e. there exists a sequence $\delta_k \searrow 0$ and a diverging sequence $N_1, N_2, \ldots$ such that

$$P_{N_k}(C^{N_k}(z) \text{ is good}) \geq 1 - \delta_k \ \forall k. \tag{21}$$

Notice that for each k the process $(X^k_z)_{z \in \mathbf{Z}^d}$ defined by

$$X^k_z := \begin{cases} 1 & \text{if } C^{N_k}(z) \text{ is good}, \\ 0 & \text{otherwise}, \end{cases} \tag{22}$$

is 1-dependent, i.e. X^k_z and $X^k_{z'}$ are independent when $\|z - z'\| > 1$. By a Theorem of Liggett, Stacey and Schonmann [21], $(X^k_z)_{z \in \mathbf{Z}^d}$ stochastically dominates an independent Bernoulli process $(Z_z^k)_{z\in\mathbf{Z}^d}$ of parameter $\rho_k > 0$, and (21) implies $\lim_{k\to\infty}\rho_k = 1$. Take k large enough such that $\rho_k > p_c(\mathbf{Z}^d, \text{site})$, where $p_c(\mathbf{Z}^d, \text{site})$ is the critical threshold of Bernoulli nearest-neighbour site percolation. For such k, the process $(Z_z^k)_{z\in\mathbf{Z}^d}$ is supercritical and by domination there exists an infinite cluster of good boxes. It is easy to see that any infinite connected component of good boxes yields an infinite connected component of sites of the original lattice. By taking N large one can thus make $P_N(0 \leftrightarrow \infty)$ arbitrarily close to 1. $\square$

Remark 4. Criterium (18) concerns the behaviour of the sum $\sum_{x\in B_N} p(x)$ for large N, *and not the details of the function* $p(\cdot)$. This has an interesting consequence, as the following discussion shows. Assume p is such that (18) holds. Then Theorem 4 guarantees the existence of some N such that the system whose edges $e = \{x, y\}$ are all of size at most $\|x-y\| \le N$, with edge probabilities $p(\cdot)$, satisfies $P_N(0 \leftrightarrow \infty) > 0$. As seen in the proof, the integer N is fixed once the sum of the edge probabilities passes a given value K_N, i.e.

$$\sum_{x\in B_N} p(x) \ge K_N\,.$$

Now, observe that the function p can be modified inside B_N, but as long as the *sum* is preserved, the percolation probability remains positive. For example, if $\pi : B_N \to B_N$ is any permutation preserving the symmetry $\|\pi(x)\| = \|\pi(y)\|$ for $\|x\| = \|y\|$, then

$$\sum_{x\in B_N} p(\pi(x)) = \sum_{x\in B_N} p(x) \ge K_N\,,$$

and so the truncated system with edge probability $p(\pi(\cdot))$ also percolates. This comment suggests that the sum $\sum_{x\in B_N} p(x)$, rather than the individual edge probabilities, is the relevant parameter in the study of percolation in the truncated model. This property can also be easily verified for long range percolation on trees.

Sparse Connections. We now give two results concerning systems where connections are not isotropic, and we will consider the case where a connection can be opened between two sites x, y only if these lie on a same coordinate axis. That is, $p(x) \ne 0$ only if x is parallel to one of the coordonate axis e_i, $i = 1, \dots, d$. In this case the probabilities $p(x)$ are determined by a sequence $(p_n)_{n\ge1}$, $p_n \in [0, 1]$, and

$$p(x) = \begin{cases} p_{\|x\|} & \text{if } x \text{ is parallel to some } e_i\,, i = 1, 2, \dots, d\,, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

We expect that

$$\sum_{n\ge1} p_n = +\infty \tag{24}$$

implies $\lim_N P_N(0 \leftrightarrow \infty) = 1$. A particular case of (24) in which nothing is assumed about the speed of divergence of the series is the following.

Theorem 5. $(d \ge 2)$ *If*

$$\limsup_{n\to\infty} p_n > 0 \tag{25}$$

then $\lim_N P_N(0 \leftrightarrow \infty) = 1$.

Since this result has already appeared in [11] we shall only remind the strategy of the proof, which is very different, in spirit, from that of Theorem 4. We consider the two dimensional case $d = 2$. Nevertheless, the core of the proof is to use properties of nearest-neighbour percolation in high dimensions d_*. Denote by $p_c(\mathbf{Z}^{d_*})$ the percolation threshold of nearest-neighbour Bernoulli edge percolation on $\mathbf{Z}^{d_*}$. It was shown by Kesten [19] that

$$p_c(\mathbf{Z}^{d_*}) \to 0 \text{ when } d_* \to \infty . \tag{26}$$

Then, let $\{1, 2, \ldots, L\}^{d_*-2} \times \mathbf{Z}^2$ denote the slab of thickness L in $\mathbf{Z}^{d_*}$. It was shown by Grimmett and Marstrand [16] that the slab percolation threshold satisfies

$$p_c(\{1, 2, \ldots, L\}^{d_*-2} \times \mathbf{Z}^2) \to p_c(\mathbf{Z}^{d_*}) \text{ when } L \to \infty . \tag{27}$$

Now call 2ϵ the $\limsup$ in (25), and consider some diverging sequence $n_1, n_2, \ldots$ for which $p_{n_k} \geq \epsilon$. By (26) and (27) there exists a dimension d_* and an integer L (both depending on ϵ) such that for all k,

$$p_{n_k} \geq \epsilon > p_c(\{1, 2, \ldots, L\}^{d_*-2} \times \mathbf{Z}^2) . \tag{28}$$

It is then clear how to pursue: we *embed* the slab $\{1, 2, \ldots, L\}^{d_*-2} \times \mathbf{Z}^2$ in $(\mathbf{Z}^2, \mathcal{E}^2)$, using edges of sizes taken in the set $\{n_1, n_2, \ldots\}$. Here, N must be taken large enough. The point is that we only need a *finite* number of sizes, and that by (28) each edge of the embedded graph has probability at least $\epsilon > p_c(\{1, 2, \ldots, L\}^{d_*-2} \times \mathbf{Z}^2)$ of being open. This guarantees that the independent truncated process on this graph is supercritical, hence contains with probability one an infinite cluster: $P_N(0 \leftrightarrow \infty) > 0$. The simplest case for which the embedding can be easily understood is when $d_* = 3$ and $L = 2$, which we illustrated on Figure 2. For higher dimension, for

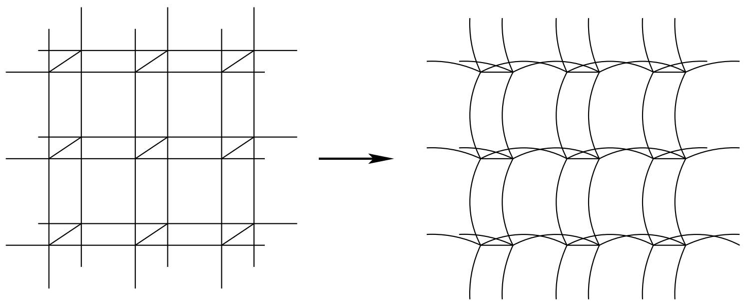

Figure 2: The embedding of the slab $\{1, 2\} \times \mathbf{Z}^2 \subset \mathbf{Z}^3$ in $(\mathbf{Z}^2, \mathcal{E}^2)$.

example in the case where $d_* = 5$, we have represented on Figure 3 an embedding of the cube $\{1, 2\}^3$ in $(\mathbf{Z}^2, \mathcal{E}^2)$, which shows what must be done in the general case. The formal embedding of the slab $\{1, 2, \ldots, L\}^{d_*-2} \times \mathbf{Z}^2$ can be found in [11]. We leave it as an exercise to the reader to show that (26) can be used again to show that $\lim_N P_N(0 \leftrightarrow \infty) = 1$.

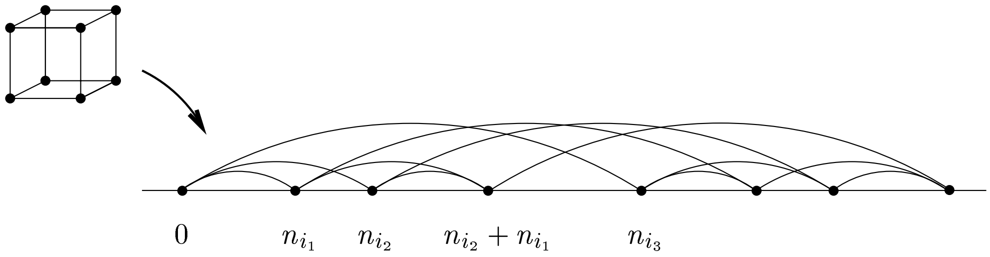

Figure 3: The embedding of the cube $\{1,2\}^3$ on the positive axis of $(\mathbf{Z}^2, \mathcal{E}^2)$, using long range edges. We have chosen $n_{i_1} := n_1$, $n_{i_2} := \min\{n_k : n_k > n_{i_1}\}$, and $n_{i_3} := \min\{n_k : n_k > n_{i_1} + n_{i_2}\}$, in order to avoid overlapping. Then, an infinite number of copies of this cube must be glued together, to form the slab $\{1,2\}^3 \times \mathbf{Z}^2$ embeded in $(\mathbf{Z}^2, \mathcal{E}^2)$.

Remark 5. In order to deduce Theorem 2 from Theorem 5, one must estimate

$$\limsup_{n\to\infty} p_n = \limsup_{n\to\infty} \frac{1 - e^{-2\beta\phi_n}}{1 + (q-1)e^{-2\beta\phi_n}} \geq C_-(\beta, q) \limsup_{n\to\infty} \phi_n \,.$$

Observe that $C_-(\beta, q)$ goes to zero when $\beta \searrow 0$. Therefore, the dimension d_* and the thickness L of the slab used in the proof above diverges for large temperatures.

Remark 6. Theorem 2 follows from Theorem 5 and Proposition 1. Recently, Bodineau [5] has proved the analog of the result of Grimmett and Marstrand [16] (which we used in (27)) for the percolation threshold of the random cluster measure associated to the Ising model ($q = 2$). Therefore, using the main result of [5] and the same embedding as above, one can obtain a more direct proof of Theorem 2 for the case $q = 2$, without using Proposition 1.

Theorem 6. *($d \geq 3$) If*

$$\sum_{n\geq 1} p_n = +\infty \tag{29}$$

then $\lim_N P_N(0 \leftrightarrow \infty) = 1$.

Proof. The proof relies on the fact that sequences p_n that satisfy (29) already imply percolation in one dimension. So for a while we consider long range percolation on $(\mathbf{Z}, \mathcal{E}^1)$, and denote by P^1 the product measure on $\{0,1\}^{\mathcal{E}^1}$ associated to the sequence $(p_n)_{n\geq 1}$. By Borel-Cantelli we have $\theta := P^1(0 \leftrightarrow \infty) = 1$. For simplicity we first assume that there exists, for all $m \in \mathbf{Z}$, a sequence $n_1 = 0, n_2, \ldots, n_k = m$ such that $p_{|n_{i+1} - n_i|} > 0$. This implies, by [2], that the infinite cluster is unique. We can thus write

$$\begin{aligned} P^1(0 \leftrightarrow 1) &\geq P^1(0 \leftrightarrow \infty, 1 \leftrightarrow \infty) \\ &\geq P^1(0 \leftrightarrow \infty) P^1(1 \leftrightarrow \infty) = \theta^2 \,, \end{aligned} \tag{30}$$

where we have used, in this order, uniqueness of the infinite cluster, FKG inequality, and translation invariance. For $a < b$ let $T_L(a, b) := [a - L, b + L]$. Then $\{0 \leftrightarrow 1 \,\mathrm{in}\, T_L(0, 1)\} \nearrow \{0 \leftrightarrow 1\}$ when $L \to \infty$. Therefore for all $\delta > 0$ there exists L_δ such that for $L \geq L_\delta$,

$$P^1(0 \leftrightarrow 1 \,\mathrm{in}\, T_L(0, 1)) \geq \theta^2 - \delta \,. \tag{31}$$

Since

$$\theta^2 = 1 > p_c(\mathcal{H}), \tag{32}$$

where $p_c(\mathcal{H})$ is the percolation threshold of the honeycomb lattice $\mathcal{H}$, we can take δ small enough such that $\theta^2 - \delta > p_c(\mathcal{H})$, and fix $L \geq L_\delta$.
Back to $d = 3$, consider the embedding of $\mathcal{H}$ in $\mathbf{Z}^3$, denoted $\mathcal{H}'$, depicted on Figure 4. To each edge $e = \{x, y\} \in \mathcal{H}'$ corresponds a one dimensional subgraph

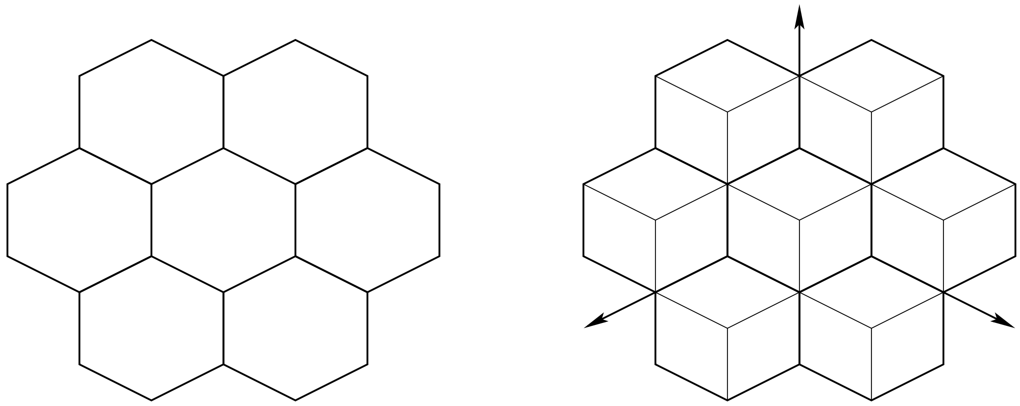

Figure 4: The honeycomb lattice $\mathcal{H}$ and its embedding in $\mathbf{Z}^3$, denoted $\mathcal{H}'$.

$(\mathbf{Z}(e), \mathcal{E}^1(e)) \subset (\mathbf{Z}^3, \mathcal{E}^3)$ which consists of all the points (and long range edges) contained in the line supported by e; $(\mathbf{Z}(e), \mathcal{E}^1(e))$ is nothing but a copy of $(\mathbf{Z}, \mathcal{E}^1)$, embedded in $(\mathbf{Z}^3, \mathcal{E}^3)$, containing e. All the previous considerations on $(\mathbf{Z}, \mathcal{E}^1)$ (for example the intervals $T_L(x, y)$) can be adapted in each subgraph $(\mathbf{Z}(e), \mathcal{E}^1(e))$. An important property of the embedding we have chosen is that *the graphs* $(\mathbf{Z}(e), \mathcal{E}^1(e))$, $(\mathbf{Z}(e'), \mathcal{E}^1(e'))$ *associated to two different edges* $e, e' \in \mathcal{H}'$ *have disjoint sets of edges.* We now define an edge $e = \{x, y\} \in \mathcal{H}'$ to be good if and only if there exists, in $(\mathbf{Z}(e), \mathcal{E}^1(e))$, a path connecting x to y in $T_L(x, y)$. Clearly, edges are good independently and for $N = 2L + 2$,

$$P_N(e \text{ is good}) = P_N^1(x \leftrightarrow y \,\text{in}\, T_L(x, y)) \geq \theta^2 - \delta > p_c(\mathcal{H}).$$

Therefore, there exist infinite paths of good edges, yielding the existence of an infinite cluster on $(\mathbf{Z}^3, \mathcal{E}^3)$ with edges of sizes smaller than N. Clearly, $\lim_N P_N(0 \leftrightarrow \infty) = 1$, which finishes the proof. When the assumption made at the beginning is not satisfied, it suffices to replace $P^1(0 \leftrightarrow 1)$, in (30), by $P^1(0 \leftrightarrow K)$ for a well chosen K. The rest of the proof can be adapted in a straightforward way. $\square$

Observe that the only place where we used the divergence of the series (29) was to obtain $P^1(0 \leftrightarrow \infty) = 1$. A variant of Theorem 6 can therefore be reformulated under a more abstract condition on the sequence $(p_n)_{n \geq 1}$, which can hold also when the series $\sum_n p_n$ converges:

Theorem 7. ($d \geq 3$). *Assume the sequence* $(p_n)_{n \geq 1}$ *is such that* $\theta := P^1(0 \leftrightarrow \infty)$ *satisfies* $\theta^2 > p_c(\mathcal{H})$ *and that the one-dimensional infinite cluster is unique. Then* $P_N(0 \leftrightarrow \infty) > 0$ *when* N *is large enough.*

4 Final Remarks

We have considered the problem of truncation in the long range Potts model with non-summable ferromagnetic interactions, via simple percolation techniques. We have shown that for various families of potentials, a phase transition occurs in the truncated model as soon as the parameter of truncation N is taken sufficiently large. Notice that by Proposition 1 all the existing results on truncation in long range percolation ([22, 24, 3]) have their counterpart in the long range Potts model. We hope that *embeddings*, as those we used in the proofs of Theorems 5 and 6, might be used for possible generalisations since they don't require any particular regularity of the potential/edge probability at infinity, and give some insight into new mechanisms of phase transitions in systems with long range interactions.

Before ending, we make two remarks concerning the problem of truncation.

A Mean Field Limit. As our results show, *infinite* systems with non-summable interactions have trivial dependence on the temperature. In the physics litterature, some methods have been used in view of understanding the properties of large but *finite* systems with non-summable interactions. These methods rely essentially on the study of the mean field version of the original model. Namely, since the energy of the finite system grows faster than its size in the limit of large volumes Λ, the model is modified ([7, 8, 25]) by dividing the total hamiltonian by a well chosen power of the volume $|\Lambda|$. In the case of Ising spins $\sigma_x = \pm 1$ with ferromagnetic interactions $\phi(x) = \|x\|^{-\alpha}$, $0 \leq \alpha \leq d$, this means considering the following formal identity:

$$\beta \sum_{x \neq y} \phi(x-y)\sigma_x\sigma_y = \beta|\Lambda|^{\delta} \sum_{x \neq y} \frac{\phi(x-y)}{|\Lambda|^{\delta}}\sigma_x\sigma_y \,. \tag{33}$$

The scaling parameter δ must be chosen in function of α in order to obtain a well defined thermodynamic limit for the potential $|\Lambda|^{-\delta}\phi(x-y)$, leading to a mean field inverse critical temperature $\beta_c^*(\alpha)$. Then, the "critical inverse temperature" $\beta_c(\alpha, \Lambda)$ of the real system in a finite volume Λ can be infered to go to zero as $\beta_c(\alpha, \Lambda) \sim \beta_c^*(\alpha)|\Lambda|^{-\delta}$. It was numerically observed ([8], [25]) that $\beta_c^*(\alpha)$ depends weakly on α, which has lead the authors to conjecture that all the systems with $0 \leq \alpha \leq d$ have the same thermodynamic behaviour, i.e. identical to the pure mean field case $\alpha = 0$. This "universal" mean field behaviour was then given an intelligible explanation by Vollmayr-Lee and Luijten [26].
We now wish to present an argument in favor of our conjecture, similar in some ways to the strategy of [26]. Apart from helping to understand our conjecture, it also sheds some light on the weak dependence in α observed numerically in [8], [25]. For simplicity we consider the case $q = 2$, i.e. the Ising model. Remember that Conjecture 1 claims that when ϕ is non-summable, then $\beta_c(\phi_N) \to 0$ as $N \to \infty$, where $\beta_c(\phi_N)$ is the critical inverse temperature of the truncated model ϕ_N. Define

$$e_N := \sum_{x \neq 0} \phi_N(x) = \sum_{x \in B_N} \phi(x)\,,$$

and consider the formal identity:

$$\begin{aligned}\beta\sum_{x\neq y}\phi_N(x-y)\sigma_x\sigma_y &= \beta e_N\sum_{x\neq y}\frac{\phi_N(x-y)}{e_N}\sigma_x\sigma_y\\ &\equiv \widehat{\beta}_N\sum_{x\neq y}\widehat{\phi}_N(x-y)\sigma_x\sigma_y\,,\end{aligned}$$

where we have defined the rescaled quantities

$$\widehat{\beta}_N := \beta e_N\,,\qquad \widehat{\phi}_N(x) := \frac{\phi_N(x)}{e_N}\,. \tag{34}$$

Since $\lim_N e_N = +\infty$ (we assume ϕ is non-summable), we have $\lim_N \widehat{\beta}_N = +\infty$. Moreover, the new potential $\widehat{\phi}_N$ has the following properties: 1) it has range at most N, 2) it is summable

$$\sum_{x\neq 0}\widehat{\phi}_N(x) = 1\,,$$

and 3) $\lim_N \widehat{\phi}_N(x) = 0$ for all x. Such properties remind those of Kac potentials, which are of the form $\Phi_\gamma(x) := \gamma^d\varphi(\gamma x)$, where $\varphi(x)\geq 0$ is bounded, supported by $[-1,+1]^d$, with $\int\varphi(x)\mathrm{d}x = 1$, and $\gamma>0$ is a small scaling parameter. Φ_γ thus has the following properties: 1) it has range γ^{-1}, 2) it is summable:

$$\int\Phi_\gamma(x)\mathrm{d}x = 1\quad \forall\gamma>0\,,$$

and 3) $\lim_{\gamma\to 0^+}\Phi_\gamma(x) = 0$ for all x. It is well known ([18], [20]) that such potentials give, in the van der Waals limit $\gamma\to 0^+$, a justification of the van der Waals-Maxwell theory of liquid-vapor equilibrium: in this limit, the properties of the system converge to those of mean field, *regardless of the details of the function* φ. Moreover, it is known ([9], [6]) that for $\gamma>0$ the model is a good approximation to mean field: a phase transition occurs *before* reaching the mean field regime, and $\sup_{\gamma>0}\beta_c(\Phi_\gamma)<+\infty$.

It is tempting to ask whether this mean field behaviour of Kac potentials Φ_γ at small γ also holds for our potential $\widehat{\phi}_N$ at large N, and to identify the parameters γ^{-1} and N. Although $\widehat{\phi}_N$ is not obtained by a rescaling of a given function, as Φ_γ is, one can expect that for a reasonable potential ϕ (for example of the type (2)), the critical temperature $\beta_c(\widehat{\phi}_N)$ is uniformly bounded in N: $\sup_N\beta_c(\widehat{\phi}_N)<+\infty$. Therefore, if N is large enough so that $\widehat{\beta}_N>\sup_N\beta_c(\widehat{\phi}_N)\geq\beta_c(\widehat{\phi}_N)$, we have a phase transition in the truncated model ϕ_N, as predicted by our conjecture. Moreover, since we expect that the properties of the system with $\widehat{\phi}_N$ converge to those of mean field when $N\to\infty$, *independently* of the fine structure of ϕ, this again is in favor of the non-dependence on α observed in [8] and [25].

Unfortunately, the unique case where this scenario can be implemented rigorously is for the constant potential: $\phi(x) = c>0$ for all $x\neq 0$ (or, equivalently, if $\alpha=0$). In this case we have $\widehat{\phi}_N(\cdot) = |B_N|^{-1}1_{\|x\|\leq N}(\cdot)$, and the correspondence $\gamma^{-1}\equiv N$ can be done (this case is treated in details in [12]).

On the General Problem of Truncation. We have seen that in the long range ferromagnetic Potts model, non-summable interactions with the symmetry (3) are pathological in the sense that in the limit $N \to \infty$,

$$\mu_{\phi_N}^{\beta,s} \Rightarrow \delta_s$$

for all $\beta > 0$ and all $s \in \{1, 2, \dots, q\}$. A natural question is to ask whether this weak convergence also occurs in the case where ϕ *is* sommable, i.e. when $\mu_{\phi}^{\beta,s}$ is well defined. Is it that

$$\mu_{\phi_N}^{\beta,s} \Rightarrow \mu_{\phi}^{\beta,s} \tag{35}$$

in the limit $N \to \infty$? The answer to this question is negative in $d = 1$. Namely, for the Ising model $q = 2$, there exist summable potentials for which $\mu_{\phi}^{\beta,+}(\sigma_0 = +1) > \frac{1}{2}$ at low temperature ([10],[13]), but $\mu_{\phi_N}^{\beta,+}(\sigma_0 = +1) = 0$ for all N as well known (see Remark 1). On can thus wonder if the weak convergence (35) occurs in dimensions $d \geq 2$. If this happens to be the case, it would lead to the following highly non-trivial fact: assume ϕ is sommable and that a phase transition occurs in the untruncated system:

$$\mu_{\phi}^{\beta,s}(\sigma_0 = s) > \frac{1}{q}\,.$$

Then, if (35) holds, there exists an integer N such that a phase transition occurs in the truncated model:

$$\mu_{\phi_N}^{\beta,s}(\sigma_0 = s) > \frac{1}{q}\,.$$

That is, the tail of the interaction plays no important role in the occurence of phase transitions in summable long range models. This question is again the analog of the general problem of truncation that has been posed in the context of long range percolation: if $P(0 \leftrightarrow \infty) > 0$, does there exist some large N such that $P_N(0 \leftrightarrow \infty) > 0$? This is believed to be true in general, with no assumption on the edge probability (other than, say, the symmetry (13), or for sparse interactions as (23)). This property is non-trivial for the following reason: the truncated measure P_N converges weakly to P, but the Portmanteau Theorem (see [4]) doesn't apply, due to the fact that in the product topology, the boundary of the event $\{0 \leftrightarrow \infty\}$ has strictly positive probability (1, in fact). One can thus not conclude that $P_N(0 \leftrightarrow \infty) \to P(0 \leftrightarrow \infty)$. Our results of Section 3 and various existing results [22, 3, 24, 11] give affirmative answer to the problem of truncation for particular cases.

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References

[1] M. Aizenman, J. T. Chayes, L. Chayes, and C. N. Newman. Discontinuity if the magnetization in one-dimensional $1/|x-y|^2$ Ising and Potts models. *Journ. Stat. Phys.*, 50:1–40, 1988.

[2] M. Aizenman, H. Kesten, and C. N. Newman. Uniqueness of the infinite cluster and continuity of the connectivity functions for short and long range percolation. *Commun. Math. Phys.*, 111:505–531, 1987.

[3] N. Berger. Transience, recurrence and critical behavior for long-range percolation. *Commun. Math. Phys.*, 226:531–558, 2002.

[4] P. Billingsley. *Convergence of Probability Measures.* Wiley, New York, 1968.

[5] T. Bodineau. Slab percolation for the Ising model. *Probab. Theory and Rel. Fields*, 132(1):83–118, 2005.

[6] A. Bovier and M. Zahradník. The low-temperature phases of Kac-Ising models. *Journ. Stat. Phys.*, 87:311–332, 1997.

[7] S. A. Cannas, A. de Magalhães, and F. A. Tamarit. Evidence of exactness of the mean field theory in the nonextensive regime of long-range spin models. *Phys. Rev. Lett.*, 122:597–607, 1989.

[8] S. A. Cannas and F. A. Tamarit. Long-range interactions and nonextensivity in ferromagnetic spin models. *Phys. Rev. B*, 54:R12661–R12664, 1986.

[9] M. Cassandro and E. Presutti. Phase transitions in Ising systems with long but finite range interactions. *Mark. Proc. Rel. Fields*, 2:241–262, 1996.

[10] F. J. Dyson. Existence of a phase-transition in a one-dimensional Ising ferromagnet. *Commun. Math. Phys.*, 12:91–107, 1969.

[11] S. Friedli, B. N. B. de Lima, and V. Sidoravicius. On long range percolation with heavy tails. *Elect. Comm. in Probab.*, 9:175–177, 2004.

[12] S. Friedli and C.-É. Pfister. Non-analyticity and the van der Waals limit. *Journ. Stat. Phys.*, 114(3/4):665–734, 2004.

[13] J. Fröhlich and T. Spencer. The phase transition in the one-dimensional Ising model with $1/r^2$ interaction energy. *Commun. Math. Phys.*, 84:87–101, 1982.

[14] H.-O. Georgii. *Gibbs Measures and Phase Transitions.* de Gruyter Studies in Mathematics, 1988.

[15] H.-O. Georgii, O. Häggström, and C. Maes. The random geometry of equilibrium phases. In C. Domb and J. Lebowitz, editors, *Phase Transitions and Critical Phenomena Vol. 18*, pages 1–142. Academic Press, London, 2001.

[16] G. Grimmett and J. M. Marstrand. The supercritical phase of percolation is well behaved. *Proc. Roy. Soc. London Ser*, A 430:439–457, 1990.

[17] P. Hertel, H. Narnhofer, and W. Thirring. Thermodynamic functions for fermions with gravostatic and electrostatic interactions. *Commun. Math. Phys.*, 28:159–176, 1972.

[18] M. Kac, G. E. Uhlenbeck, and P. C. Hemmer. On the van der Waals theory of the vapor-liquid equilibrium. *Journ. Math. Phys.*, 4:216–228, 1962.

[19] H. Kesten. Asymptotics in high dimensions for percolation. In O. S. Publ., editor, *Disorder in physical systems*, pages 219–240. Oxford Univ. Press, New York, 1990.

[20] J. L. Lebowitz and O. Penrose. Rigorous treatment of the van der Waals-Maxwell theory of the liquid-vapor transition. *Journ. Math. Phys.*, 7:98–113, 1966.

[21] T. M. Liggett, R. H. Schonmann, and A. M. Stacey. Domination by product measures. *Ann. Probab.*, 25:71–95, 1997.

[22] R. Meester and J. E. Steif. On the critical value for long range percolation in the exponential case. *Commun. Math. Phys.*, 180:483–504, 1996.

[23] S. A. Pirogov and Y. G. Sinai. Phase diagrams of classical lattice systems. *Teoreticheskaya i Matematicheskaya Fizika*, 26(1):61–76, 1976.

[24] V. Sidoravicius, D. Surgailis, and M. E. Vares. On the truncated anisotropic long-range percolation on $\mathbf{Z}^2$. *Stoch. Proc. and Appl.*, 81:337–349, 1999.

[25] F. Tamarit and C. Anteneodo. Rotators with long-range interactions: Connections with the mean-field approximation. *Phys. Rev. Lett.*, 84:208, 2000.

[26] B. P. Vollmayr-Lee and E. Luijten. Kac-potential treatment of nonintegrable interactions. *Phys.Rev. E*, 63:031108, 2001.