Remarks on the balance relations for the two-dimensional Navier–Stokes equation with random forcing.

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Abstract

We use the balance relations for the stationary in time solutions of the randomly forced 2D Navier-Stokes equations, found in [KP05], to study these solutions further. We show that the vorticity $\xi(t,x)$ of a stationary solution has a finite exponential moment, and that for any $a\in\mathbb{R}$, $t\geq 0$ the expectation of the integral of $|\nabla_x\xi|$ over the level-set $\{x\mid \xi(t,x)=a\}$, up to a constant factor equals the expectation of the integral of $|\nabla_x\xi|^{-1}$ over the same set.

1 Preliminaries

In this paper we continue to study the 2D Navier–Stokes equation (NSE) on the two-dimensional torus $\mathbb{T}^2=\mathbb{R}^2/(2\pi\mathbb{Z}^2)$, perturbed by a random force:

$$\dot u-\nu\Delta u+(u\cdot\nabla)u+\nabla p(t,x)=\sqrt{\nu}\,\tilde\eta(t,x), \tag{1.1}$$

$$u=u(t,x)\in\mathbb{R}^2,\ \ x\in\mathbb{T}^2,\ \ \operatorname{div}u=0,\ \ \int u\,dx\equiv\int\eta\,dx\equiv 0.$$

The scaling factor $\sqrt{\nu}$ in the r.h.s. of the equation is not important for us (since the force $\tilde\eta$ may depend on ν), but it makes the formulas obtained nicer.

We denote by H the space of square–integrable vector fields $u(x)$ such that $\operatorname{div}u=0$ and $\int u\,dx=0$, given the L_2-norm $\|\cdot\|$ and the L_2-scalar product $(\cdot,\cdot)$. By H^n, $n\in\mathbb{N}$, we denote the Sobolev space $H^n=H\cap H^n(\mathbb{T}^2;\mathbb{R}^2)$, given the norm $\|u\|_n=\big((-\Delta)^n u,u\big)^{1/2}$.

Let us denote by Π the Leray projector $\Pi:L_2(\mathbb{T}^2;\mathbb{R}^2)\to H$, which removes the gradient and the constant part of a vector field it operates upon (see e.g., [CF88]). Applying Π to (1.1) we write it in the usual form

$$\dot u(t,x)+\nu Lu+B(u)=\sqrt{\nu}\,\eta, \tag{1.2}$$

where we have denoted $Lu=-\Pi\Delta u$, $B(u)=\Pi(u\cdot\nabla)u$ and $\eta=\Pi\tilde\eta$.

The force $\eta(t,x)$ is assumed to be a Gaussian random field, white in time and smooth in x:

$$\eta = \frac{d}{dt}\zeta(t,x), \quad \zeta = \sum_{s\in\mathbb{Z}_0^2} b_s\beta_s(t)e_s(x). \tag{1.3}$$

Here $Z_0^2 = \mathbb{Z}^2 \setminus \{0\}$, $\{\beta_s(t) = \beta_s^\omega(t)\}$ is a set of independent standard Wiener processes, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, satisfying $\beta_s(0) = 0$. The real coefficients b_s are such that $M_1 < \infty$, where

$$M_n = \sum_{s\in\mathbb{Z}_0^2} |s|^{2n} b_s^2, \quad n \in \mathbb{Z},$$

and $(e_s, s \in \mathbb{Z}_0^2)$ is the standard trigonometric basis of H:

$$e_s(x) = \frac{\sin(s\cdot x)}{\sqrt{2}\pi|s|}\begin{bmatrix}-s_2\\ s_1\end{bmatrix} \quad \text{if} \quad s_1 + s_2\delta_{s_1,0} > 0$$

$$e_s(x) = \frac{\cos(s\cdot x)}{\sqrt{2}\pi|s|}\begin{bmatrix}-s_2\\ s_1\end{bmatrix} \quad \text{if} \quad s_1 + s_2\delta_{s_1,0} < 0.$$

The vector fields e_s are eigenvectors of L: $Le_s = |s|^2 e_s$ for each s.

We shall often interpret random fields $u(t,x)$ and $\eta(t,x)$ as random process in H (or in another space H^n) and write them as $u(t)$ and $\eta(t)$.

It is well known (see [VF88, DZ92, Fla94]) that for any random initial data u_0, independent of $\eta(\cdot)$, and such that $\mathbf{E}\|u_0\|^2 < \infty$, the equation (1.2) has a unique solution $u(t)$, belonging to $C([0,\infty), H)$ almost sure. This solution is a Markov process in H. We shall denote it as $u = u(t;u_0)$. If

$$b_s \neq 0 \quad \forall s \tag{1.4}$$

then the Markov process has a unique stationary measure μ_ν, [1] and all solutions $u(t)$ as above converge to μ_ν in distribution:

$$\mathcal{D}u(t) \rightharpoonup \mu_\nu \quad \text{as} \quad t \to \infty.$$

Here and below $\mathcal{D}$ signifies the distribution of a random variable. If we choose for u_0 a random vector in H, distributed as μ_ν, then the corresponding solution, which will be denoted as $u_\nu(t), t \geq 0$, is stationary:

$$\mathcal{D}(u_\nu(t)) = \mu_\nu \quad \forall t \geq 0.$$

Now let us assume that, in addition to (1.4), the coefficients b_s are symmetric in s:

$$b_s = b_{-s} \neq 0 \quad \forall s. \tag{1.5}$$

[1] See [KS00, EMS01, BKL02, KS02] and see [Kuk02] for discussion of this result and its development. Recently it was announced in [HM04] that the stationary measure is unique if (1.4) holds for all $|s| \leq 2$ (previously it was known that the result is true if (1.4) holds for $|s| \leq N$, where $N = N(\nu) < \infty$ is sufficiently large).

Then the random field $\zeta(t,x)$ is translationary invariant. That is, its distribution is invariant under the translation T_h of the torus $\mathbb{T}^2$, $T_h x = x+h$ ($h\in\mathbb{T}^2$). Due to the uniqueness, this implies that the stationary measure μ_ν and the stationary process $u_\nu(t,x)$ also are translationary invariant:

$$T_h\mu_\nu=\mu_\nu,\quad \mathcal{D}u_\nu(\cdot,\cdot+h)=\mathcal{D}u_\nu(\cdot,\cdot)\quad\forall\, h\in\mathbb{T}^2,$$

see [Kuk02, KP05]. It is proved in [KP05] that under the assumption (1.5) the stationary measure μ_ν satisfies infinitely many *balance relations*. To state them let us denote

$$\xi_\nu(t,x)=\operatorname{curl} u_\nu=\frac{\partial u_\nu^2}{\partial x_1}-\frac{\partial u_\nu^1}{\partial x_2}\,.$$

The measure $m_\nu=\mathcal{D}\xi_\nu(t)$ is time independent, and is the image of the measure μ_ν under the map $u\to\operatorname{curl} u$. Moreover, since

$$u_\nu=D\xi_\nu,\qquad D=\left(\frac{\partial}{\partial x_2},-\frac{\partial}{\partial x_1}\right)^t(-\Delta)^{-1},\tag{1.6}$$

where $-(\Delta)^{-1}$ is the inverse of the minus-Laplacian, defined on the functions on $\mathbb{T}^2$ with zero mean-value, then μ_ν is the image of m_ν under the map D. So a study of the process ξ_ν and the measure m_ν is equivalent to the study of u_ν and μ_ν.

The balance relations reed as follows:

$$\mathbf{E}\Big(g(\xi_\nu(t,x))\mid\nabla\xi_\nu(t,x)|^2\Big)=B_1\mathbf{E}(g(\xi_\nu(t,x)),\tag{1.7}$$

for any t and x. Here $\nabla=\nabla_x$, $B_1=\frac12(2\pi)^{-2}M_1$ and g is any continuous function which has at most a polynomial growth, i.e. $|g(v)|\le C(1+|v|^k)$ for all v, where C and k are some fixed constants. Equivalently,

$$\mathbf{E}(|\nabla\xi_\nu(t,x)|^2\mid\mathcal{F}_{\xi_\nu(t,x)})=B_1,$$

where $\mathbf{E}(\cdot\mid\mathcal{F}_{\xi_\nu(t,x)})$ denotes the conditional expectation with respect to the σ-algebra, generated by the random variable $\xi_\nu(t,x)$.

It is known (e.g., see [VF88, Fla94, KS03]) that if $M_k<\infty$ ($k\ge1$), and $u_0\in H$ is a non-random vector, then for any $T\ge2$ the solution $u=u(t;u_0)$ satisfies

$$u\in C([1,T],H^k)\quad\text{and}\quad u-\sqrt{\nu}\,\zeta\in C^1([1,T],H^{k-2})\quad\text{a.s.}\,.\tag{1.8}$$

The stationary solutions $u_\nu(t,x)$ also satisfy (1.8). Moreover, in this case the relations hold with $[1,T]$ replaced by any finite segment $[T_1,T_2]$, $T_1\ge0$.

For $u_0\in H$, let us denote $\xi(t,x;u_0)=\operatorname{curl} u(t,x;u_0)$. Noting that for any $x\in\mathbb{T}^2$ the map $H^4\to\mathbb{R}^2$, $u(\cdot)\mapsto(\operatorname{curl} u(x),|\nabla\operatorname{curl} u(x)|^2)$, is continuous, we see that

$$\frac1T\int_1^{T+1}g(\xi(t,x;u_0))|\nabla\xi(t,x;u_0)|^2\,dt\to\mathbf{E}\big(g(\xi_\nu(x)|\nabla\xi_\nu(x)|^2\big)\quad\text{a.s.},$$

by the Strong Law of Large Numbers (see [Kuk02, Shi05]), if

$$M_5 < \infty, \quad \mathbf{E}|u_0|^2 < \infty, \quad g(\xi) \text{ is a polynomial.} \tag{1.9}$$

Similar result holds for the functional $\xi \to g(\xi(t,x))$, so *the equality* (1.7) *still holds if we replace* $\mathbf{E}$ *by* $\lim T^{-1} \int_1^{T+1} \dots dt$ *and replace* ξ_ν *by any* $\xi(t,x;u_0)$, *provided that* (1.9) *is satisfied.*

2 The co-area form of the balance relations

From now on we assume that

$$M_6 < \infty.$$

Then, by (1.8) and ii), there exists a null-set Ω_0 such that for $\omega \notin \Omega_0$ we have

$$\xi_\nu \in C(0,T; C^3(\mathbb{T}^2)), \quad \xi_\nu - \sqrt{\nu}\,\mathrm{curl}\,\zeta \in C^1([0,T] \times \mathbb{T}^2). \tag{2.1}$$

We re-define ξ_ν and ζ to vanish for $\omega \in \Omega_0$. Now (2.1) holds for all ω.

Lemma 2.1. *For any t, x and ν we have*

$$\mathbf{P}\{\nabla \xi_\nu(t,x) = 0\} = 0. \tag{2.2}$$

The lemma is proved in Appendix.

Let us fix any $t \geq 0$, abbreviate $\xi_\nu(t,x) = \xi(x)$ and integrate the balance relation over $\mathbb{T}^2$:

$$\mathbf{E} \int_{\mathbb{T}^2} (g(\xi(x)|\nabla \xi(x)|^2)\, dx = B_1 \mathbf{E} \int_{\mathbb{T}^2} g(\xi(x))\, dx. \tag{2.3}$$

For a continuous function g such that $|g| \leq 1$, we denote

$$I_1 = \mathbf{E} \int_{\mathbb{T}^2} \left(g(\xi(x)|\nabla \xi(x)|^2 \right) dx\,, \quad I_2 = \mathbf{E} \int_{\mathbb{T}^2} g(\xi(x))\, dx.$$

Then

$$I_1 = B_1 I_2. \tag{2.4}$$

For $\varepsilon > 0$ let us consider the set $K_\varepsilon \subset \mathbb{T}^2 \times \Omega$,

$$K_\varepsilon = \{(x,\omega) \mid |\nabla \xi| \geq \varepsilon\}.$$

Clearly,

$$I_1 = I_1^\varepsilon + O(\varepsilon^2), \quad I_1^\varepsilon = \mathbf{E} \int_{\mathbb{T}^2} (g(\xi(x)|\nabla \xi(x)|^2) I_{K_\varepsilon}(x,\omega)\, dx.$$

For the quantity I_2^ε, obtained from I_2 by multiplying the integrand by the factor I_{K_ε}, we have

$$|I_2 - I_2^\varepsilon| \leq \int_\Omega \int_{T^2} I_{\{|\nabla \xi| < \varepsilon\}}(x,\omega)\, dx P(d\omega).$$

Since the integrand converges to $I_{\{\nabla\xi=0\}}$ when $\varepsilon \to 0$, then by Lemma 2.1 we have

$$I_2^\varepsilon \to I_2 \quad \text{as} \quad \varepsilon \to 0\,.$$

Let us denote $K_\varepsilon(\omega) = \{x \mid (x,\omega) \in K_\varepsilon\}$. Then by the implicit function theorem, for every ω, every $\tau \in \mathbb{R}$ and any $\varepsilon > 0$, the set

$$\Gamma_\varepsilon(\tau,\omega) = \{x \in K_\varepsilon(\omega) \mid \xi(x) = \tau\}$$

is a finite union of C^3-smooth curves of finite length (here 'curve' means 'one-dimensional manifold with a boundary'). We shall denote by γ points of a set $\Gamma_\varepsilon(\tau,\omega)$ and denote by $d\gamma$ the length-element. Performing the 'co-area change of variables'

$$K_\varepsilon(\omega) \ni x \to (\tau,\gamma), \quad \tau = \xi(x),\ \gamma \in \Gamma_\varepsilon(\tau,\omega),$$

we have $d\tau\, d\gamma = |\nabla\xi|\, dx_1\, dx_2$. So

$$I_1^\varepsilon = \mathbf{E}\int_{\mathbb{R}} g(\tau) \int_{\Gamma_e(\tau,\omega)} |\nabla\xi|\, d\gamma\, d\tau\,,$$

and

$$I_2^\varepsilon = \mathbf{E}\int_{\mathbb{R}} g(\tau) \int_{\Gamma_\varepsilon(\tau,\omega)} |\nabla\xi|^{-1}\, d\gamma\, d\tau\,.$$

Since the map ξ is C^3-smooth, then the Sard lemma applies, so for every ω and almost every $\tau \in \mathbb{R}$ the level-set

$$\Gamma(\tau,\omega) = \{x \in \mathbb{T}^2 \mid \xi(x) = \tau\} \tag{2.5}$$

is a C^3-smooth manifold. So

$$\int_{\Gamma_e(\tau,\omega)} |\nabla\xi|\, d\gamma \nearrow \int_{\Gamma(\tau,\omega)} |\nabla\xi|\, d\gamma < \infty \ \text{ as } \ \varepsilon \to 0\,, \tag{2.6}$$

for every ω and a.e. τ. Hence,

$$I_1^\varepsilon \to \mathbf{E}\int_{\mathbb{R}} g(\tau) \int_{\Gamma(\tau,\omega)} |\nabla\xi|\, d\gamma\, d\tau \ \text{ as } \ \varepsilon \to 0\,,$$

by the monotone convergence theorem.

Similar,

$$\int_{\Gamma_e(\tau,\omega)} |\nabla\xi|^{-1}\, d\gamma \nearrow \int_{\Gamma(\tau,\omega)} |\nabla\xi|^{-1}\, d\gamma \le \infty \ \text{ as } \ \varepsilon \to 0\,,$$

for every ω and a.e. τ, where we accept the following agreement: $0^{-1} = 0$. By this convergence,

$$I_2^\varepsilon \to \mathbf{E}\int_{\mathbb{R}} g(\tau) \int_{\Gamma(\tau,\omega)} |\nabla\xi|^{-1}\, d\gamma\, d\tau \ \text{ as } \ \varepsilon \to 0\,.$$

From the other hand, we know that $I_1^\varepsilon \to I_1$ and $I_2^\varepsilon \to I_2$. Therefore (2.4) implies that

$$\mathbf{E}\int_{\mathbb{R}} g(\tau)\int_{\Gamma(\tau,\omega)} |\nabla\xi|\,d\gamma\,d\tau = B_1\,\mathbf{E}\int_{\mathbb{R}} g(\tau)\int_{\Gamma(\tau,\omega)} |\nabla\xi|^{-1}\,d\gamma\,d\tau\,, \tag{2.7}$$

where the integral in the r.h.s. is finite and equals $B_1 I_2$. Since for each ω the set of critical values τ of ξ has zero measure, then we can arbitrarily re-define $|\nabla\xi|^{-1}$ in critical points of ξ, without changing the integral in the r.h.s.. Below we adopt the following natural agreement:

$$\int_{\Gamma(\tau,\omega)} |\nabla\xi|^{-1}\,d\gamma = \infty \;\text{ if }\; \tau \;\text{ is a critical value of }\; \xi \tag{2.8}$$

(i.e., if $\Gamma(\tau,\omega)$ is not a finite union of smooth curves). Finally, if τ is a critical value, we define the integral $\int_{\Gamma(\tau,\omega)} |\nabla\xi|\,d\gamma \le \infty$ using the limit (2.6). Now the internal integrals in the both sides of (2.7) are defined for all ω and all τ.

Since (2.7) holds for any continuous function g, $|g| \le 1$, then we have proved the following result:

Theorem 2.2. *Let $\xi_\nu(t,x)$ be the vorticity of the stationary solution $u_\nu(t,x)$. Then for any $t \ge 0$ and $\nu > 0$ we have*

$$\mathbf{E}\int_{\Gamma(\tau,\omega)} |\nabla\xi_\nu|\,d\gamma = B_1\mathbf{E}\int_{\Gamma(\tau,\omega)} |\nabla\xi_\nu|^{-1}\,d\gamma, \tag{2.9}$$

for a.a. $\tau \in \mathbb{R}$. Here $\Gamma(\tau,\omega)$ is defined in (2.5) *(where $\xi(x) = \xi_\nu(t,x)$), and we assume* (2.8).

In other words, the l.h.s. and the r.h.s. of (2.9) define the same L_1-function of τ. Choosing in (2.7) $g = 1$ we see that the integral of this function over $\mathbb{R}$ equals $(2\pi)^2$.

If τ is a regular value of ξ_ν as a function of x, then $\Gamma(\tau,\omega)$ is a finite union of smooth curves, and in any point $x \in \Gamma(\tau,\omega)$ we have $\nabla\xi_\nu = \pm\frac{\partial}{\partial n}\xi_\nu$, where n is a unit normal to the curve. Hence, in (2.9) $|\nabla\xi_\nu|$ can be replaced by $|\frac{\partial}{\partial n}\xi_\nu|$. So the integral $\int |\nabla\xi_\nu|\,d\gamma$ is a sum of the moduli of the flows of $\nabla\xi_\nu$ through the curves, forming the set Γ. The integral $\int |\nabla\xi_\nu|^{-1}\,d\gamma$ can be interpreted similar.

3 Bounds for some exponential moments

As before, we abbreviate $\xi_\nu(t,x) = \xi(x)$. Choosing in (2.3) $g(v) = v^{2m}$, where m is a natural number, and denoting $\xi(x)^{m+1} = w(x)$, we get:

$$\mathbf{E}\int_{\mathbb{T}^2} |\nabla w(x)|^2\,dx = B_1(m+1)^2\,\mathbf{E}\int_{\mathbb{T}^2} |w(x)|^{\frac{2m}{m+1}}\,dx. \tag{3.1}$$

By the Hölder inequality,

$$\int |w(x)|^{\frac{2m}{m+1}}\,dx \le (2\pi)^{\frac{2}{m+1}} \Big(\int w(x)^2\,dx\Big)^{\frac{m}{m+1}}. \tag{3.2}$$

By (2.1), $w(x) \in C^1(\mathbb{T}^2)$ for all ω. We wish to estimate the integral of $w(x)^2$ in the r.h.s. of (3.2) by its integral Dirichlet. If m was zero, then $w(x) = \xi(x)$ would be a C^1-smooth function with zero mean-value, and the estimate would follow from the Poincaré inequality. Since $m \ge 1$, then what we need is the 'nonlinear Poincaré inequality', given below. A short and nice proof of this result, suggested by M. Struwe and S. Brandle, is given in Appendix.

Lemma 3.1. *There exists a constant $C \ge 1$ such that for any $k \in \mathbb{N}$ and any C^1-smooth function $u(x)$ on $\mathbb{T}^2$ with zero mean-value we have*

$$\int u^{2k}\,dx \le C \int |\nabla(u^k)|^2\,dx. \tag{3.3}$$

Combining (3.1)–(3.3) we get:

$$\mathbf{E}\int |\nabla w(x)|^2\,dx \le B_1(m+1)^2 (2\pi)^{\frac{2}{m+1}} C\mathbf{E}\Big(\int |\nabla w|^2\,dx\Big)^{\frac{m}{m+1}}.$$

Therefore,

$$\mathbf{E}\int |\nabla w(x)|^2\,dx \le C_1^{m+1}(m+1)^{2(m+1)}.$$

Using (3.3) (with $k = m+1$) once again we see that

$$\mathbf{E}\int \xi(x)^{2(m+1)}\,dx \le C_2^{m+1}(m+1)^{2(m+1)}. \tag{3.4}$$

Since ξ is a homogeneous random field, then the l.h.s. equals $(2\pi)^2\mathbf{E}\xi(x)^{2(m+1)}$. Thus, we have proved that

$$\Big(\mathbf{E}|\xi(x)|^j\Big)^{1/j} \le Cj \tag{3.5}$$

for $j = 2r$, $r = 2, 3, \dots$. Since for any random variable η the function

$$(0,1] \ni t \to \ln(\mathbf{E}|\eta|^{1/t})^t \le \infty$$

is convex by the Hölder inequality, then (3.5) holds for all $j \in \mathbb{N}$.

For $\sigma > 0$ we have $\mathbf{E}e^{\sigma|\xi(x)|} = \sum_m (\sigma^m/m!)\mathbf{E}|\xi(x)|^m$. As $m! > (m/e)^m$ by the Stirling formula, then (3.5) implies that

$$\mathbf{E}e^{\sigma|\xi(x)|} \le \sum_m (\sigma e C)^m.$$

We have got

Theorem 3.2. *There exists $\sigma > 0$ and $C \geq 1$ such that for any $t \geq 0$, $x \in \mathbb{T}^2$ and $\nu > 0$ we have*

$$\mathbf{E}e^{\sigma|\xi_\nu(t,x)|} \leq C. \tag{3.6}$$

Remark 1. The operator D (see (1.6)), defined on functions with zero mean-value, is bounded as a linear operator in $L_2(\mathbb{T}^2)$ and as an operator in $L_\infty(T^2)$. Hence, by the interpolation theorem its norm as an operator in $L_p(\mathbb{T}^2)$, $2 \leq p \leq \infty$, is bounded by a p-independent constant C'. Due to this observation, (1.6) and (3.4),

$$\mathbf{E}\int u_\nu(t,x)^{2(m+1)}\,dx \leq C'^{2(m+1)} C_2^{m+1}(m+1)^{2(m+1)}.$$

Therefore

$$\mathbf{E}e^{\sigma_1|u_\nu(t,x)|} \leq C_1 \tag{3.7}$$

for some $\sigma_1, C_1 > 0$.

Remark 2. Let us abbreviate $u_\nu(t,x) = u(x)$. Due to (1.6), $\nabla u = \nabla D\xi$. So $\nabla u(x)$ is obtained from $\xi(x)$ by applying a singular integral operator. Hence, $|\nabla u|_{L_p} \leq C_p|\xi|_{L_p}$ for $1 < p < \infty$ by the Calderón–Zydmund theorem. For $2 \leq p < \infty$ the constant C_p can be chosen $C_p = Cp$, e.g., see [Ste70], section II.6. Due to this estimate with $p = 2(m+1)$ and (3.4) we have

$$\mathbf{E}\int |\nabla u(x)|^{2(m+1)}\,dx \leq C_3^{m+1}(m+1)^{4(m+1)}.$$

As before, this inequality implies that $(\mathbf{E}|\nabla u(x)|^{j/2})^{1/j} \leq Cj$ for all $j \in \mathbb{N}$ and all x. Therefore,

$$\mathbf{E}e^{\sigma|u_\nu(t,x)|^{1/2}} \leq C_2 \tag{3.8}$$

for any t, x, with some $\sigma_2, C_2 > 0$.

Remark 3. It is shown in [Kuk04] that, along sequences $\nu_j \to 0$, the processes $u_{\nu_j}(t,x)$ converge in distribution to limiting stationary processes $u_0(t,x)$ such that a.a. their trajectories satisfy the Euler equation. Using the Fatou lemma it is easy to check that $u_0(t,x)$ and $\xi_0(t,x) = \operatorname{curl} u_0(t,x)$ satisfy (3.5) and (3.6) (as before, first one has to establish analogies of the inequalities (3.4)). Similar, u_0 satisfy (3.7) and (3.8).

4 Appendix.

4.1 Proof of Lemma 2.1.

Let us consider the space $\mathfrak{A} = [0,1] \times \mathbb{T}^2 \times \Omega$, given the product sigma-algebra and the product-measure $\mathcal{P} = L_t \times L_x \times \mathbf{P}$, where L_t is the Lebesgue measure on [0,1] and L_x is the normalised Lebesgue measure on $\mathbb{T}^2$. Re-defining ζ and ξ_ν on a null-set as at the beginning of Section 2 we achieve that (2.1) hold for all ω.

We set $Q = \{(t,x,\omega) \in \mathfrak{A} \mid \nabla\xi_\nu = 0\}$ (as before, $\nabla = \nabla_x$) and $Q(x) = \{(t,\omega) \mid (t,x,\omega) \in Q\}$, $Q(t,x) = \{\omega \mid (t,x,\omega) \in Q\}$, $Q(x,\omega) = \{t \mid (t,x,\omega) \in Q\}$. Since the random function $\xi_\nu(t,x)$ is stationary in t and x, then $p := \mathbf{P}(Q(t,x))$ is independent of (t,x) and $\mathcal{P}(Q) = p$. We have to prove that $p = 0$. Assume that, on the contrary, $p > 0$.

Let us fix any $x_0 \in \mathbb{T}^2$ and denote by q^ω the set of points of density of $Q(x_0,\omega) \subset [0,1]$. For $t \in [0,1]$ we denote $\pi(t) = \mathbf{P}\{t \in q^\omega\}$. The set $Q^+(x_0) = \{(t,\omega) \mid \omega \in \Omega, t \in q^\omega\}$ is measurable as well as the set $Q(x_0) = \{(t,\omega) \mid \omega \in \Omega, t \in Q(x_0,\omega)\}$ (since the former may be obtained from the latter in a constructive way). Therefore we have

$$\begin{aligned} \int_0^1 \pi(t)\,dt &= \int_0^1 \int_\Omega I_{Q^+(x_0)}\,d\mathbf{P}\,dt = \int_\Omega \left(\int_0^1 I_{q^\omega}\,dt \right) d\mathbf{P} \\ &= \int_\Omega \left(\int_0^1 I_{Q(x_0,\omega)}\,dt \right) d\mathbf{P} = (L_t \times \mathbf{P})Q(x_0) = p. \end{aligned}$$

In particular, there exists $t_0 < 1$ such that $\pi(t_0) > 0$.

Lemma 4.1. *In $[0,1]$ there exists a converging sequence $t_n \searrow t_0$ $(n \geq 1)$ such that*

$$\mathbf{P}\{\omega \mid (t_n, x_0, \omega) \in Q \ \ \forall\, n \geq 0\} > 0. \tag{4.1}$$

Proof. Since $\pi(t_0) > 0$, then $\delta := \mathbf{P}(S) > 0$, where $S = \{\omega \mid t_0 \in q^\omega\}$. For $0 < \tau \leq 1 - t_0$ we denote $f^\omega(\tau) = \tau^{-1} L_t(q^\omega \cap [t_0, t_0 + \tau]) I_S(\omega)$. Then

$$f^\omega(\tau) \to 1 \quad \text{when} \quad \tau \to 0\,, \quad \text{for each } \omega \in S.$$

Hence, $\mathbf{E} f^\omega(\tau) \to \delta$ as $\tau \to 0$, and for any $\varepsilon > 0$ there exists $\tau_\varepsilon > 0$ such that

$$\mathbf{E} f^\omega(\tau_\varepsilon) = \int_{t_0}^{t_0+\tau_\varepsilon} \pi_S(\tau)\,\frac{d\tau}{\tau_\varepsilon} \geq \delta(1-\varepsilon), \quad \pi_S(\tau) = \mathbf{P}\{\tau \in q^\omega, \omega \in S\}.$$

Accordingly, for any $\varepsilon > 0$ we can find $\tau'_\varepsilon \in (0, \tau_\varepsilon]$ which satisfies $\pi_S(t_0 + \tau'_\varepsilon) \geq \delta(1-\varepsilon)$. Now we can use induction to construct a sequence $t_n \searrow t_0$ with the property $\pi_S(t_n) \geq \delta(1 - 2^{-n-1})$. Then $\mathbf{P}\{\omega \in S \mid (t_n, x_0, \omega) \in Q \ \ \forall\, n\} \geq \frac{1}{2}\,\delta$, and (4.1) follows. $\square$

Let us denote

$$\beta(t) = \nabla \operatorname{curl} \zeta(t, x_0), \qquad w(t) = \nabla \xi_\nu(t, x_0) - \sqrt{\nu} \nabla \operatorname{curl} \zeta(t, x_0).$$

By (1.3) and (1.4) $\beta(t)$ is a Brownian motion. By (2.1), $|w(t) - w(t_0)| \leq C^\omega |t - t_0|$ for each ω. So

$$\sqrt{\nu}|\beta(t) - \beta(t_0)| \leq C^\omega |t - t_0| + |\nabla \xi_\nu(t, x_0) - \nabla \xi_\nu(t_0, x_0)|.$$

This inequality and (4.1) imply that the event

$$\{|\beta(t_n) - \beta(t_0)| \leq \nu^{-1/2} C^\omega (t_n - t_0), \quad n = 1, 2, \dots\} \tag{4.2}$$

has a positive probability. But this is impossible. Indeed, let us consider the event

$$\bigcap_N \bigcup_{n\ge N} \{|\beta(t_n)-\beta(t_0)| \ge \sqrt{t_n-t_0}\}. \tag{4.3}$$

We claim that its probability is one. Since (4.2) and (4.3) do not intersect, then the probability of (4.2) must be zero.

It remains to verify the Borel–Cantelli like claim we have made. It suffive to prove that the probability one has the event, obtained from (4.3) by replacing the sequence $\{t_n\}$ by any its subsequence $\{t'_n\}$. To do this it suffices to check that

$$\mathbf{P}\Big\{ \bigcap_{n\ge k} \{|\beta(t'_n)-\beta(t_0)| \le \sqrt{t'_n-t_0}\} \Big\} = 0\,, \tag{4.4}$$

for any $k\ge 1$. The l.h.s. is bounded by

$$\mathbf{E} \prod_{n=k}^{n=K} I_{\{|\beta(t'_n)-\beta(t_0)|\le \sqrt{t'_n-t_0}\}}\,, \tag{4.5}$$

for any $K>k$. If $t_{n+1}-t_0 \ll t_n-t_0$, then

$$\mathbf{E}\Big(I_{\{|\beta(t'_n)-\beta(t_0)|\le \sqrt{t'_n-t_0}\}} \mid \beta(t'_{n+1})\Big) \le \mathbf{P}\{|\beta(t'_n)-\beta(t_0)| \le 2\sqrt{t'_n-t_0}\} =: c\,,$$

for each $\beta(t'_{n+1})$, satisfying $|\beta(t'_{n+1})-\beta_{t_0}| \le \sqrt{t'_{n+1}-t_0}$. Now, choosing for $\{t'_n\}$ a subsequnce, converging to t_0 fast enough and using the Markov property, we see that (4.5) $\le c^{K-k+1}$. Since $c<1$ and K is any number $>k$, then (4.4) follows.

4.2 Proof of Lemma 3.1.

Assume that the lemma's assertion is wrong. Then there exists a sequence of functions $\{u_l\}\subset C^1(\mathbb{T}^2)$ with zero mean-value, and a sequence of integers $\{p_l\ge 1\}$ such that for $v_l(x)=\operatorname{sgn} u_l(x)\,|u_l(x)|^{p_l}$ we have

$$\int v_l^2\,dx \ge l\int |\nabla v_l|^2\,dx. \tag{4.6}$$

Without loss of generality we may assume that

$$\int v_l^2\,dx = (2\pi)^2 \quad \forall\, l. \tag{4.7}$$

Due to (4.6) and (4.7) the sequence $\{v_l\}$ is bounded in $H^1(\mathbb{T}^2)$. So it contains a converging subsequence $\{v_{l_j}\}$:

$$v_{l_j}\to v \;\text{ weakly in }\; H^1(T^2) \;\text{ and }\; v_{l_j}\to v \;\text{ strongly in }\; L_2(T^2). \tag{4.8}$$

By (4.7), $\|v\|_{L_2}=2\pi$. Due to (4.6), $\|\nabla v_l\|_{L_2}\le 2\pi/\sqrt{l}$. Therefore $\|\nabla v\|_{L_2}\le \liminf \|\nabla v_{l_j}\|_{L_2}=0$, and $v(x)\equiv 1$.

Now let us consider the function

$$\phi(p,u) = \Big| \frac{|u|^p \operatorname{sgn} u - 1}{u-1} \Big|, \quad p \geq 1,\ u \neq 1.$$

We claim that

$$\phi(p,u) > 1/2 \quad \forall\, p \geq 1,\ \forall\, u \neq 1. \tag{4.9}$$

Indeed, note first that $\phi(1,u) \equiv 1$. It is easy to check that as a function of $p \geq 1$, $\phi(p,u)$ increases with p if $|u| > 1$ and if $u \in (0,1)$, so for such u the estimate holds. It remains to consider the case $u \in [-1,0]$. But now

$$\phi = \frac{|u|^p + 1}{|u| + 1} > \frac{1}{2}\,.$$

By (4.9), $|u_l(x) - 1| \leq 2|v_l(x) - 1|$. Due to this inequality and (4.8) (where $v \equiv 1$), we have that $u_{l_j} \to 1$ in L_1. This contradicts the normalisation $\int u_l\,dx = 0$.

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