MEAN CURVATURE PROPERTIES FOR p-LAPLACE PHASE TRANSITIONS

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Abstract. This paper deals with phase transitions corresponding to an energy which is the sum of a kinetic part of p-Laplacian type and a double well potential h_0 with suitable growth conditions. We prove that level sets of global solutions of $\Delta_p u = h_0'(u)$ with some uniform limit properties satisfy a mean curvature equation in a suitable viscosity sense. From this, we show that, if the above level sets approach uniformly a hypersurface, the latter has zero mean curvature.

Keywords: Ginzburg-Landau-Allen-Cahn phase transition models, p-Laplacian operator, sliding methods, geometric and qualitative properties of solutions.

Contents

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1. Introduction

Given a domain $\Omega \subseteq \mathbb{R}^N$, we define the following functional on $W^{1,p}(\Omega)$:

$$\mathcal{F}_\Omega(u) = \int_\Omega \frac{|\nabla u(x)|^p}{p} + h_0(u(x))\, dx\,.$$

Here above and in the sequel, we suppose $1 < p < \infty$ and $h_0 \in C^0([-1,1]) \cap C^{1,1}((-1,1))$. We will also assume that, for some $0 < c < 1 < C$ and some $\theta^* \in (0,1)$, we have

$$h_0(\zeta) > 0 \text{ for any } \zeta \in (-1,1), \tag{1.1}$$

$$\begin{aligned} &\text{for any } \theta \in [0,1],\ c\,\theta^p \le h_0(-1+\theta) \le C\,\theta^p \text{ and} \\ &c\,\theta^p \le h_0(1-\theta) \le C\,\theta^p, \end{aligned} \tag{1.2}$$

$$\text{for any } \theta \in [0,\theta^*),\ h_0'(-1+\theta) \ge c\theta^{p-1} \text{ and } h_0'(1-\theta) \le -c\theta^{p-1}. \tag{1.3}$$

Quantities depending only on the constants above will be referred to as "universal constants". As a model example for a potential h_0 satisfying the conditions stated here above, one may consider

$$h_0(\zeta) := (1-\zeta^2)^p$$

In the literature, h_0 is often referred to as a "double-well" potential.

In light of (1.1) and (1.2), we have that, with no loss of generality, possibly reducing the size of θ^*, we may and do assume that

$$\begin{aligned} &\text{for any } \zeta \in [-1+\theta^*, 1-\theta^*], \\ &h_0(\zeta) \ge \max\nolimits_{[-1,-1+\theta^*]\cup[1-\theta^*,1]} h_0, \end{aligned} \tag{1.4}$$

Notice that, if $u \in W^{1,p}(\Omega)$, $|u| < 1$, is critical for $\mathcal{F}_\Omega$, then u satisfies in the weak sense the following singular/degenrate elliptic equation of p-Laplacian type:

$$\Delta_p u(x) \;=\; h_0'(u(x))\,, \tag{1.5}$$

for any $x \in \Omega$. Here and in what follows, we make use of the standard notation

$$\Delta_p u \;:=\; \operatorname{div}\Big(|\nabla u|^{p-2}\,\nabla u\Big)\,.$$

Notice also that, if h_0 can be extended to a function which is C^1 in a neighborhood of $[-1,1]$, then (1.5) holds for any u critical for $\mathcal{F}_\Omega$ so[1] that $|u| \le 1$.

Functional $\mathcal{F}_\Omega$ here above has been widely studied both for pure mathematical reasons and for physical applications.

For $p = 2$, the model was introduced by [7] in order to approximate the behavior of minimal surfaces and a famous conjecture concerning level sets of solutions of (1.5) was presented (see also [2], [1] and [17]).

In [12] and [13] the connection of $\mathcal{F}_\Omega$ with minimal surfaces was developed, proving, with Γ–convergence methods, that if u is a Class A minimizer for the functional (i.e., if $\mathcal{F}_K(u) \le \mathcal{F}_K(u+\phi)$ for any compact set $K \subset \mathbb{R}^N$ and any $\phi \in C_0^\infty(K)$), and $u_\varepsilon(x) := u(x/\varepsilon)$, then u_ε possesses a subsequence, say u_{ε_j} which

[1]For other comments on the relation between conditions $|u| \le 1$ and $|u| < 1$, see the footnote on page 8.

converges to a step function[2] $\chi_E - \chi_{\mathbb{R}^N \setminus E}$, where ∂E is a minimal surface (in the sense of [10]). A more geometric version of this was obtained in [5], showing that the level sets $\{u_{\varepsilon_j} = 0\}$ converge locally uniformly[3] to E. Remarkably, the proof in [5] also showed a "density estimate" property for such level sets, stating that they somehow "behave in measure as they were codimension 1" (see also [20] for further details).

The results in [12] and [13] have been extended to the case $p \neq 2$ in [4], while the ones in [5] have been extended to the case $p \neq 2$ in [14] and [15].

The above functional has also some physical relevance, since it appears in the study of the equilibrium of elastic rods under tension (see [3]) and in the van der Waals-Allen-Cahn-Hilliard and Ginzburg-Landau theories of phase transition (see, for instance, [16]). In the latter situation, the term $|\nabla u|^p$ in the functional represents a penalization to the total energy, which keeps under control the formation of interfaces (see [11]).

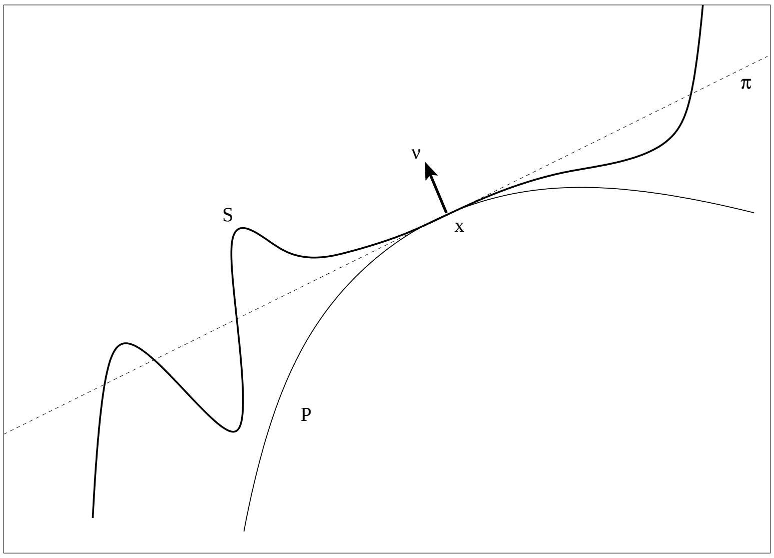
2. Statement of results

To state our result, we need to recall some standard convention about the sign of the mean curvature of a paraboloid. Let us consider a hyperplane $\pi \subset \mathbb{R}^N$ with normal vector ν. Let S be a hypersurface and P a paraboloid with vertex at some point x, and let us assume that they are tangent to each other and to π at x. We say that P is touching S from below at x if there exists $\rho > 0$ such that, if $y \in S$ and $z \in P$ with y, $z \in B_\rho(x)$ and $y - z$ in the same direction as ν, then $(y - z) \cdot \nu \geq 0$. An analogous definition can be given for a paraboloid touching by below.

[2]We freely use the standard notation for the characteristic function of a set $X \subseteq \mathbb{R}^N$:

$$\chi_X(x) := \begin{cases} 1 & \text{if} \quad x \in X\,, \\ 0 & \text{if} \quad x \in \mathbb{R}^N \setminus X\,. \end{cases}$$

[3]For a formal definition of such locally uniform convergence, see (2.2) here below.



P touches S by below at x

Of course, up to a suitable choice of coordinates, one may assume that $x = 0$, $\pi = \{x_N = 0\}$ and $\nu = e_N$. In this set of coordinates, the paraboloid P takes the form

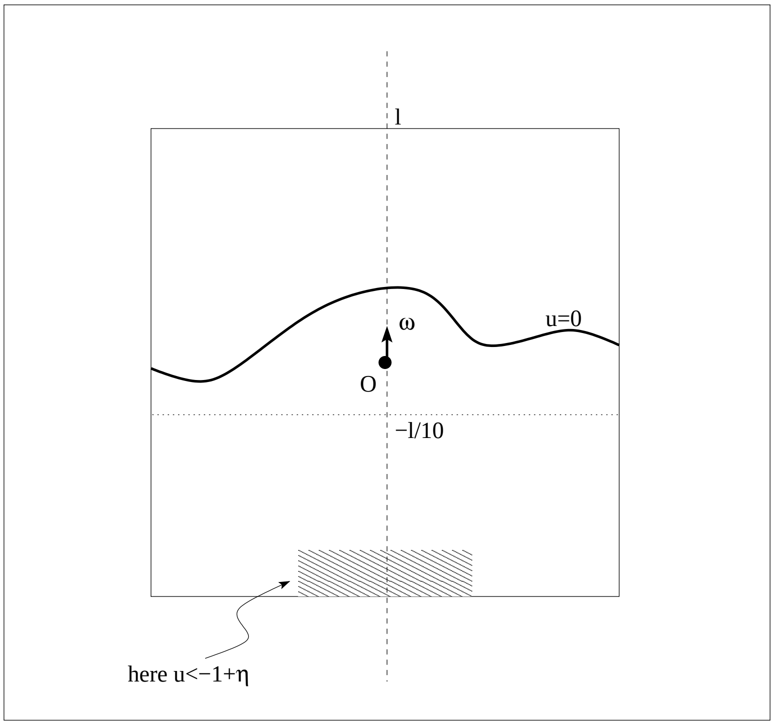
$$\left\{ (x', x) \in \mathbb{R}^{N-1} \times \mathbb{R} \;:\; x_N = \frac{1}{2} x' \cdot M x' \right\} ,$$

for some $M \in \mathrm{Mat}((N-1) \times (N-1))$. We say that P has non-negative mean curvature if $\operatorname{tr} M \geq 0$. Analogously, one may define positive, negative, non-negative and zero mean curvature. Obviously, the sign of the mean curvature depends on the orientation of ν, i.e., changing ν into $-\nu$ a positive mean curvature would be sent into a negative one, and so on. Similarly, changing ν into $-\nu$, touching by below would be transformed into touching by above.

The solutions u of (1.5) we deal with are assumed to satisfy the following uniform limit property: for any $\eta > 0$ there exists $L_\eta > 0$ so that, for any $l \geq L_\eta$ and any $\omega \in \mathrm{S}^{N-1}$,

$$\begin{aligned} &\text{if } \{u = 0\} \cap \{|x - (\omega \cdot x)\omega|_\infty \leq l\} \subset \{\omega \cdot x \geq -l/10\}, \text{ then } u(x) < -1 + \eta \\ &\text{for any } x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \text{ satisfying} \\ &\omega \cdot x < -3l/5 \text{ and } |x - (\omega \cdot x)\omega|_\infty < l/2; \\ &\text{if } \{u = 0\} \cap \{|x - (\omega \cdot x)\omega|_\infty \leq l\} \subset \{\omega \cdot x \leq l/10\}, \text{ then } u(x) > 1 - \eta \\ &\text{for any } x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \text{ satisfying} \\ &\omega \cdot x \geq 3l/5 \text{ and } |x - (\omega \cdot x)\omega|_\infty < l/2. \end{aligned} \tag{2.1}$$

In § 9, we will give further details and motivation for assumption (2.1), showing also that it is fulfilled by Class A minimizers of $\mathcal{F}$, under an additional hypothesis on the convexity of h_0 near ± 1.



Condition (2.1)

Quantities depending only on N, p and on the quantities introduced in (1.2), (1.3) and (2.1) will be often referred to with the name of "universal constants".

With these conventions, we are ready to state our main result:

Theorem 2.1. *Let $u \in W^{1,p}_{\rm loc}$ be a Sobolev weak solution of (1.5) in the whole $\mathbb{R}^N$ satisfying (2.1), with $|u| \le 1$. Let $\mathcal{S} = \partial\mathcal{E}$ be a continuous hypersurface in R^N. Let $u_\varepsilon(x) := u(x/\varepsilon)$. Assume that u_ε converges in $L^1_{\rm loc}$ to $\chi_{\mathcal{E}} - \chi_{R^N\setminus\mathcal{E}}$ and that $\{u_\varepsilon = 0\}$ converges locally uniformly to $\mathcal{S}$, i.e., that for any compact set $K \subset \mathbb{R}^N$,*

$$\lim_{\varepsilon\longrightarrow 0} \sup_{x\in\{u_\varepsilon=0\}\cap K} \mathrm{dist}\,(x,\mathcal{S}) \,=\, 0\,. \tag{2.2}$$

Then, $\mathcal{S}$ satisfies the zero mean curvature equation in the viscosity sense.

More explicitly, let $x^\star \in \mathcal{S}$ be so that, for any $r > 0$

$$\mathfrak{L}\Big(B_r(x^\star)\cap(\mathbb{R}^N\setminus\mathcal{E})\Big) > 0 \ \text{ and } \ \mathfrak{L}\Big(B_r(x^\star)\cap\mathcal{E}\Big) > 0\,, \tag{2.3}$$

where $\mathfrak{L}$ denotes the N-dimensional Lebesgue measure. Assume also that $\mathcal{S}$ admits a tangent hyperplane in $x^\star$.

Then:

- *if a paraboloid with vertex in $x^\star$ touches $\mathcal{S}$ by below at $x^\star$, then its mean curvature at $x^\star$ must be non-positive;*
- *if a paraboloid with vertex in $x^\star$ touches $\mathcal{S}$ by above at $x^\star$, then its mean curvature at $x^\star$ must be non-negative.*

In particular, if $x^\star \in \mathcal{S}$ is so that $\mathcal{S}$ is C^2 in a neighborhood of $x^\star$, then the mean curvature of $\mathcal{S}$ at $x^\star$ is zero in the classical sense.

The proof deeply relies on the ideas of [17], which deals with the case $p = 2$, and it is geometric in nature. The technique presented seems very flexible, and it may be also appropriate for further interesting extensions (such as more general functionals related with non flat metrics on manifolds, operators in non-divergence form, fully-nonlinear equations, etc.). In the case $p \neq 2$, some additional care is needed in the sliding procedure, since, due to the singularity or degeneracy of the p-Laplacian operator at points where ∇u vanishes, no general maximum and comparison results are available in the literature. Also, a careful choice of parameters is necessary to deal with the more severe nonlinearities provided by the p-Laplace equation.

More precisely, Theorem 2.1 will follows from a stronger result concerning a mean curvature property, in a weak viscosity sense, for level sets of rescaled solutions. Such result is the following:

Theorem 2.2. *Let $u \in W^{1,p}(\mathbb{R}^N)$ be a Sobolev weak solution of (1.5) in the whole $\mathbb{R}^N$, satisfying (2.1), so that $|u| \le 1$ and $u(0) = 0$. Let $\mathfrak{d} \in (0,1)$ and $M \in \mathrm{Mat}((N-1)\times(N-1))$ with*

$$\operatorname{tr} M > \mathfrak{d}\|M\| \qquad \text{and} \qquad \|M\| \le \mathfrak{d}^{-1} .$$

Let $u_\varepsilon(x) := u(x/\varepsilon)$ and

$$\Gamma := \left\{ x = (x', x_n) \in \mathbb{R}^{N-1} \times \mathbb{R} \ \ s.t. \ \ x_N = \frac{1}{2} x' \cdot M x' \right\} .$$

Then, there exist a universal $\mathfrak{d}^\star > 0$ and a function $\sigma_0 : (0,1) \longrightarrow (0,1)$ such that if $\varepsilon \in (0, \sigma_0(\mathfrak{d}))$ and $\mathfrak{d} \in (0, \mathfrak{d}^\star)$, then Γ cannot touch $\{u_\varepsilon = 0\}$ by below in $B_{\mathfrak{d}/\sqrt{\operatorname{tr} M}}$: more explicitly,

$$\{u_\varepsilon = 0\} \cap \left\{ x_N < \frac{1}{2} x' \cdot M x' \right\} \cap \left\{ |x| < \frac{\mathfrak{d}}{\sqrt{\operatorname{tr} M}} \right\} \neq \emptyset . \tag{2.4}$$

We remark that, under an additional hypothesis on the convexity of h_0 near ± 1, Class A minimizers of $\mathcal{F}$ are particular solutions satisfying the assumptions of Theorems 2.1 and 2.2. More explicitly, we have:

Theorem 2.3. *Let u be a Class A minimizers of $\mathcal{F}$ with $|u| \le 1$, and let h_0' be monotone increasing in $(-1, -1+\theta^*) \cup (1-\theta^*, 1)$. Then, the claims of Theorems 2.1 and 2.2 hold true.*

The paper is organized as follows. In § 3 we recall some standard PDE notions, such as the definition of viscosity solutions and some comparison/maximum principles that will be of use in this paper. For making the proof of the main results more readable, we collected some technical lemmata, mostly elementary in nature, in § 4. Of course, the expert reader may skip § 3 and § 4 and dedicate herself to the proof of the main results, the core of which is contained in § 5 and § 6. In particular, § 5 is devoted to the construction of suitable barriers, built via the one-dimensional solution, which will be used in § 6 for the sliding method. Such geometric construction is an extension of the one presented in [17]. The proof of Theorem 2.2 will be completed in § 7, while the one of Theorem 2.1 is in § 8. In § 9 we make comments on the assumption in (2.1) and prove Theorem 2.3.

3. PDE TOOLS

We recall here the definition of viscosity supersolution (and subsolution, and solution) for p-Laplacian type operators. Roughly, the notion of viscosity supersolution requires a pointwise evaluation of the p-Laplacian of smooth functions touching by below. However, since the p-Laplacian diverges at critical points for $1 < p < 2$, we need to *exclude this case* from the following definition:

Definition 3.1. Let $\Omega \subseteq \mathbb{R}^N$ be an open domain and let $u \in C^0(\Omega)$. If $p \geq 2$, we say that u is a viscosity supersolution of (1.5) (or that $\Delta_p u \leq h_0'(u)$ in the viscosity sense) if, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $u(x_0) = \phi(x_0)$, $u(x) \geq \phi(x)$ in Ω, we have

$$\Delta_p \phi(x_0) \leq h_0'(\phi(x_0)) .$$

If $1 < p < 2$, we say that u is a viscosity supersolution of (1.5) if, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $\nabla\phi(x_0) \neq 0$, $u(x_0) = \phi(x_0)$, $u(x) \geq \phi(x)$ in Ω, we have

$$\Delta_p \phi(x_0) \leq h_0'(\phi(x_0)) .$$

Analogously, if $p \geq 2$, we say that u is a viscosity subsolution of (1.5) (or that $\Delta_p u \geq h_0'(u)$ in the viscosity sense) if, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $u(x_0) = \phi(x_0)$, $u(x) \leq \phi(x)$ in Ω, we have

$$\Delta_p \phi(x_0) \geq h_0'(\phi(x_0)) ;$$

if $1 < p < 2$, we say that u is a viscosity subsolution of (1.5) if, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $\nabla\phi(x_0) \neq 0$, $u(x_0) = \phi(x_0)$, $u(x) \leq \phi(x)$ in Ω, we have

$$\Delta_p \phi(x_0) \geq h_0'(\phi(x_0)) .$$

If u is both a supersolution and a subsolution in the viscosity sense, we say that u is a viscosity solution.

Of course, if $u \in C^2(\Omega)$ and $p \geq 2$, then u is a viscosity solution of (1.5) if, and only if,

$$|\nabla u|^{p-4} \Big(|\nabla u|^2 \Delta u + (p-2) \langle D^2 u \, \nabla u \, , \, \nabla u \rangle \Big) = h_0'(u) \tag{3.1}$$

pointwise. However, if $1 < p < 2$, the expression above may be ill defined even for smooth functions, due to the vanishing of the gradient. Therefore, if $1 < p < 2$, for a function $u \in C^2(\Omega)$, being a viscosity solution of (1.5) is equivalent for (3.1) to hold at points where $\nabla u \neq 0$.

One of the greatest difficulty when dealing with p-Laplace equations is that the solutions belongs generally only to the class $C^{1,\alpha}$ with $\alpha < 1$ (see [8] and [19]). Also, the p-Laplace operator is singular or degenerate elliptic (respectively if $1 < p < 2$ or $p > 2$). A consequence of such pathologies is that there is no general comparison theorem for solutions in case $p \neq 2$. Therefore, no complete analogy is possible, in general, between the cases $p = 2$ and $p \neq 2$.

In this paper, we will need to compare weak Sobolev solutions of (1.5) with viscosity supersolutions of (1.5). Even if there are not general results in the literature dealing with this problem, we will succeed in doing this by exploiting some results obtained in [6], together with Hopf's Lemma for p-Laplace equations [21], and thanks to some geometric properties of barriers that we will introduce.

We recall now the maximum and comparison principles needed in our proofs. First of all, in [6] (see in particular Theorem 1.4 there) the following result is obtained:

Theorem 3.2 (Strong Comparison Principle I)**.** *Let Ω be an open (not necessarily bounded nor connected) subset of $\mathbb{R}^N$, $\Lambda \in \mathbb{R}$ and $u, v \in C^1(\Omega)$ satisfy*

$$-\Delta_p u + \Lambda u \le -\Delta_p(v) + \Lambda v, \qquad u \le v \text{ in } \Omega. \tag{3.2}$$

Define $Z_{u,v} = \{x \in \Omega : |Du(x)| + |Dv(x)| = 0\}$ if $p \ne 2$, $Z_{u,v} = \emptyset$ if $p = 2$. If $x_0 \in \Omega \setminus Z_{u,v}$ and $u(x_0) = v(x_0)$, then $u = v$ in the connected component of $\Omega \setminus Z_{u,v}$ containing x_0.

An easy consequence of the above result is the following one, which will be suitable for our applications[4]:

Corollary 3.3 (Strong Comparison Principle II)**.** *Let Ω be an open (not necessarily bounded nor connected) subset of $\mathbb{R}^N$, and $u, v \in C^1(\Omega)$ satisfy*

$$-\Delta_p u + f(u) \le -\Delta_p(v) + f(v), \qquad u \le v \text{ in } \Omega\,, \tag{3.3}$$

with f locally Lipschitz continuous. Define $Z_{u,v} = \{x \in \Omega : |Du(x)| + |Dv(x)| = 0\}$ if $p \ne 2$, $Z_{u,v} = \emptyset$ if $p = 2$. If $x_0 \in \Omega \setminus Z_{u,v}$ and $u(x_0) = v(x_0)$, then $u = v$ in the connected component of $\Omega \setminus Z_{u,v}$ containing x_0.

Proof. Let $\epsilon > 0$ be so that $B_\epsilon(x_0) \subset \Omega$ and let $M_{u,v} = \max\{|u|_{L^\infty(B_\epsilon(x_0))}, |v|_{L^\infty(B_\epsilon(x_0))}\}$. Define

$$\Lambda = \sup_{\substack{U \ne V \\ |U|,\ |V| \le M_{u,v}}} \frac{|f(U) - f(V)|}{|U - V|}\,.$$

Then,

$$\begin{aligned} -\Delta_p u + \Lambda u &\le -\Delta_p v + f(v) - f(u) + \Lambda u \\ &\le -\Delta_p v + \Lambda\,|v - u| + \Lambda u \\ &= -\Delta_p v + \Lambda\,(v - u) + \Lambda u \\ &= -\Delta_p v + \Lambda v\,, \end{aligned}$$

hence the result follows from Theorem 3.2. ☐

We recall that a similar result had been previously proved by P. Tolksdorf in [18], under stronger assumptions.

We now state a result which will allow us of take care of the points in which solutions of (1.5) may have a vanishing gradient. To this aim, we recall the following version of a more general result proved[5] by J. L. Vazquez in [21]:

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[4] As a technical remark, we point out that we will use Corollary 3.3 on pages 25 and 30 here below, with $f := h_0'$. In our case, h_0 is not assumed to be $C^{1,1}$ in the closed inteval $[-1, 1]$, but only in the open one $(-1, 1)$. Nevertheless, we will be able to exclude touching points at ± 1 with a direct argument, hence we will apply Corollary 3.3 in the domain where h_0 is $C^{1,1}$.

[5] Although we do not explicitly assume $|u| < 1$ here, but only $|u| \le 1$, we think it is appropriate to notice that, in many cases of interest, the two conditions are equivalent, thanks to the results mentioned in these pages. Indeed, the condition $|u| < 1$, under suitable assumptions, is fulfilled by any solution u such that $|u| \le 1$ with $|u|$ not identically equal to ± 1. For instance, let us suppose that for any $\theta \in [0, \theta^*)$, $h_0'(-1 + \theta) \le c'\theta^{p-1}$ and $h_0'(1 - \theta) \ge -c'\theta^{p-1}$ (this condition is in particular verified by $h_0(\zeta) := (1 - \zeta^2)^p$). Let us assume that there are points where $u = 1$ and let us show that this is not possible (the case $u = -1$ follows in the same way). Since $|u|$ is not identically equal to 1, there exists a ball $B_r(x)$ with $u < 1$ in the interior of $B_r(x)$ and $u(y) = 1$

Theorem 3.4 (Strong Maximum Principle and Hopf's Lemma)**.** *Let Ω be an open connected (not necessarily) bounded set in $\mathbb{R}^N$ and suppose that $u \in C^1(\Omega)$, $u \geq 0$ in Ω, weakly solves*

$$-\Delta_p u + cu^q = g \geq 0 \quad in \quad \Omega$$

with $q \geq p-1$, $c \geq 0$ and $g \in L^\infty_{loc}(\Omega)$. If u is not identically zero, then $u > 0$ in Ω. Moreover, for any point $x_0 \in \partial\Omega$ where the interior sphere condition is satisfied, and such that u is C^1 in a neighborhood of $\Omega \cup \{x_0\}$ and $u(x_0) = 0$, we have that $\frac{\partial u}{\partial s} > 0$ for any inward directional derivative.

Remark 3.5. The proof of Theorem 3.4 follows at once exploiting Theorem 5 in [21] with $\beta(s) = cs^q$, $q \geq p-1$ and $c \geq 0$. In particular the condition $q \geq p-1$ ensure that condition (13) and (13') in [21] are fulfilled. Moreover, the condition $c \geq 0$ causes β to be nondecreasing with $\beta(0) = 0$.

4. Technical and elementary lemmata

This section, which may be skipped by the expert reader, collects some elementary lemmata that will be of use in the course of the proofs of the main results.

Lemma 4.1. *There exists a positive constant C, depending only on p, such that*

$$a^{1/p} - b^{1/p} \leq C \, \frac{a-b}{a^{\frac{p-1}{p}} + b^{\frac{p-1}{p}}} \, ,$$

for any $a \geq b \geq 0$, $a \neq 0$.

Proof. For $t \in [0,1)$, set

$$f(t) \quad := \quad \frac{(1-t^{1/p})(1+t^{(p-1)/p})}{1-t} \, .$$

Notice that $f \in C^0([0,1))$, $f(0) = 1$ and

$$\lim_{t \longrightarrow 1} f(t) = \frac{2}{p} \, .$$

Hence, $f(t) \leq C$ for any $t \in [0,1)$ and so

$$\frac{(a^{1/p} - b^{1/p})(a^{\frac{p-1}{p}} + b^{\frac{p-1}{p}})}{a-b} = f(b/a) \leq C \, .$$

□

Lemma 4.2. *For any $0 \leq s \leq t \leq \theta^*$,*

$$h_0(-1+t) - h_0(-1+s) \geq c(t^p - s^p) \, ,$$

for a suitable universal constant $c > 0$.

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with $y \in \partial B_r(x)$. If now we consider $w := 1 - u$ it follows that w is not identically zero in the interior of $B_r(x)$ and $w(y) = 0$. Moreover, we have

$$-\Delta_p w = h_0'(u) = \frac{h_0'(1-w)}{w^{p-1}} w^{p-1} \geq -c' w^{p-1}$$

i.e. $-\Delta_p w + c' w^{p-1} \geq 0$ weakly. We can therefore exploit the Strong Maximum Principle for p-Laplace equations and prove that $u < 1$. In the same way we also get $u > -1$.

Proof. From (1.3),

$$\begin{aligned} h_0(-1+t)-h_0(-1+s) &= \int_s^t h_0'(-1+\theta)\,d\theta \\ &\geq \text{const}\int_s^t \theta^{p-1}\,d\theta \ , \end{aligned}$$

which implies the desired result. □

Lemma 4.3. *There exists a positive constant $\tilde{C}$, depending only on p, so that*

$$\int_{-1+\mathfrak{b}}^{0}\frac{d\zeta}{\Big((1+\zeta)^p-\mathfrak{a}^p\Big)^{1/p}}\leq \tilde{C}\,\log\frac{1}{\mathfrak{b}}\,,$$

for any $0<\mathfrak{a}\leq\mathfrak{b}\leq 1$.

Proof. Let

$$\mathfrak{x}:=\begin{cases}1 & \text{if } \mathfrak{b}<1/2,\\ 0 & \text{if } \mathfrak{b}\geq 1/2.\end{cases}$$

Using the substitution $\tau:=(1+\zeta)/\mathfrak{b}$, we bound the integral above by

$$\begin{aligned} \int_1^{1/\mathfrak{b}}\frac{d\tau}{\big(\tau^p-\big(\frac{\mathfrak{a}}{\mathfrak{b}}\big)^p\big)^{1/p}} &\leq \int_1^{1/\mathfrak{b}}\frac{d\tau}{(\tau^p-1)^{1/p}} \\ &\leq \int_1^{2}\frac{d\tau}{(\tau^p-1)^{1/p}}+\mathfrak{x}\int_2^{1/\mathfrak{b}}\frac{d\tau}{(\tau^p-1)^{1/p}}\,. \end{aligned}$$

Noticing that

$$\tau^p-1\geq\frac{2^p-1}{2^p}\,\tau^p$$

if $\tau\geq 2$ and that

$$\tau^p-1\geq\tau-1$$

if $\tau\geq 1$, we bound the quantity here above by

$$\text{const}\left(\int_1^2\frac{d\tau}{(\tau-1)^{1/p}}+\int_2^{1/\mathfrak{b}}\frac{d\tau}{\tau}\right)\leq\text{const}\left(1+\log\frac{1}{2\mathfrak{b}}\right)\,,$$

which proves the desired result. □

Lemma 4.4. *Let U be an open subset of $\mathbb{R}$. Let $g\in C^2(U)$ and assume that g has no critical points. Define*

$$\Psi^{y,l}(x)\ :=\ g(|x-y|-l) \tag{4.1}$$

Then, for $t=|x-y|-l\in U$ and $x\neq y$, we have

$$\Delta_p(\Psi^{y,l}(x))=(p-1)g''(t)g'^{(p-2)}(t)+g'^{(p-1)}(t)\frac{N-1}{|x-y|} \tag{4.2}$$

Proof. Let us note that

$$\nabla\Psi^{y,l}(x)=g'(t)\frac{(x-y)}{|x-y|} \tag{4.3}$$

and

$$\Psi^{y,l}_{ij}(x)=g''(t)\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}+g'(t)\left(\frac{\delta_{ij}}{|x-y|}-\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^3}\right) \tag{4.4}$$

Since

$$\Delta_p(\Psi^{y,l}(x)) = |\nabla\Psi^{y,l}(x)|^{p-2}\Delta\Psi^{y,l}(x) + (p-2)|\nabla\Psi^{y,l}(x)|^{p-4}\Psi^{y,l}_{ij}(x)\Psi^{y,l}_i(x)\Psi^{y,l}_j(x)\,,$$

we get

(4.5)

$$\begin{aligned}\Delta_p(\Psi^{y,l}(x)) &= g'^{(p-2)}(t)g''(t) + (p-2)\sum_{ij} g'^{(p-2)}g''(t)\frac{(x_i-y_i)^2(x_j-y_j)^2}{|x-y|^4} + \\ &+ g'^{(p-1)}\frac{N}{|x-y|} + (p-2)g'^{(p-1)}\frac{1}{|x-y|} + \\ &- g'^{(p-1)}\frac{1}{|x-y|} - (p-2)\sum_{ij} g'^{(p-1)}\frac{(x_i-y_i)^2(x_j-y_j)^2}{|x-y|^5} = \\ &= (p-1)g'^{(p-2)}(t)g''(t) + g'^{(p-1)}(\frac{N-1}{|x-y|})\end{aligned}$$

□

Lemma 4.5. *Let U be an open subset of $\mathbb{R}$ and assume that $g \in C^2(U)$. Let Γ be a smooth hypersurface in $\mathbb{R}^n$ and consider $d_\Gamma(x)$ the signed distance function to Γ with $x \in \Omega$ such that $d_\Gamma(x) \in U$. Then we have*

(4.6)

$$\Delta_p g(d_\Gamma(x)) = (p-1)|g'(d_\Gamma(x))|^{p-2}g''(d_\Gamma(x)) + |g'(d_\Gamma(x))|^{p-2}g'(d_\Gamma(x))\Delta d_\Gamma(x)\,.$$

Proof. Easy calculations shows that

$$\nabla g(d_\Gamma(x)) = g'(d_\Gamma(x))\nabla d_\Gamma(x)\,. \tag{4.7}$$

and

$$g_{ij}(d_\Gamma(x)) = g''(d_\Gamma(x))(d_\Gamma)_i(x)(d_\Gamma)_j(x) + g'(d_\Gamma(x))(d_\Gamma)_{ij}(x)\,. \tag{4.8}$$

By the rotation invariance, it is not restrictive to consider a particular system of coordinates for which we have

$$\nabla d_\Gamma(x) = (0,0,\ldots,0,1)$$

and

$$D^2 d_\Gamma = \operatorname{diag}\left(\frac{-k_1}{1-d_\Gamma k_1},\ldots,\frac{-k_{N-1}}{1-d_\Gamma k_{N-1}},0\right) \in \operatorname{Mat}(N\times N)\,,$$

where the k_i's are the principal curvatures of Γ at the point where the distance is realized (see, for instance, §14.6 of [9] for details). Therefore, taking into account (4.7) and (4.8), we get

$$\Delta_p g(d_\Gamma(x)) = (p-1)|g'(d_\Gamma(x))|^{p-2}g''(d_\Gamma(x)) + |g'(d_\Gamma(x))|^{p-2}g'(d_\Gamma(x))\Delta d_\Gamma(x)\,.$$

□

Lemma 4.6. *Let $I \ni 0$ be an interval of $\mathbb{R}$ and let $h \in C^1(I)$ satisfy $h(s) > 0$ for any $s \in I$. Let*

$$H(s) := \int_0^s \frac{(p-1)^{1/p}\,d\zeta}{(p\,h(\zeta))^{1/p}}\,, \qquad \forall s \in I\,.$$

Define also g as the inverse of H, that is $g(t) := H^{-1}(t)$ for any $t \in H(I)$. Then, $g \in C^2(H(I))$ and

$$\begin{aligned} g'(t) &= \Big(\frac{p}{p-1}\, h(g(t))\Big)^{1/p} \\ g''(t) &= \frac{\Big(p\, h(g(t))\Big)^{(2-p)/p}}{(p-1)^{2/p}}\, h'(g(t)), \end{aligned}$$

for any $t \in H(I)$.

Proof. The first identity easily follows by differentiating $H(g(t)) = t$. For the second claim, notice that, using the first identity twice,

$$\begin{aligned} g''(t) &= \frac{d}{dt}\Big(\frac{p}{p-1}\, h(g(t))\Big)^{1/p} \\ &= \frac{\Big(p\, h(g(t))\Big)^{\frac{1}{p}-1}}{(p-1)^{1/p}}\, h'(g(t))\, g'(t) \\ &= \frac{\Big(p\, h(g(t))\Big)^{\frac{2}{p}-1}}{(p-1)^{2/p}}\, h'(g(t)). \end{aligned}$$

□

Lemma 4.7. *Let Ω be an open domain in $\mathbb{R}^N$ and let $x_0 \in \Omega$. Let $w \in C^1(\Omega)$. Assume that there exist $v \in \mathbb{R}^N$, $\varpi \in \mathbb{R}^N \setminus \{0\}$ such that*

$$w(x_0 + x) \le v \cdot x + w(x_0),$$

for any $x \in \mathbb{R}^N$ so that $x + x_0 \in \Omega$ and $\varpi \cdot x \ge 0$. If $P \in C^2(\Omega)$ is a quadratic function touching w from below at x_0, then $\Delta_p P(x_0) \le 0$ in the viscosity sense. Analogously, if

$$w(x_0 + x) \ge v \cdot x + w(x_0),$$

for any $x \in \mathbb{R}^N$ so that $x + x_0 \in \Omega$ and $\varpi \cdot x \ge 0$, and $P \in C^2(\Omega)$ is a quadratic function touching w from above at x_0, then $\Delta_p P(x_0) \ge 0$ in the viscosity sense.

Proof. We prove the first claim, the second one being analogous. Since P touches w at x_0, $w \in C^1(\Omega)$ and $\nabla w(x_0) = v$, then

$$P(x) = \frac{1}{2} M(x - x_0) \cdot (x - x_0) + v \cdot (x - x_0) + w(x_0),$$

for some $M \in \mathrm{Mat}(N \times N)$. Notice that M must be non-positive definite: indeed, if M had a positive eigenvalue λ with corresponding eigenvector e and $|e| = 1$, possibly changing e into $-e$ we may assume that $\varpi \cdot e \ge 0$ and therefore, for a small $\varepsilon > 0$,

$$\begin{aligned} \varepsilon v \cdot e + r &\ge w(x_0 + \varepsilon e) \\ &\ge P(x_0 + \varepsilon e) \\ &= \frac{\varepsilon^2}{2} M e \cdot e + \varepsilon v \cdot e + r \\ &\ge \frac{\varepsilon^2 \lambda}{2} + \varepsilon v \cdot e + r \\ &> \varepsilon v \cdot e + r, \end{aligned}$$

which is a contradiction. Hence, M is non-positive definite and thus $\Delta P = \mathrm{tr}\, M \leq 0$. Now, if $p \geq 2$,

$$\begin{aligned} \Delta_p P &= |\nabla P|^{p-4} \Big(|\nabla P|^2 \Delta P + (p-2) \langle D^2 P \, \nabla P \, , \, \nabla P \rangle \Big) \\ &= |v|^{p-4} \Big(|v|^2 \mathrm{tr}\, M + (p-2)\, Mv \cdot v \Big) \leq 0 \, , \end{aligned}$$

at x_0, which proves the claim for $p \geq 2$.
Let now assume $1 < p < 2$. If $v = 0$, by Definition 3.1, there is nothing to check; we may then assume $v \neq 0$. Let $\lambda_1, \ldots, \lambda_N \leq 0$ be the eigenvalues of M. Then,

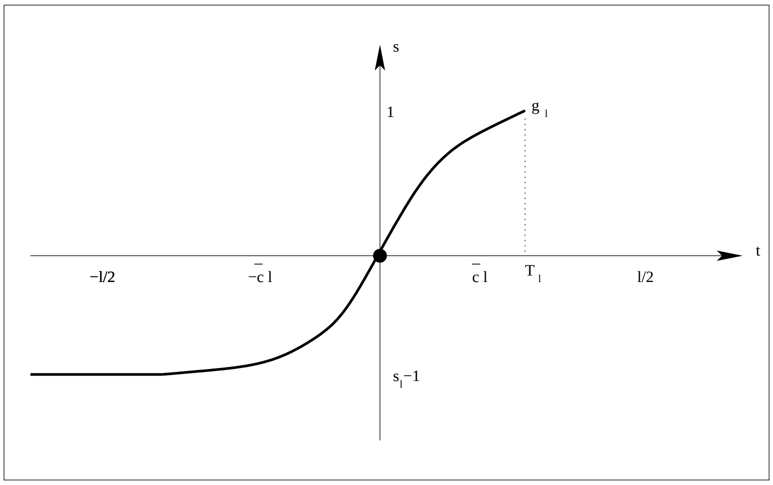
$$\begin{aligned} \Delta_p P &= \frac{1}{|v|^{2-p}} \Big(\mathrm{tr}\, M - (2-p)\, Mv \cdot \frac{v}{|v|} \cdot \frac{v}{|v|} \Big) \\ &\leq \frac{1}{|v|^{2-p}} \Big(\lambda_1 + \cdots + \lambda_N - (2-p)\, \lambda_1 \Big) \\ &= \frac{1}{|v|^{2-p}} \left((p-1)\lambda_1 + \lambda_2 + \cdots + \lambda_N \right) \, \leq \, 0 \, , \end{aligned}$$

at x_0, which ends the proof for $1 < p < 2$. □

5. Useful barriers

The core of the proof of our results begins here. Before going into the details of the arguments, which will be quite technical, we would like to point out some heuristic ideas that are underneath the construction here below. Roughly, the crucial idea, which goes back to De Giorgi, is that one-dimensional solutions are the ones which encode much information of the system. Following this belief, we will construct two barriers, which are a suitable modification of one-dimensional solutions. The first barrier, built in Lemma 5.1 here below, is radially simmetric. More precisely, it is flat in a ball and then radially increasing. Clearly, since the solution we consider does not possess such symmetry, this barrier may provide good bounds in some direction, but poor bounds in another direction. Therefore, in § 6, we will have to slide this barrier to obtain information in all the domain we are interested in. In a very non-rigorous way, we may think that the radial growth of this barrier will provide information on "how the solution grows from -1 to $+1$", while the flat part of the barrier controls the "directions along which the level sets of the solution lie". The second barrier we need is constructed in Lemma 5.2 here below. This is a modification of a one-dimensional solution which takes into account the distance from the level sets. Equation (1.5) will then relate second derivatives of this barrier with the mean curvature of the level sets of our rescaled solution, leading to the estimates we need.

We now construct the first comparison function that will be used in the proof of the main result. The function needed to construct such barrier is sketched in the picture.



The function g_l introduced in Lemma 5.1

Lemma 5.1. *There exist universal constants $\bar{l} > 1$ and $0 < \bar{c} \le 1/2$, so that, if $l \ge \bar{l}$, we can find $T_l \in [\bar{c}l, l/2]$ and a nondecreasing function*

$$g_l \in C^0(-\infty, T_l) \cap C^{1,1}(-\infty, 0) \cap C^2((-\bar{c}l, T_l) \setminus \{0\})$$

which is constant in $(-\infty, -l/2]$, with $g_l' > 0$ in $[-\bar{c}l, T_l]$, satisfying $g_l(0) = 0$, $g_l(T_l) = 1$ and such that, if we define

$$\Psi^{y,l}(x) := g_l(|x-y| - l)\,, \tag{5.1}$$

then $\Psi^{y,l}(x)$ is a strict supersolution of (1.5) in the viscosity sense, in $B_{T_l+l}(y) \setminus \partial B_l(y)$.

More precisely, g_l is constructed as follows. There exists a suitable constants $0 < c_1 < \overline{C}_1, \overline{C}_2$ so that, if we define

$$
\begin{aligned}
s_l &:= e^{-\bar{c}_1 l}\,,\\
h_l(s) &:= \begin{cases} h_0(s) - h_0(s_l - 1) - \frac{\overline{C}_2}{l}\left((1+s)^p - s_l^p\right) & \text{if} \quad (s_l - 1) < s < 0 \\ h_0(s) + h_0(1 - s_l) + \frac{\overline{C}_2}{l}\left((1-s)^p + s_l^{(p-1)}(1-s)\right) & \text{if} \quad 0 \le s < 1, \end{cases}\\
H_l(s) &:= \int_0^s \frac{(p-1)^{\frac{1}{p}}}{(p\,h_l(\zeta))^{\frac{1}{p}}}\,d\zeta\,,\\
H_0(s) &:= \int_0^s \frac{(p-1)^{\frac{1}{p}}}{(p\,h_0(\zeta))^{\frac{1}{p}}}\,d\zeta\,, \qquad \text{for any } s \in (-1, 1),
\end{aligned}
$$

then the following holds:

(i) *$h_l(s) > 0$ in $(s_l - 1) < s < 1$; in particular, H_l is well defined and strictly increasing for $(s_l - 1) < s < 1$ and thus we may define $g_l(t) := H_l^{-1}(t)$ for $t \in H_l(s_l - 1, 1)$;*
(ii) *$g_l(t)$ is defined to be constantly equal to $s_l - 1$ for $t \le H_l(s_l - 1)$;*
(iii) *the following estimates on H_l hold:*

$$H_l(1) \le \frac{l}{2} \tag{5.2}$$

$$H_l(s_l-1) \geq -\frac{l}{2}\,; \tag{5.3}$$

$$H_0(s) \leq H_l(s) - \frac{\overline{C}_1}{l}\log(1-|s|) \qquad \forall |s| < 1-e^{-\frac{\overline{c}_1 l}{2}}\,; \tag{5.4}$$

$$H_l(1-e^{-\frac{\overline{c}_1 l}{2}}) \geq \bar{c}\, l\,; \tag{5.5}$$

$$H_l(e^{-\frac{\overline{c}_1 l}{2}}-1) \leq -\bar{c}\, l\,. \tag{5.6}$$

Proof. The idea of the proof is that, once H_l is well defined, estimates (5.2), (5.3), (5.4) and the viscosity supersolution property for $\Psi^{y,l}$ are the core of the matter. Indeed, (5.3) says, in particular, that, by construction, g_l is constant in $(-\infty, -l/2]$. Also, estimates (5.4) and (1.2) imply that

$$\begin{aligned} H_l(1-e^{-\frac{\overline{c}_1 l}{2}}) &\geq \text{const} \int_0^{1-e^{-\frac{\overline{c}_1 l}{2}}} \frac{d\zeta}{1-\zeta} - \text{const} \\ &= \text{const}\, l - 1 \\ &\geq \bar{c}\, l\,, \end{aligned}$$

provided l big enough and $\bar{c}$ small enough, and, analogously,

$$H_l(e^{-\frac{\overline{c}_1 l}{2}}-1) \leq -\bar{c}\, l\,,$$

proving (5.5) and (5.6). Estimate (5.6) also implies that g_l is strictly increasing, at least, in $(-\bar{c}\, l, \bar{c}\, l)$.

Also, if $T_l := H_l(1)$, by (5.2) and (5.5), we have that $T_l \in [\bar{c}l, l/2]$. Some careful computation will be needed to show $\Psi^{y,l}$ to be a strict viscosity supersolution of (1.5) at any point where it is defined, except possibly on the sphere $\{|x-y| = l\}$ (the fact that h_l may be discontinuous at 0 make g_l not necessarily smooth at 0 and deprive us of information on the value of $\Delta_p \Psi^{y,l}$ on the aforesaid sphere). We also remark that the extension in (ii) is a $C^{1,1}$-extension, since, using Lemma 4.6, if $t = H_l(s_l-1)$,

$$\begin{aligned} g_l'(t) &= \Big(\frac{p}{p-1}\, h_l(g_l(t))\Big)^{1/p} \\ &= \Big(\frac{p}{p-1}\, h_l(s_l-1)\Big)^{1/p} \\ &= 0\,. \end{aligned}$$

We now deal with the actual proof of Lemma 5.1: in light of the arguments above, we will focus on proving (5.2), (5.3), (5.4) and the viscosity supersolution property for $\Psi^{y,l}$.

The proof will consider separately the cases $(s_l-1) < s < 0$ and $0 \leq s < 1$. Let us first consider the case $(s_l-1) < s < 0$. From (1.4),

$$h_l(s) \geq \frac{h_0(\theta^*-1)}{2} \tag{5.7}$$

if $\theta^* - 1 \le s < 0$, provided l is suitably large. Also, in light of Lemma 4.2, we get

$$h_0(s) - h_0(s_l - 1) \ge c\left((1+s)^p - s_l^p\right), \tag{5.8}$$

for $(s_l - 1) < s < \theta^* - 1$, therefore

$$\frac{p-1}{p}(h_0(s) - h_0(s_l - 1)) \le h_l(s), \tag{5.9}$$

if $(s_l - 1) < s < \theta^* - 1$, provided l is sufficiently large. This, (1.4) and (5.7) say that $h_l(s) > 0$ in $(s_l - 1) < s < 0$, showing that H_l is well defined and strictly increasing in this case. Also, from the definition of H_l, (5.7), (5.9) and (5.8), we gather

$$\begin{aligned}
-H_l(s_l - 1) &= \text{const} \int_{s_l-1}^{0} \frac{d\zeta}{(h_l(\zeta))^{1/p}} \\
&= \text{const}\left(\int_{\theta^*-1}^{0} \frac{d\zeta}{(h_l(\zeta))^{1/p}} + \int_{s_l-1}^{\theta^*-1} \frac{d\zeta}{(h_l(\zeta))^{1/p}}\right) \\
&\le \text{const}\left(1 + \int_{s_l-1}^{\theta^*-1} \frac{d\zeta}{(h_l(\zeta))^{1/p}}\right) \\
&\le \text{const}\left(1 + \int_{s_l-1}^{\theta^*-1} \frac{d\zeta}{\left((1+\zeta)^p - s_l^p\right)^{1/p}}\right) \\
&\le \text{const}\left(1 + \int_{s_l-1}^{0} \frac{d\zeta}{\left((1+\zeta)^p - s_l^p\right)^{1/p}}\right),
\end{aligned}$$

hence, from Lemma 4.3, we get

$$H_l(s_l - 1) \ge -\frac{l}{2},$$

provided $\bar{c}_1$ is suitably small, proving (5.3).

We now show that $\Psi^{y,l}$ is a viscosity supersolution of (1.5) when $|x-y| < l$ (i.e., when $s = g_l(t) < 0$; here and in what follows, we often use the notation $t = |x-y| - l$ and $s = g_l(t) = \Psi^{y,l}(x)$).
Of course, if $|x-y| < l/2$, then $\Psi^{y,l}(x) = s_l - 1$ by definition of g_l, and therefore

$$\Delta_p \Psi^{y,l}(x) = 0 < h_0'(s_l - 1) = h_0'(\Psi^{y,l}(x)), \tag{5.10}$$

showing that the viscosity supersolution property of $\Psi^{y,l}$ holds for $|x-y| < l/2$. Hence, we can now concentrate on the case $l/2 \le |x-y| < l$. In light of Lemma 4.6,

$$\begin{aligned}
g_l'(t) &= \left(\frac{p}{p-1}\, h_l(g(t))\right)^{\frac{1}{p}} \qquad \text{and} \\
g_l''(t) &= \frac{(p\, h_l(g(t)))^{\frac{2-p}{p}}}{(p-1)^{\frac{2}{p}}}\, h_l'(g_l(t)).
\end{aligned}$$

Thus, by Lemma 4.4, we have

$$\begin{aligned} \Delta_p(\Psi^{y,l}(x)) &= (p-1)g''(t)g'^{(p-2)}(t) + g'^{(p-1)}(t)\frac{N-1}{|x-y|} \\ &\le h_l'(g_l(t)) + K(N-1)(h_l(g_l(t)))^{\frac{p-1}{p}}\frac{1}{|x-y|} \\ &\le h_l'(g_l(t)) + \frac{2K(N-1)(h_l(g_l(t)))^{\frac{p-1}{p}}}{l} \end{aligned} \tag{5.11}$$

for $|x-y| \ge \frac{l}{2}$, provided $K > 0$ is suitably large.
Hence, by definition of h_l we get (using again the notation $s = g_l(t)$) that

$$\begin{aligned} h_l(s) &\le h_0(s) - h_0(s_l - 1) \le h_0(s) \\ \text{and} \qquad h_l'(s) &= h_0'(s) - \frac{p\overline{C}_2}{l}(1+s)^{(p-1)} \end{aligned}$$

in $(s_l - 1) < s < 0$, hence

$$\Delta_p(\Psi^{y,l}(x)) < h_0'(s) - \frac{p\overline{C}_2}{l}(1+s)^{(p-1)} + \frac{2K(N-1)}{l}(h_0(s))^{\frac{p-1}{p}}, \tag{5.12}$$

for $(s_l - 1) < s < 0$. By (1.2), we get, for $\overline{C}_2$ suitably large, that

$$\frac{p\overline{C}_2}{l}(1+s)^{(p-1)} \ge \frac{2K(N-1)}{l}(h_0(s))^{\frac{p-1}{p}} \tag{5.13}$$

and therefore

$$\Delta_p(\Psi^{y,l}(x)) < h_0'(\Psi^{y,l}(x)) \tag{5.14}$$

for $s_l - 1 < g_l(t)$ and $|x-y| \ge \frac{l}{2}$.
Estimates (5.10) and (5.14) show $\Psi^{y,l}$ to be a strict viscosity supersolution of (1.5) at any points x so that $|x-y| < l$, provided $\Psi^{y,l}(x) \neq s_l - 1$.
Hence, we now deal with the case in which $|x-y| < l$ and $\Psi^{y,l}(x) = g_l(t) = s_l - 1$: in such circumstance, since Ψ is C^1 we have, by Lemma 4.7, that

$$\Delta_p(\Psi^{y,l}(x)) \le 0 < h_0'(s_l - 1) \tag{5.15}$$

in the viscosity sense. This completes the proof of the desired supersolution property of $\Psi^{y,l}$ at any points x so that $|x-y| < l$.

Let us now prove (5.4) for $e^{-\frac{\overline{c}_1 l}{2}} - 1 < s \le 0$. Observe that, by definition of h_l, recalling (1.2),

$$\begin{aligned} h_0(s) - h_l(s) &\le h_0(s_l - 1) + \frac{\overline{C}_2}{l}\Big((1+s)^p - s_l^p\Big) \\ &\le Cs_l^p + \frac{\overline{C}_2}{l}\Big((1+s)^p - s_l^p\Big) \\ &\le \frac{\overline{C}_2}{l}(1+s)^p\,, \end{aligned} \tag{5.16}$$

provided l is sufficiently large.
Also, using Lemma 4.1,

$$\begin{aligned}
H_0(s) - H_l(s) &= \int_s^0 \frac{1}{(\frac{p}{p-1} h_l(\zeta))^{\frac{1}{p}}} - \frac{1}{(\frac{p}{p-1} h_0(\zeta))^{\frac{1}{p}}} \, d\zeta \\
&= \text{const} \int_s^0 \frac{(h_0(\zeta))^{1/p} - (h_l(\zeta))^{1/p}}{\Big(h_0(\zeta)\, h_l(\zeta)\Big)^{1/p}} \, d\zeta \\
&\le \text{const} \int_s^0 \frac{\Big(h_0(\zeta) - h_l(\zeta)\Big)\, d\zeta}{\Big((h_0(\zeta))^{\frac{p-1}{p}} + (h_l(\zeta))^{\frac{p-1}{p}}\Big)\Big(h_0(\zeta)\, h_l(\zeta)\Big)^{\frac{1}{p}}} \\
&\le \text{const} \int_s^0 \frac{h_0(\zeta) - h_l(\zeta)}{h_0(\zeta)\, (h_l(\zeta))^{\frac{1}{p}}} \, d\zeta \,.
\end{aligned}$$

Consequently, from (5.7), (5.9), (5.8) and (1.2),

$$\begin{aligned}
H_0(s) - H_l(s) &\le \text{const} \left(\int_{\theta^*-1}^0 \frac{h_0(\zeta) - h_l(\zeta)}{h_0(\zeta)\, (h_l(\zeta))^{\frac{1}{p}}} \, d\zeta + \int_s^{\theta^*-1} \frac{h_0(\zeta) - h_l(\zeta)}{h_0(\zeta)\, (h_l(\zeta))^{\frac{1}{p}}} \, d\zeta \right) \\
&\le \text{const} \Big(\int_{\theta^*-1}^0 (h_0(\zeta) - h_l(\zeta)) \, d\zeta \\
&\quad + \int_s^{\theta^*-1} \frac{h_0(\zeta) - h_l(\zeta)}{(1+\zeta)^p \, ((1+\zeta)^p - s_l^p)^{1/p}} \, d\zeta \Big) \,.
\end{aligned}$$

Recalling (5.16), we deduce that

$$\begin{aligned}
H_0(s) - H_l(s) &\le \frac{\text{const}}{l} \Big(1 + \int_s^{\theta^*-1} \frac{d\zeta}{((1+\zeta)^p - s_l^p)^{1/p}} \Big) \\
&\le \frac{\text{const}}{l} \Big(1 + \int_s^0 \frac{d\zeta}{((1+\zeta)^p - s_l^p)^{1/p}} \Big) \,.
\end{aligned}$$

By Lemma 4.3, we can estimate the last integral and get

$$H_0(s) - H_l(s) \le -\frac{c}{l} \log(1+s) \,,$$

proving (5.4) for $e^{-\frac{\bar{c}_1 l}{2}} - 1 < s \le 0$. This completes the proof in the case $s_l - 1 < s < 0$.

Let us now consider the case $0 \le s < 1$. In this case, $h_l > 0$ by inspection, thus H_l is well defined and strictly increasing in [0,1). Denoting $t = |x-y| - l$ and $s = g_l(t) = \Psi^{y,l}(x)$, we notice that $s > 0$ corresponds to $|x-y| \ge l$, therefore, arguing as in (5.11), we have:

$$\Delta_p(\Psi^{y,l}(x)) < h_l'(s) + \frac{K(N-1)}{l} (h_l(s))^{\frac{p-1}{p}} \tag{5.17}$$

if $|x-y| \ge l$, provided K is large enough. Since, by definition of h_l and (1.2),

$$h_l(s) \le \text{const}\, (h_0(s) + h_0(1-s_l)) \,,$$

for $\overline{C}_2$ large enough, it follows that

$$\begin{aligned}\Delta_p(\Psi^{y,l}(x)) &< h_0'(s)-\frac{p\overline{C}_2}{l}\Big((1-s)^{p-1}+s_l^{p-1}\Big)\\ &\quad +\frac{K(N-1)}{l}(h_0(s)+h_0(1-s_l))^{\frac{p-1}{p}}\\ &\leq h_0'(s)\,,\end{aligned}$$

where, in the last estimate, (1.2) has been used once more together with the simple inequality $(a+b)^q \leq 2^q(a^q+b^q)$ with $q := (p-1)/p$. Thus $\Psi^{y,l}(x)$ is a strict viscosity supersolution of (1.5) for $|x-y|>l$, provided $\Psi^{y,l}(x)$ is well defined.

We need now to prove (5.4) in the case $0 \leq s < 1-e^{\frac{-\overline{c}_1 l}{2}}$. To this end, first notice that, if $0 \leq s < 1-e^{\frac{-\overline{c}_1 l}{2}}$, we have $1-s > \sqrt{s_l}$ and therefore

$$s_l^p \leq s_l^{p/2}(1-s)^p \leq \frac{1}{l}\,(1-s)^p\,, \tag{5.18}$$

if l is large enough. The definition of h_l, (1.2) and (5.18) imply that

$$h_l(s)-h_0(s) \leq s_l^p+\frac{\text{const}}{l}\,(1-s)^p \leq \frac{\text{const}}{l}\,(1-s)^p\,, \tag{5.19}$$

for $0 \leq s < 1-e^{\frac{-\overline{c}_1 l}{2}}$. On the other hand, the definition of h_l and (1.2) lead to

$$h_l(s) \geq h_0(s)+h_0(1-s_l) \geq \text{const}\,((1-s)^p+s_l^p) \tag{5.20}$$

for $0 \leq s < 1-e^{\frac{-\overline{c}_1 l}{2}}$. Also, from Lemma 4.1,

$$\begin{aligned}H_0(s)-H_l(s) &= \int_0^s \frac{1}{(\frac{p}{p-1}h_0(\zeta))^{\frac{1}{p}}}-\frac{1}{(\frac{p}{p-1}h_l(\zeta))^{\frac{1}{p}}}\,d\zeta\\ &\leq \text{const}\int_0^s \frac{h_l(\zeta)-h_0(\zeta)}{\Big((h_l(\zeta))^{\frac{p-1}{p}}+(h_0(\zeta))^{\frac{p-1}{p}}\Big)\Big(h_l(\zeta)\,h_0(\zeta)\Big)^{\frac{1}{p}}}\,d\zeta\\ &\leq \text{const}\int_0^s \frac{h_l(\zeta)-h_0(\zeta)}{h_0(\zeta)\,(h_l(\zeta))^{\frac{1}{p}}}\,d\zeta\,.\end{aligned} \tag{5.21}$$

Thus, from (5.21), (5.20), (5.19) and (1.2),

$$\begin{aligned}H_0(s)-H_l(s) &\leq \frac{\text{const}}{l}\int_0^s \frac{d\zeta}{\Big((1-\zeta)^p+s_l^p\Big)^{1/p}}\\ &= \frac{\text{const}}{l}\int_{-s}^0 \frac{d\xi}{\Big((1+\xi)^p+s_l^p\Big)^{1/p}}\,,\end{aligned}$$

if $0 \leq s < 1-e^{\frac{-\overline{c}_1 l}{2}}$ and, therefore, by Lemma 4.3,

$$H_0(s)-H_l(s) \leq -\frac{\text{const}}{l}\,\log(1-s)\,.$$

This indeed proves (5.4) in the case $0 \leq s < 1-e^{\frac{-\overline{c}_1 l}{2}}$.

Let us now prove (5.2). Using the definitions of H_l, h_l and (1.2),

$$\begin{aligned} H_l(1) &\leq \text{const} \int_0^1 \frac{d\zeta}{\Big(h_0(\zeta)+h_0(1-s_l)\Big)^{1/p}} \\ &\leq \text{const} \int_0^1 \frac{d\zeta}{\Big((1-\zeta)^p+s_l^p\Big)^{1/p}} \\ &\leq \text{const} \int_0^1 \frac{d\zeta}{1-\zeta+s_l} \\ &\leq \text{const} \log\frac{1}{s_l}\,. \end{aligned}$$

This proves (5.2), provided $\bar{c}_1$ is chosen to be suitably small, and ends the proof of Lemma 5.1. □

We now introduce an appropriate modification of the barrier in Lemma 5.1, in order to deal with the distance function. Given a smooth hypersurface $\mathcal{S}$ we define $d_{\mathcal{S}}(x)$ to be the signed distance from x to $\mathcal{S}$, with the assumption that $d_{\mathcal{S}}$ is positive *above* $\mathcal{S}$. For some properties of the distance function, see [9]. With this definition, we are now in position to introduce another barrier:

Lemma 5.2. *Let $0<\varepsilon\leq\sigma\leq\delta<1$, $\xi\in\mathbb{R}^{N-1}$, $M\in\mathrm{Mat}((N-1)\times(N-1))$. Let Γ be the hypersurface defined as*

$$\Gamma := \{\, x_n = \frac{\varepsilon}{2}x'\cdot Mx' + \sigma\xi\cdot x'\,\}\cap\{\,|x'|<\frac{\sigma}{\varepsilon}\,\}$$

and assume that

$$\mathrm{tr}\, M\geq\delta\,,\qquad \|M\|\leq\frac{2}{\delta}\,,\qquad |\xi|\leq\frac{1}{\delta}\,.$$

Then, there exist functions $\sigma_0:(0,+\infty)\longrightarrow(0,1)$ and $C_0:(0,+\infty)\longrightarrow[1,+\infty)$ and $T_{\varepsilon,\delta}\in\Big[0,C_0(\delta)\log\frac{1}{\varepsilon}\Big]$ such that, if $\varepsilon\leq\sigma\leq\sigma_0(\delta)$, we can find a non-decreasing function $g_\Gamma\in C^{1,1}(-\infty,T_{\varepsilon,\delta})$, constant in $\Big(-\infty,-C_0(\delta)\log\frac{1}{\varepsilon}\Big]$, such that $g_\Gamma(0)=0$, $g_\Gamma(T_{\varepsilon,\delta})=1$, and for which $g_\Gamma(d_\Gamma(x))$ is a strict viscosity supersolution of (1.5) *provided $d_\Gamma(x)<T_{\varepsilon,\delta}$.*

More precisely, g_Γ is constructed as follows. Let $c_1>0$ be suitably small and let $\rho\in C^1(\mathbb{R})$ be a non-decreasing function so that $\rho(0)=0$, $\rho(s)=-1$ for $s\leq-1/2$ and $\rho(s)=1$ for $s\geq1/2$. For any $s\in(0,1)$, define

$$h_\Gamma(s) := \max\{\,0,h_0(s)+c_1\delta\varepsilon\rho(s)\}\,. \tag{5.22}$$

Let $s_{\delta,\varepsilon}$ be the point near -1 with for which $h_0(s_{\delta,\varepsilon})=c_1\delta\varepsilon$. Define also

$$H_\Gamma(s) := \int_0^s \frac{(p-1)^{1/p}\,d\zeta}{(p\,h_\Gamma(\zeta))^{1/p}}\,.$$

Then, we have:

(i) *There exist a constant $c^\sharp\in(0,1)$ so that*

$$c^\sharp\,(\delta\varepsilon)^{\frac{1}{p}}\leq 1+s_{\delta,\varepsilon}\leq\frac{1}{c^\sharp}\,(\delta\varepsilon)^{\frac{1}{p}}\,; \tag{5.23}$$

(ii) *for any for* $s_{\delta,\varepsilon} < s \le 1$,

$$h_\Gamma(s) > 0 \tag{5.24}$$

in particular, H_Γ *is well defined and strictly increasing in* $[s_{\delta,\varepsilon}, 1]$ *and thus we may define* $g_\Gamma(t) := H_\Gamma^{-1}(t)$*, for any* $t \in [H_\Gamma(s_{\delta,\varepsilon}), H_\Gamma(1)]$ *and extend* $g_\Gamma(t)$ *to be constantly equal* $s_{\delta,\varepsilon}$ *for* $t \le H_\Gamma(s_{\delta,\varepsilon})$*. In particular, if* $g_\Gamma(t) > s_{\delta,\varepsilon}$*, then* $g'_\Gamma(t) > 0$.

Proof. First observe that (5.23) follows from (1.2): indeed, if c and C are as in (1.2),

$$\left(\frac{c_1}{C}\right)^{1/p} (\delta\varepsilon)^{1/p} \le 1 + s_{\delta,\varepsilon} \le \left(\frac{c_1}{c}\right)^{1/p} (\delta\varepsilon)^{1/p}.$$

Also, with no loss of generality, we may assume $s_{\delta,\varepsilon} < -1+\theta^*$, in order to use (1.3). Note that, since by (1.3), h_0 is increasing in $[s_{\delta,\varepsilon}, \theta^*)$, we get $h_0(s_{\delta,\varepsilon}) > c_1\delta\varepsilon$ in $(s_{\delta,\varepsilon}, \theta^*)$. Moreover, from (1.2), if c_1 is small enough, we may suppose $h_0(s) > c_1\delta\varepsilon$ for $s_{\delta,\varepsilon} < s < 0$. From the above discussions, (5.24) follows.

Notice that the constant extension of g_Γ is $C^{1,1}$ since, by Lemma 4.6, if $t = H_\Gamma(s_{\delta,\varepsilon})$,

$$\begin{aligned} g'_\Gamma(t) &= \left(\frac{p}{p-1}\, h_\Gamma(g_\Gamma(t))\right)^{1/p} \\ &= \left(\frac{p}{p-1}\, h_\Gamma(s_{\delta,\varepsilon})\right)^{1/p} \\ &= 0. \end{aligned}$$

To estimate the domain on which g_Γ is strictly increasing we have therefore to estimate $H_\Gamma(s_{\delta,\varepsilon})$ and $H_\Gamma(1)$. Using Lemma 4.2, one obtains

$$\begin{aligned} h_\Gamma(s) &\ge h_0(s) - c_1\delta\varepsilon \\ &= h_0(s) - h_0(s_{\delta,\varepsilon}) \\ &\ge \text{const}\Big((1+s)^p - (1+s_{\delta,\varepsilon})^p\Big)^{1/p} \end{aligned} \tag{5.25}$$

for any $s \in [s_{\delta,\varepsilon}, -1+\theta^*]$. On the other hand, for any $s \in [-1+\theta^*, 0]$, (1.4) implies that

$$h_\Gamma(s) \ge h_0(-1+\theta^*) - c_1\delta\varepsilon \ge \frac{h_0(-1+\theta^*)}{2}. \tag{5.26}$$

Therefore, using the definition of H_Γ, (5.23), (5.25) and (5.26) and making use of Lemma 4.3, we get

$$\begin{aligned} -H_\Gamma(s_{\delta,\varepsilon}) &= \int_{s_{\delta,\varepsilon}}^0 \frac{(p-1)^{1/p}\, d\zeta}{(p\, h_\Gamma(\zeta))^{1/p}} \\ &= \int_{-1+\theta^*}^0 \frac{(p-1)^{1/p}\, d\zeta}{(p\, h_\Gamma(\zeta))^{1/p}} + \int_{s_{\delta,\varepsilon}}^{-1+\theta^*} \frac{(p-1)^{1/p}\, d\zeta}{(p\, h_\Gamma(\zeta))^{1/p}} \\ &\le \text{const}\left(1 + \int_{s_{\delta,\varepsilon}}^{-1+\theta^*} \frac{d\zeta}{((1+\zeta)^p - (1+s_{\delta,\varepsilon})^p)^{1/p}}\right) \\ &\le C_0(\delta)\log\frac{1}{\varepsilon}, \end{aligned}$$

or, equivalently,

$$H_\Gamma(s_{\delta,\varepsilon}) \ge C_0(\delta)\log\varepsilon. \tag{5.27}$$

This completes the desired estimate on $H_\Gamma(s_{\delta,\varepsilon})$.

Let us now estimate $H_\Gamma(1)$: from the definition of h_Γ and (1.2),

$$\begin{aligned}
H_\Gamma(1) &= \int_0^1 \frac{(p-1)^{1/p}\,d\zeta}{(p\,h_\Gamma(\zeta))^{1/p}} \le \\
&\le \int_0^{1/2} \frac{\mathrm{const}\,d\zeta}{(1-\zeta)} + \int_{1/2}^1 \frac{\mathrm{const}\,d\zeta}{\Big(c(1-\zeta)^p + c_1\delta\varepsilon\Big)^{\frac{1}{p}}} \\
&\le \int_0^{1/2} \frac{\mathrm{const}\,d\zeta}{(1-\zeta)} + \int_{1/2}^1 \frac{\mathrm{const}\,d\zeta}{1-\zeta+(\delta\varepsilon)^{\frac{1}{p}}} \\
&\le \mathrm{const}\,(1-\log(\delta\varepsilon)) \le -C_0(\delta)\log\varepsilon\,.
\end{aligned} \tag{5.28}$$

The claims on the domain of g_Γ are thus a consequence of (5.27) and (5.28).

Now we deal with the proof of the viscosity supersolution property of g_Γ. First of all, notice that, in an appropriate system of coordinates we have

$$D^2 d_\Gamma = \mathrm{diag}\left(\frac{-k_1}{1-d_\Gamma k_1},\dots,\frac{-k_{N-1}}{1-d_\Gamma k_{N-1}},0\right) \in \mathrm{Mat}\,(N\times N)\,,$$

where the k_i's are the principal curvatures of Γ at the point where the distance is realized (see §14.6 in [9] for further details). We also define P as the paraboloid describing Γ, i.e.,

$$P(x') := \frac{\varepsilon}{2}x'\cdot Mx' + \sigma\xi\cdot x'\,.$$

Notice that, by construction, $|\nabla P|\le 1$; thus, by the mean curvature equation (see, for instance, equation (14.103) of [9]), it follows that

$$\begin{aligned}
\sum_{i=1}^{N-1} k_i &= \sum_{1\le i\le N-1} \partial_i\left(\frac{\partial_i P}{\sqrt{1+|\nabla P|^2}}\right) \\
&= \frac{\Delta P}{\sqrt{1+|\nabla P|^2}} - \frac{(D^2P\,\nabla P)\cdot\nabla P}{(1+|\nabla P|^2)^{3/2}} \\
&\ge \Delta P - \mathrm{const}\,|\nabla P|^2\|D^2P\|\,.
\end{aligned}$$

Consequently, if x is so that $|d_\Gamma(x)|\le C_0(\delta)\log\frac{1}{\varepsilon}$, since $|k_i|\le C_1(\delta)\varepsilon$, we have

$$\begin{aligned}
\Delta d_\Gamma &\le \sum_{i=1}^{N-1}\frac{-k_i}{1-d_\Gamma k_i} \le -\sum_{i=1}^{N-1}k_i + C_1(\delta)\varepsilon^2\log\frac{1}{\varepsilon} \le \\
&\le -\Delta P + \mathrm{const}\,|\nabla P|^2\|D^2P\| + C_1(\delta)\varepsilon^{\frac{3}{2}} \le \\
&\le -\varepsilon\delta + C_2(\delta)(\varepsilon\sigma^2+\varepsilon^{\frac{3}{2}}) \le -\varepsilon\delta + C_3(\delta)\varepsilon\sigma^{\frac{1}{2}}
\end{aligned} \tag{5.29}$$

Therefore, if $d_\Gamma(x)\in\Big(H_\Gamma(s_{\delta,\varepsilon}),H_\Gamma(1)\Big)$ (and thus, by (5.27) and (5.28), $|d_\Gamma(x)|\le C_0(\delta)\log(1/\varepsilon)$ and $g'_\Gamma(d_\Gamma(x))>0$), we have by Lemma 4.5 that

$$\begin{aligned}
\Delta_p(g_\Gamma(t)) &= (p-1)g'_\Gamma(t)^{p-2}g''_\Gamma(t) + g'_\Gamma(t)^{p-1}\Delta d_\Gamma(t) \le \\
&\le (p-1)g'_\Gamma(t)^{p-2}g''_\Gamma(t) - \varepsilon(\delta - C_4(\delta)\sigma^{\frac{1}{2}})g'_\Gamma(t)^{p-1}
\end{aligned} \tag{5.30}$$

where we are using the notation $t = d_\Gamma(x)$. Taking into account Lemma 4.6, by (5.30) we get

$$\Delta_p(g_\Gamma(t)) \le h'_\Gamma(s) - \varepsilon\left(\delta - C_4(\delta)\sigma^{\frac{1}{2}}\right)\left(\frac{p}{p-1}h_\Gamma(s)\right)^{\frac{p-1}{p}}.$$

where we are using the notation $s = g_\Gamma(d_\Gamma(x))$.
Now we chose $\sigma_0(\delta)$ small such that $\delta - C_4(\delta)\sigma^{\frac{1}{2}} \ge \frac{\delta}{2}$ for $\sigma \le \sigma_0(\delta)$. Thus, if $|d_\Gamma(x)| \le C_0(\delta)\log(1/\varepsilon)$ (and so $s = g_\Gamma(d_\Gamma(x)) > s_{\delta,\varepsilon}$), we gather (recall also (5.24)) that

$$\begin{aligned}\Delta_p(g_\Gamma(t)) &\le h'_\Gamma(s) - \mathrm{const}\,\delta\varepsilon\,(h_\Gamma(s))^{\frac{p-1}{p}}\\ &\le h'_0(s) + c_1\delta\varepsilon\rho'(s) - \mathrm{const}\,\delta\varepsilon\,(h_0(s)+c_1\delta\varepsilon\rho(s))^{\frac{p-1}{p}}\end{aligned} \tag{5.31}$$

We now claim that

$$c_1\rho'(s) - \mathrm{const}\,(h_0(s)+c_1\delta\varepsilon\rho(s))^{\frac{p-1}{p}} < 0\,, \tag{5.32}$$

for any $s \in (s_{\delta,\varepsilon}, 1)$, provided c_1 is small enough. Indeed, if $s \le -1/2$ or $s \ge 1/2$, $\rho'(s) = 0$ and therefore the left hand side of (5.32) is under control. On the other hand, if $s \in (-1/2, 1/2)$, by denoting $c^* := \inf_{s\in[-1/2,1/2]} h_0(s)$ (which is strictly positive on account of (1.2)), we bound the left hand side of (5.32) by

$$c_1\|\rho'\|_\infty - \mathrm{const}\,(c^*)^{\frac{p-1}{p}}\,,$$

which is negative for suitably small c_1. This proves (5.32).
Therefore, by virtue of (5.31) and (5.32), if $d_\Gamma(x) \in \Big(H_\Gamma(s_{\delta,\varepsilon}), H_\Gamma(1)\Big)$, we get

$$\Delta_p(g_\Gamma(t)) < h'_0(g_\Gamma(t))\,.$$

If else $d_\Gamma(x) < H_\Gamma(s_{\delta,\varepsilon})$, we have

$$\Delta_p(g_\Gamma(t)) = 0 < h'_0(s_{\delta,\varepsilon}) = h'_0(g_\Gamma(t))\,.$$

Finally, in the case $d_\Gamma(x) = H_\Gamma(s_{\delta,\varepsilon})$, since g_Γ is $C^{1,1}$, we get the thesis by exploiting Lemma 4.7 as we did in Lemma 5.1 above. ☐

6. Sliding methods

We now use the barriers introduced in §5 and an appropriate sliding techniques to deduce an estimate on the curvature of touching paraboloids for solutions of (1.5):

Lemma 6.1. *Let $l, \theta, \delta > 0$ and $M_1 \in \mathrm{Mat}((N-1)\times(N-1))$. Let u be a weak Sobolev solution of (1.5) in the whole $\mathbb{R}^N$, satisfying (2.1). Assume that $|u| < 1$ in $[-l,l]^N$, $u(0) = 0$ and $u(x) < 0$ for any $x = (x', x_n) \in [-l,l]^N$ so that*

$$x_n < \frac{\theta}{2l^2}x'\cdot M_1 x' + \frac{\theta}{l}\xi\cdot x'\,.$$

Then, there exist a universal constant $\delta_0 > 0$ and a function $\sigma : (0,1) \longrightarrow (0,1)$, so that, if

$$\delta \in (0,\delta_0]\,,\quad \delta \le \theta \le 2\delta\,,\quad \frac{\theta}{l} \in \Big(0, \sigma(\delta)\Big]\,,\quad \|M_1\| \le \frac{1}{\delta}\quad \textit{and}\quad |\xi| \le \frac{1}{\delta}\,,$$

then

$$\mathrm{tr} M_1 \le \delta\,.$$

Proof. Let g_l and $\Psi^{y,l}$ be the functions defined in Lemma 5.1. Define also

$$\Gamma_1 := \left\{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \text{ s.t. } x_N = \frac{\theta}{2l^2} x' \cdot M_1 x' + \frac{\theta}{l} \xi \cdot x' \right\} .$$

Notice that, by construction, u is negative below Γ_1 in $[-l,l]^N$. Furthermore, by our assumptions,

$$\frac{\theta \|M_1\|}{l^2} \le \frac{\theta}{l^2 \delta} \le \frac{\sigma(\delta)}{l\delta} .$$

Fixed $\kappa_1 \in (0, 1/2)$, to be better specified in the sequel, the above bound on the curvature of Γ_1 implies that, if $\sigma(\delta)/\delta$ is sufficiently small, then, given any x such that there is $X \in \Gamma_1$ with $|X - x| = d_{\Gamma_1}(x) \le \kappa_1 l$, there is a ball of radius $\kappa_1 l$ which touches Γ_1 by below at X and whose center X' lies in the direction $X - x$. Therefore, since we chose the signed distance d_{Γ_1} to be positive above Γ_1,

$$|x - X'| - d_{\Gamma_1}(x) = \kappa_1 l . \tag{6.1}$$

We claim that, for suitable universal positive constants κ_1 and κ_2, we have

$$u(x) \le g_{\kappa_1 l}(d_{\Gamma_1}(x)) \tag{6.2}$$

for any $x \in [-\kappa_2 l, \kappa_2 l]^N$.

To prove (6.2), first recall that $\Psi^{y,\kappa_1 l}$ is defined in $B_{\kappa_1 l + T_{\kappa_1 l}}(y)$, with $T_{\kappa_1 l} \in [\bar{c} l, \kappa_1 l/2]$, and that $\Psi^{y,\kappa_1 l} \ge -1 + e^{-\bar{c} l}$ in its domain of definition (see the statement of Lemma 5.1). Therefore, since

$$\begin{aligned} B_{(\kappa_1 + \bar{c}) l}(0, \dots, 0, -l/2) &\subseteq B_{\kappa_1 l + T_{\kappa_1 l}}(0, \dots, 0, -l/2) \\ &\subseteq B_{2\kappa_1 l}(0, \dots, 0, -l/2) \subset [-l, l]^N \end{aligned}$$

if κ_1 is conveniently small, we deduce from (2.1) that

$$u(x) < \Psi^{(0,\dots,0,-l/2),\kappa_1 l}(x) , \quad \forall x \in B_{\kappa_1 l + T_{\kappa_1 l}}(0, \dots, 0, -l/2) .$$

Given $X \in \Gamma_1$ let ν_X be the normal direction of Γ_1 at X pointing downwards and define

$$X' := X + \kappa_1 l \nu_X ;$$

we now slide the surface $\Psi^{(0,\dots,0,-l/2),\kappa_1 l}$ in the direction of the vector

$$v := X' - (0, \dots, 0, -l/2) ,$$

that is, we will consider the surface $\Psi^t := \Psi^{(0,\dots,0,-l/2)+tv,\kappa_1 l}$ for $t > 0$.

We will show that

$$\Psi^t(\mathfrak{x}) > u(\mathfrak{x}) \text{ for any } t \in [0,1) \text{ and any } \mathfrak{x} \in B_{\kappa_1 l + T_{\kappa_1 l}}\Big((0, \dots, 0, -l/2) + tv\Big). \tag{6.3}$$

Indeed, let $t \in [0,1)$ be the first time on which Ψ^t touches u. First of all, note that, since $t < 1$, we have $u < 0$ on $\partial B_{\kappa_1 l}\Big((0, \dots, 0, -l/2) + tv\Big)$, while $\Psi^t = 0$ there. Therefore, u cannot be equal to Ψ^t and touching points does not occur on $\partial B_{\kappa_1 l}\Big((0, \dots, 0, -l/2) + tv\Big)$.

Touching points cannot lie on $\partial B_{\kappa_1 l + T_{\kappa_1 l}}\Big((0, \dots, 0, -l/2) + tv\Big)$ either, as we are now going to show.

Assume, indeed, by contradiction that $u \le \Psi^t$ in $B_{\kappa_1 l + T_{\kappa_1} l}\Big((0,\dots,0,-l/2)+tv\Big)$ and $u(z) = \Psi^t(z)$ for some $z \in \partial B_{\kappa_1 l + T_{\kappa_1} l}\Big((0,\dots,0,-l/2)+tv\Big)$. Let us consider the "radial direction"

$$w := \frac{z - \Big((0,\dots,0,-l/2)+tv\Big)}{\Big|z - \Big((0,\dots,0,-l/2)+tv\Big)\Big|}.$$

Then, by the construction in Lemma 5.1, $\Psi^t(z) = 1$ and $\nabla \Psi^t(z) \cdot w > 0$. On the other hand, $u \le 1$ and, since $u(z) = 1$, we have that $\nabla u(z) = 0$ (thanks to the regularity theory in [8] or [19]). Let $\hat{u} := u - \Psi^t$. From the above discussion, $\hat{u} \le 0$ in $B_{\kappa_1 l + T_{\kappa_1} l}\Big((0,\dots,0,-l/2)+tv\Big)$ and $\hat{u}(z) = 0$, therefore

$$\nabla \hat{u}(z) \cdot w \ge 0.$$

But then

$$0 \le \nabla\Big(u - \Psi^t\Big)(z) \cdot w = -\nabla \Psi^t(z) \cdot w < 0,$$

which is a contradiction.

This shows that no touching point occurs on $\partial B_{\kappa_1 l + T_{\kappa_1} l}\Big((0,\dots,0,-l/2)+tv\Big)$.

Moreover, by construction, Ψ^t has no points with vanishing gradient outside from the region where it is not flat: more precisely, we recall that Ψ^t is flat in $B_{\tau(l)}\Big((0,\dots,0,-l/2)+tv\Big)$ for some $\tau(l) \in [-l/2, \bar{c}l]$, that $\Psi^t = -1 + e^{-\tilde{c}l}$ in $B_{\tau(l)}\Big((0,\dots,0,-l/2)+tv\Big)$ and that, if

$$Z_{u,\Psi_t} := \{|Du| + |D\Psi^t| = 0\},$$

we have that Z_{u,Ψ^t} is inside $B_{\tau(l)}\Big((0,\dots,0,-l/2)+tv\Big)$. Therefore, by means of Corollary 3.3, touching points for $t \in [0,1)$ may only occur in the closure of $B_{\tau(l)}\Big((0,\dots,0,-l/2)+tv\Big)$.

With these considerations, we have that the proof of (6.3) will follow by showing that touching points do not occur in $B_{\tau(l)}\Big((0,\dots,0,-l/2)+tv\Big)$ either.

To this aim, let us assume, by contradiction, that $u(y) = \Psi^t(y)$, with y in the closure of $B_{\tau(l)}\Big((0,\dots,0,-l/2)+tv\Big)$. We first prove that, given any $r > 0$, u cannot agree with Ψ^t in $B_r(y) \cap B_{\tau(l)}\Big((0,\dots,0,-l/2)+tv\Big)$. For this, assume the converse; then, u would be flat in $B_r(y) \cap B_{\tau(l)}\Big((0,\dots,0,-l/2)+tv\Big)$ and hence, taking a positive test function φ supported in $B_r(y) \cap B_{\tau(l)}\Big((0,\dots,0,-l/2)+tv\Big)$, we deduce from (1.5) that

$$\begin{aligned} 0 &= \int |\nabla u|^{p-1} \, |\nabla \varphi| \\ &\geq \int h_0'(u) \, \varphi \\ &= \int h_0'(-1 + e^{-\tilde{c}l}) \, \varphi \\ &> 0 \, . \end{aligned}$$

This contradiction proves that u cannot agree with Ψ^t in the domain $B_r(y) \cap B_{\tau(l)}\Big((0, \dots, 0, -l/2) + tv\Big)$.

Therefore, in order to rule out any touching in $B_{\tau(l)}\Big((0, \dots, 0, -l/2) + tv\Big)$, we may assume that $u < \Psi^t$ in the interior of $\hat{B} \subseteq B\Big((0, \dots, 0, -l/2) + tv\Big)$ and $u(y') = \Psi^t(y')$ for some $y' \in \partial \hat{B}$. Since $\nabla \Psi^t(y') = 0$ and $u \in C^1$ (see [8] or [19]), we have that

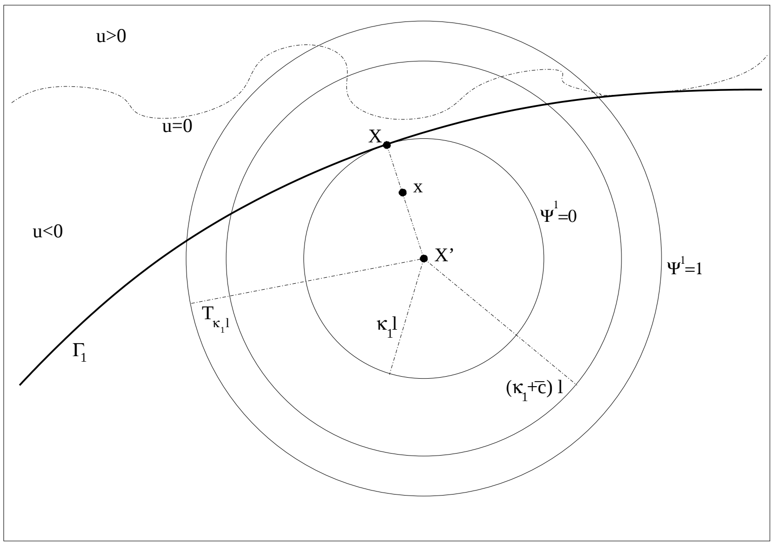
$$\nabla u(y') = 0 \, . \tag{6.4}$$

Setting $u^\star := -u + 1 - e^{-\tilde{c}l}$, by (1.3), if l is sufficiently large, we have that

$$-\Delta_p u^\star = \Delta_p u = h_0'(u) \geq c \, (1+u)^{p-1} \geq c \, e^{-\tilde{c}l} > 0$$

in $\hat{B}$; also, $u^\star > 0$ in the interior of $\hat{B}$ and $u^\star(y') = 0$: consequently, by Theorem 3.4 (with $c = 0$ and $g = 0$) it would follow that $\nabla u^\star(y') \neq 0$, and so $\nabla u(y') \neq 0$, contradicting (6.4).

Therefore, a touching point may only occur on the zero level set of Ψ^t (due to the lack of supersolution property). That is, a touching point may only occur on the zero level set of u. Since, by assumption, the zero level set of u is above Γ_1, we can slide Ψ^t at least until its zero level sets touches Γ_1 and still have $\Psi^t > u$. This proves (6.3). With this, we are now in position to complete the proof of (6.2), by arguing as follows.



Touching between Ψ^1 and Γ_1

Indeed, we deduce from (6.3) that $\Psi^1(\mathfrak{x}) \geq u(\mathfrak{x})$ for any $\mathfrak{x} \in B_{(\kappa_1+\bar{c})l}(X')$. Therefore, using (6.1),

$$(6.5)\quad g_{\kappa_1 l}\Big(d_{\Gamma_1}(x)\Big) = g_{\kappa_1 l}(|x - X'| - \kappa_1 l) = \Psi^{X',\kappa_1 l}(x) = \Psi^1(x) \geq u(x)\,.$$

By our assumptions and an easy covering argument over Γ_1, the reader may easily convince herself that

$$[-\kappa_2 l, \kappa_2 l]^N \subseteq \bigcup_{X\in\Gamma_1} B_{(\kappa_1+\bar{c})l}(X + \kappa_1 l \nu_X)\,,$$

provided κ_2 is suitably small with respect to κ_1 and $\bar{c}$. Therefore, (6.2) follows from (6.5).

We now complete the proof of the desired result arguing by contradiction and supposing that $\mathrm{tr}M_1 > \delta$: with this assumption, by Lemma 5.2, we get that $g_{\Gamma_2}(d_{\Gamma_2})$ is a strict supersolution of (1.5), where

$$\begin{aligned} \Gamma_2 \quad &:= \quad \Big\{ x = (x', x_n) \in \mathbb{R}^{N-1} \times \mathbb{R} \;\text{ s.t.} \\ &\qquad\quad x_n = \frac{\theta}{2l^2} x' \cdot M_1 x' + \frac{\theta}{l} \xi \cdot x' - \frac{\varepsilon\delta}{4}|x'|^2 \Big\} \\ \text{and } \varepsilon \quad &:= \quad \frac{2\theta}{l^2}\,. \end{aligned}$$

Moreover, if θ/l and δ are sufficiently small, we claim that

$$(6.6)\qquad \text{if } s_{\delta,\varepsilon} < s \leq -1 + (\delta\theta l^{-2})^{1/p}, \qquad h_{\Gamma_2}(s) \;\leq\; h_{\kappa_1 l}(s) \qquad \text{and}$$

$$(6.7)\qquad \text{if } 1 - (\delta\theta l^{-2})^{1/p} \leq s \leq 1, \qquad h_{\Gamma_2}(s) \;\geq\; h_{\kappa_1 l}(s)\,.$$

Indeed, if $h_{\Gamma_2}(s) = 0$, (6.6) follows from item (i) in Lemma 5.1. If, on the contrary, $h_{\Gamma_2}(s) > 0$ and $s \in [s_{\delta,\varepsilon}, -1 + (\delta\varepsilon)^{1/p}]$, then, by the definitions of h_{Γ_2} and $h_{\kappa_1 l}$, (1.2) and (5.23),

$$\begin{aligned}
h_{\Gamma_2}(s) - h_{\kappa_1 l}(s) &\le -\mathrm{const}\,\delta\varepsilon + \frac{\mathrm{const}}{l}(1+s)^p - \frac{\mathrm{const}}{l}s^p_{\kappa_1 l} + h_0(-1+s_{\kappa_1 l})\\
&\le -\mathrm{const}\,\frac{\delta\theta}{l^2} + \frac{\mathrm{const}}{l}(1+s)^p + \mathrm{const}\,s^p_{\kappa_1 l}\\
&\le -\mathrm{const}\,\frac{\delta\theta}{l^2} + \mathrm{const}\,\frac{\delta\varepsilon}{l} + \mathrm{const}\,s^p_{\kappa_1 l}\\
&= -\mathrm{const}\,\frac{\delta\theta}{l^2} + \mathrm{const}\,\frac{\delta\theta}{l^3} + \mathrm{const}\,e^{-\mathrm{const}\,l}\,,
\end{aligned}$$

which is negative for conveniently large l, completing the proof of (6.6). To prove (6.7), use (1.2), (5.23) and the definitions of h_{Γ_2} and $h_{\kappa_1 l}$ to deduce that, if $1-(\delta\theta l^{-2})^{1/p} \le s \le 1$,

$$\begin{aligned}
h_{\Gamma_2}(s) - h_{\kappa_1 l}(s) &\ge h_0(s) + \mathrm{const}\,\delta\varepsilon - h_{\kappa_1 l}(s)\\
&\ge \mathrm{const}\,\delta\varepsilon - h_0(1-s_{\kappa_1 l}) - \frac{\mathrm{const}}{l}\Big((1-s)^p + s^p_{\kappa_1 l}\Big)\\
&\ge \mathrm{const}\,\delta\varepsilon - \mathrm{const}\,s^p_{\kappa_1 l} - \frac{\mathrm{const}}{l}\Big((1-s)^p + s^p_{\kappa_1 l}\Big)\\
&\ge \mathrm{const}\,\delta\varepsilon - \mathrm{const}\,\frac{\delta\theta}{l^3} - \mathrm{const}\,e^{-\mathrm{const}\,l}\\
&= \mathrm{const}\,\frac{\delta\theta}{l^2} - \mathrm{const}\,\frac{\delta\theta}{l^3} - \mathrm{const}\,e^{-\mathrm{const}\,l}\,.
\end{aligned}$$

This, taking l big enough, provide the proof (6.7).

According to (6.6) and (6.7), the function $s \mapsto H_{\Gamma_2}(s) - H_{\kappa_1 l}(s)$ is increasing for $s \le -1 + (\delta\theta l^{-2})^{1/p}$ and decreasing for $s \ge 1-(\delta\theta l^{-2})^{1/p}$, therefore its maximum occurs in $[-1+(\delta\theta l^{-2})^{1/p}, 1-(\delta\theta l^{-2})^{1/p}]$, i.e.,

$$\max_{s\in[s_{\delta,\varepsilon},1]} H_{\Gamma_2}(s) - H_{\kappa_1 l}(s) = \max_{s\in[-1+(\delta\theta l^{-2})^{1/p},1-(\delta\theta l^{-2})^{1/p}]} H_{\Gamma_2}(s) - H_{\kappa_1 l}(s) \tag{6.8}$$

Also, recalling the definition of H_0 in Lemma 5.1, if $s \in [0, 1-(\delta\theta l^{-2})^{1/p}]$,

$$H_{\Gamma_2}(s) = \int_0^s \frac{(p-1)^{1/p}\,d\zeta}{(p\,h_{\Gamma_2}(\zeta))^{1/p}} \le \int_0^s \frac{(p-1)^{1/p}\,d\zeta}{(p\,h_0(\zeta))^{1/p}} = H_0(s) \tag{6.9}$$

and analogously, if $s \in [-1+(\delta\theta l^{-2})^{1/p}, 0]$,

$$-H_{\Gamma_2}(s) = \int_s^0 \frac{(p-1)^{1/p}\,d\zeta}{(p\,h_{\Gamma_2}(\zeta))^{1/p}} \ge \int_0^s \frac{(p-1)^{1/p}\,d\zeta}{(p\,h_0(\zeta))^{1/p}} = -H_0(s)\,. \tag{6.10}$$

Hence, from (6.9) and (6.10),

$$H_{\Gamma_2}(s) \le H_0(s)$$

for any $s \in [-1+(\delta\theta l^{-2})^{1/p}, 1-(\delta\theta l^{-2})^{1/p}]$. Consequently, from (5.4), if $s \in [-1+(\delta\theta l^{-2})^{1/p}, 1-(\delta\theta l^{-2})^{1/p}]$, then

$$H_{\Gamma_2}(s) \le H_{\kappa_1 l}(s) + \frac{\mathrm{const}}{l}\log\frac{l^2}{\delta\theta}\,.$$

Therefore, by (6.8),

$$H_{\Gamma_2}(s) \le H_{\kappa_1 l}(s) + \frac{\mathrm{const}}{l}\log\frac{l^2}{\delta\theta} \tag{6.11}$$

for any $s \in [s_{\delta,\varepsilon}, 1]$.
Furthermore, by definition of Γ_1 and Γ_2, if $|x'|_\infty = \kappa_2 l$, then

$$d_{\Gamma_2}(x) \geq d_{\Gamma_1}(x) + c(\delta)$$

for a suitable $c(\delta) \in (0,1)$. Hence, using (6.11) and taking l appropriately large, with $s = g_{\kappa_1 l}(d_{\Gamma_1}(x))$,

$$\begin{aligned} H_{\Gamma_2}\Big(g_{\kappa_1 l}(d_{\Gamma_1}(x))\Big) &< H_{\kappa_1 l}\Big(g_{\kappa_1 l}(d_{\Gamma_1}(x))\Big) + \frac{\text{const}}{l}\log\frac{l^2}{\delta\theta} \\ &= d_{\Gamma_1}(x) + \frac{\text{const}}{l}\log\frac{l^2}{\delta\theta} \leq d_{\Gamma_2}(x)\,, \end{aligned}$$

provided $g_{\kappa_1 l}(d_{\Gamma_1}(x)) \geq s_{\delta,\varepsilon}$ and $|x'|_\infty = \kappa_2 l$; therefore, since H_{Γ_2} is strictly increasing in $[s_{\delta,\varepsilon}, 1]$,

$$g_{\kappa_1 l}(d_{\Gamma_1}(x)) < g_{\Gamma_2}(d_{\Gamma_2}(x))\,, \tag{6.12}$$

for any x so that $g_{\kappa_1 l}(d_{\Gamma_1}(x)) \geq s_{\delta,\varepsilon}$ and $|x'|_\infty = \kappa_2 l$. Of course, if $g_{\kappa_1 l}(d_{\Gamma_1}(x)) < s_{\delta,\varepsilon}$, (6.12) holds since $g_{\Gamma_2} \geq s_{\delta,\varepsilon}$ by construction (recall item (ii) of Lemma 5.2). Thus,

$$g_{\kappa_1 l}(d_{\Gamma_1}(x)) < g_{\Gamma_2}(d_{\Gamma_2}(x))\,, \qquad \forall |x'|_\infty = \kappa_2 l\,, \tag{6.13}$$

provided that $d_{\Gamma_1}(x)$ is in the domain of $g_{\kappa_1 l}$ and $d_{\Gamma_2}(x)$ in the domain of g_{Γ_2}. Notice, however, that the first of these conditions is fulfilled for free in $[-\kappa_2 l, \kappa_2 l]^N$, provided κ_2 is chosen to be suitably small, since, by Lemma 5.1, $T_l \geq \bar{c}l$ and so, if $x \in [-\kappa_2 l, \kappa_2 l]^N$,

$$d_{\Gamma_1}(x) \leq 2\kappa_2 l \leq T_l\,.$$

Then, (6.13) and (6.2) imply that

$$u(x) < g_{\Gamma_2}(d_{\Gamma_2}(x))\,, \tag{6.14}$$

for any for any $x \in [-\kappa_2 l, \kappa_2 l]^N$ so that $|x'|_\infty = \kappa_2 l$ and $d_{\Gamma_2}(x)$ is in the domain of g_{Γ_2}.

With these estimates, we are now ready to deduce the contradiction that will finish the proof of the desired result. To this end, we slide $g_{\Gamma_2}(d_{\Gamma_2})$ in the e_n-direction till we touch u, i.e., we consider, for $t \in \mathbb{R}$,

$$g^t(x) := g_{\Gamma_2}\Big(d_{\Gamma_2}(x - te_n)\Big)\,.$$

If we denote by $\mathcal{D}_0$ the domain of $g_{\Gamma_2}(d_{\Gamma_2})$, we have by Lemma 5.2 that

$$\mathcal{D}_0 \subseteq \left\{ x_n \leq C_0(\delta)\log\frac{l^2}{\theta} \right\}\,,$$

and so, in particular,

$$\mathcal{D}_0 \subseteq \left\{ x_n \leq \frac{\kappa_2 l}{8} \right\}\,,$$

if l is sufficiently large. Notice that, with this notation, g^t is defined in

$$\mathcal{D}_t := \mathcal{D}_0 + te_n \subseteq \left\{ x_n \leq \frac{\kappa_2 l}{8} + t \right\}\,. \tag{6.15}$$

Thus, if $t \leq -\kappa_2 l/4$, then $\mathcal{D}_t \subseteq \{x_n \leq -(\kappa_2 l)/8\}$ and therefore, $g^t > u$ in $\mathcal{D}_t \cap [-\kappa_2 l, \kappa_2 l]$ because of (2.1), provided l is large enough. On the other hand,

$$g^0(0) = g_{\Gamma_2}(d_{\Gamma_2}(0)) = g_{\Gamma_2}(0) = 0 = u(0)\,,$$

therefore, there is a time $t \in [-\kappa_2 l/4, 0]$ of first touching between g^t and u in $\mathcal{D}_t \cap [-\kappa_2 l, \kappa_2 l]$. Since $g_{\Gamma_2}(d_{\Gamma_2})$ is a strict supersolution of (1.5) exploiting Corollary 3.3 and Lemma 4.7 as above we prove that contact points in the interior of $\mathcal{D}_t \cap [-\kappa_2 l, \kappa_2 l]$ may only happen at critical points of u in the region where g^t is flat. Arguing exactly as above we can exploit Theorem 3.4 and rule out this possibility (see the arguments on page 26).

Hence, the first touching point must occur at the boundary of $\mathcal{D}_t \cap [-\kappa_2 l, \kappa_2 l]$. We first observe that no contact point may lie in $\partial\mathcal{D}_t$. Indeed, if $x \in \partial\mathcal{D}_t$, then, from Lemma 5.2, we have that $\nabla g^t(x) \neq 0$, $g^t(x) = 1$, while $u \leq 1$. Hence, since u is C^1 (see [8] or [19]), if a contact point $\bar{x}$ occur on $\partial\mathcal{D}_t$ we would have

$$0 = \nabla u(\bar{x}) = \nabla g^t(\bar{x}) \neq 0\,.$$

This contradiction shows that contact points may only happen either on $|x'|_\infty = \kappa_2 l$ or on $x_N = -\kappa_2 l$ (the case $x_N = \kappa_2 l$ cannot hold for $t \leq 0$ because of (6.15)).

We now exclude the possibility of touching at $x_N = -\kappa_2 l$. Indeed, if $x^\sharp$ is so that $x^\sharp_n = -\kappa_2 l$ and $u(x^\sharp) = g^t(x^\sharp)$ for some $t \in [-\kappa_2 l/4, 0]$, then

$$x^\sharp_n - t \leq -\kappa_2 l + \frac{\kappa_2 l}{4} = -\frac{3}{4}\kappa_2 l\,; \tag{6.16}$$

on the other hand, by construction, $\Gamma_2 \subseteq \{|x_n| \leq \mathrm{const}\}$, therefore, (6.16) leads to

$$d_{\Gamma_2}(x^\sharp - te_n) \leq -\frac{1}{2}\kappa_2 l \leq -C_0(\delta)\log\frac{1}{\varepsilon}\,,$$

if l is large enough, for our choices of the parameters and therefore, recalling Lemma 5.2, $x^\sharp$ is a critical point for g^t. Since $u \in C^1$ (see [8] or [19]), $x^\sharp$ is also critical for u: but this possibility may be excluded using Theorem 3.4 and arguing by contradiction exactly as above.

Therefore, a contact point $x^\star \in \mathcal{D}_t \cap [-\kappa_2 l, \kappa_2 l]$ does occur when $|x'|_\infty = \kappa_2 l$. Notice now that since $t \leq 0$ and d_{Γ_2} is defined to be positive above Γ_2,

$$d_{\Gamma_2}(x^\star - te_n) \geq d_{\Gamma_2}(x^\star)\,.$$

But then, since g_{Γ_2} is non-decreasing, we deduce from (6.14) that

$$g_{\Gamma_2}(d_{\Gamma_2}(x^\star - te_n)) = g^t(x^\star) = u(x^\star) < g_{\Gamma_2}(d_{\Gamma_2}(x^\star)) \leq g_{\Gamma_2}(d_{\Gamma_2}(x^\star - te_n))\,.$$

This contradiction concludes the proof of Lemma 6.1. □

7. A viscosity solution property for the limiting equation and proof of Theorem 2.2

We now show that rescaled solutions of (1.5) satisfy a mean curvature equation in a weak viscosity sense.

Proposition 7.1. *Let u be a Sobolev weak solution of (1.5) in the whole $\mathbb{R}^N$, satisfying (2.1), so that $|u| < 1$ and $u(0) = 0$. Let $\mathfrak{d} \in (0,1)$, $V \in \mathbb{R}^{N-1}$ and $M \in \mathrm{Mat}((N-1)\times(N-1))$ with*

$$|V| \leq \operatorname{tr} M\,, \qquad \operatorname{tr} M > \mathfrak{d}\|M\| \qquad \text{and} \qquad \|M\| \leq \mathfrak{d}^{-1}\,. \tag{7.1}$$

Let $u_\varepsilon(x) := u(x/\varepsilon)$ and

$$\Gamma := \left\{ x = (x', x_n) \in \mathbb{R}^{N-1}\times\mathbb{R} \;\; s.t. \;\; x_N = \frac{1}{2}x'\cdot Mx' + V\cdot x' \right\}\,.$$

Then, there exist a universal $\mathfrak{d}^\star > 0$ and a function $\sigma_0 : (0,1) \longrightarrow (0,1)$ such that if $\varepsilon \in (0, \sigma_0(\mathfrak{d}))$ and $\mathfrak{d} \in (0, \mathfrak{d}^\star)$, then Γ cannot touch $\{u_\varepsilon = 0\}$ by below in $B_{\mathfrak{d}/\sqrt{\operatorname{tr} M}}$: more explicitly,

$$\{u_\varepsilon = 0\} \cap \left\{ x_N < \frac{1}{2} x' \cdot M x' + V \cdot x' \right\} \cap \left\{ |x| < \frac{\mathfrak{d}}{\sqrt{\operatorname{tr} M}} \right\} \neq \emptyset . \tag{7.2}$$

Proof. We will apply Lemma 6.1 by making use of the following choice of parameters:

$$l := \frac{\mathfrak{d}}{\varepsilon \sqrt{\operatorname{tr} M}} , \quad \delta := \theta := \mathfrak{d}^2 , \quad M_1 := \frac{1}{\operatorname{tr} M} M , \quad \xi := \frac{\varepsilon l}{\theta} V .$$

If, by contradiction, the claim of Proposition 7.1 were false, by scaling back and using the above parameters, we would have that Γ_1 touches by below u inside $[-l, l]^N$, where

$$\Gamma_1 = \left\{ x = (x', x_n) \in \mathbb{R}^{N-1} \times \mathbb{R} \text{ s.t. } x_n = \frac{\theta}{2l^2} x' \cdot M_1 x' + \frac{\theta}{l} \xi \cdot x' \right\} .$$

By Lemma 6.1, we gather that $\delta \geq \operatorname{tr} M_1 = 1$, which is the desired contradiction $\square$

Notice that Theorem 2.2 follows at once from Proposition 7.1 by taking $V := 0$.

8. Proof of Theorem 2.1

We are now in position to prove Theorem 2.1. Let $x^\star$ be a point where $\mathcal{S}$ admits a tangent plane. With no loss of generality, we may assume that the normal vector of $\mathcal{S}$ at $x^\star$ is e_N. Let be P a paraboloid touching by below. We will show that the mean curvature of P is non-positive: an analogous proof would give that, if P is a paraboloid touching by above, then its mean curvature is non-negative, and this would end the proof of Theorem 2.1.

By construction, if P touches $\mathcal{S}$ by below at $x^\star$, we have that

$$P = \left\{ (x', x) \in \mathbb{R}^{N-1} \times \mathbb{R} \ : \ x_N = \frac{1}{2} (x' - (x^\star)') \cdot \mathcal{M} (x' - (x^\star)') + x_N^\star \right\} ,$$

for some $\mathcal{M} \in \operatorname{Mat}((N-1) \times (N-1))$. Let $\mathbb{I}_{N-1}$ be the $(N-1) \times (N-1)$ identity matrix and set

$$\widehat{\mathcal{M}} := \mathcal{M} - \frac{\operatorname{tr} \mathcal{M}}{2(N-1)} \mathbb{I}_{N-1} . \tag{8.1}$$

Notice that

$$\operatorname{tr} \widehat{\mathcal{M}} = \frac{1}{2} \operatorname{tr} \mathcal{M} . \tag{8.2}$$

Let us also define

$$\mathfrak{P} := \left\{ (x', x) \in \mathbb{R}^{N-1} \times \mathbb{R} \ : \ x_N = \frac{1}{2} (x' - (x^\star)') \cdot \widehat{\mathcal{M}} (x' - (x^\star)') + x_N^\star \right\} ,$$

Let us assume, by contradiction, that $\operatorname{tr} \mathcal{M} > 0$. Then, $\operatorname{tr} \widehat{\mathcal{M}} > 0$ in light of (8.2), and $\mathfrak{P}$ is also touching $\mathcal{S}$ from below at $x^\star$. Let $r \in (0,1)$, to be chosen suitably small in the sequel, and let us consider the cube $Q := \{|x - x^\star|_\infty \leq r\}$ Notice that, by the tangency between $\{x_N = x_N^\star\}$ and $\mathcal{S}$, if r is small enough, we have that

$$\mathcal{S} \cap Q \subseteq \left\{ |x_N - x_N^\star| \leq \frac{r}{4} \right\} , \tag{8.3}$$

Also, by (2.2), fixed any $\eta > 0$, we have that

$$\operatorname{dist}(x, \mathcal{S}) \leq \eta\,, \tag{8.4}$$

for any $x \in \{u_\varepsilon = 0\} \cap B_1(x^\star)$, provided ε is small enough. We also claim that

$$B_\eta(x^\star) \cap \{u_\varepsilon = 0\} \neq \emptyset\,, \tag{8.5}$$

if ε is small enough. To prove this, let us argue by contradiction and assume, say, that $u_\varepsilon < 0$ in $B_\eta(x^\star)$. Then, since u_ε converges in L^1_{loc} to $\chi := \chi_{\mathcal{E}} - \chi_{\mathbb{R}^N \setminus \mathcal{E}}$, we have

$$\begin{aligned} 0 &\geq \lim_{\varepsilon \longrightarrow 0^+} \int_{B_\eta(x^\star)} |u_\varepsilon - \chi| \\ &= \lim_{\varepsilon \longrightarrow 0^+} \int_{B_\eta(x^\star)\cap\mathcal{E}} |u_\varepsilon - 1| + \int_{B_\eta(x^\star)\cap(\mathbb{R}^N\setminus\mathcal{E})} |u_\varepsilon + 1| \\ &\geq \mathfrak{L}(B_\eta(x^\star) \cap \mathcal{E})\,, \end{aligned}$$

where we denoted by $\mathfrak{L}$ the N-dimensional Lebesgue meausure, from which

$$\mathfrak{L}(B_\eta(x^\star) \cap \mathcal{E}) = 0\,;$$

this contradicts (2.3) and proves (8.5).

We will now consider the touching of a suitable sliding of $\mathfrak{P}$ with $\{u_\varepsilon = 0\}$ in Q. In order to formalize the argument, let us define

$$\mathfrak{P}_t := \left\{ (x', x) \in \mathbb{R}^{N-1} \times \mathbb{R} \,:\, x_N = \frac{1}{2}\,(x' - (x^\star)') \cdot \widehat{\mathcal{M}}\,(x' - (x^\star)') + x^\star_N - t \right\}\,,$$

and let $t \in \mathbb{R}$ so that $\mathfrak{P}_t \cap \{u_\varepsilon = 0\} \cap Q \neq \emptyset$, while $\mathfrak{P}_s \cap \{u_\varepsilon = 0\} \cap Q = \emptyset$ for any $s > t$. Notice that, from (8.4) and (8.5),

$$|t| \leq 2\eta\,. \tag{8.6}$$

Let $x^\sharp \in \mathfrak{P}_t \cap \{u_\varepsilon = 0\} \cap Q$. We now show that $x^\sharp$ is in the interior of Q; more precisely, we will show that

$$x^\sharp \in \left\{ |x - x^\star|_\infty \leq \frac{r}{2} \right\}\,. \tag{8.7}$$

First of all, by (8.3) and (8.4),

$$\{u_\varepsilon = 0\} \cap Q \subseteq \left\{ |x_N - x^\star_N| \leq \frac{r}{2} \right\}\,, \tag{8.8}$$

provided ε is small enough. Hence, to prove (8.7), it remains to show that $|x^\sharp_i - x^\star_i| \leq r/2$ for $1 \leq i \leq N-1$.

For this, recalling (8.4), let $\tilde{x} \in \mathcal{S}$ be so that

$$|\tilde{x} - x^\sharp| \leq \eta\,. \tag{8.9}$$

Then, since $\mathcal{S}$ is above P,

$$\tilde{x}_N \geq \frac{1}{2}(\tilde{x}' - (x^\star)') \cdot \mathcal{M}(\tilde{x}' - (x^\star)') + x^\star_N\,. \tag{8.10}$$

Since, on the other hand, $x^\sharp \in \mathfrak{P}_t$,

$$x^\sharp_N = \frac{1}{2}((x^\sharp)' - (x^\star)') \cdot \widehat{\mathcal{M}}((x^\sharp)' - (x^\star)') + x^\star_N - t\,,$$

which, from (8.1) and (8.6), implies

$$(8.11)\quad x_N^\sharp \le -\frac{\operatorname{tr}\mathcal{M}}{4(N-1)}|(x^\sharp)'-(x^\star)'|^2+\frac{1}{2}((x^\sharp)'-(x^\star)')\cdot\mathcal{M}((x^\sharp)'-(x^\star)')+x_N^\star+2\eta\,.$$

Subtracting (8.10) and (8.11), and making use of (8.9), one gets that

$$\begin{aligned}\operatorname{const}\eta &\ge \tilde{x}_N - x_N^\sharp\\ &\ge \frac{\operatorname{tr}\mathcal{M}}{4(N-1)}|(x^\sharp)'-(x^\star)'|^2\\ &\quad +\frac{1}{2}(\tilde{x}'-(x^\star)')\cdot\mathcal{M}(\tilde{x}'-(x^\star)')\\ &\quad -\frac{1}{2}((x^\sharp)'-(x^\star)')\cdot\mathcal{M}((x^\sharp)'-(x^\star)')-2\eta\\ &\ge \frac{\operatorname{tr}\mathcal{M}}{4(N-1)}|(x^\sharp)'-(x^\star)'|^2-\operatorname{const}(1+\|\mathcal{M}\|)\,\eta\,.\end{aligned}$$

Since $\operatorname{tr}\mathcal{M}>0$, this shows in particular that $|x_i^\sharp-x_i^\star|\le r/2$ for $1\le i\le N-1$, provided η (and so ε) is suitably small, and thus completing the proof of (8.7).

With no loss of generality, we may now assume that $x^\sharp=0$. Notice that, with this assumption, by (8.7), $\mathfrak{P}_t$ touches $\{u_\varepsilon=0\}$ from below in $\{|x|_\infty\le r/2\}$. Then, either $u_\varepsilon<0$ or $u_\varepsilon>0$ below $\mathfrak{P}_t$: we will consider the first possibility, the second one being analogous. Also, since $0\in\mathfrak{P}_t$, the equation of $\mathfrak{P}_t$ takes the form

$$x_N=\frac{1}{2}\,x'\cdot\widehat{\mathcal{M}}x'+V\cdot x'\,,$$

with

$$|V|\le\operatorname{const}\|\widehat{\mathcal{M}}\|\,r\le\operatorname{tr}\widehat{\mathcal{M}}$$

provided

$$(8.12)\qquad r\le\operatorname{const}\frac{\operatorname{tr}\widehat{\mathcal{M}}}{\|\widehat{\mathcal{M}}\|}\,.$$

We now take $\mathfrak{d}>0$ appropriately small and define

$$r:=\frac{2\mathfrak{d}}{\sqrt{\operatorname{tr}\widehat{\mathcal{M}}}}\,.$$

Notice that if $\Gamma:=\mathfrak{P}_t$ and $M:=\widehat{\mathcal{M}}$, the hypotheses of Proposition 7.1 are fulfilled. However, the result of Proposition 7.1 is in contradiction with the fact that $\mathfrak{P}_t$ touches $\{u_\varepsilon=0\}$ from below in $\{|x|_\infty\le r/2\}$.

This contradiction, caused by the assumption that $\operatorname{tr}\mathcal{M}>0$, shows that the mean curvature of P must be non-positive.

This completes the proof of Theorem 2.1.

9. Comments on the uniform limit property (2.1) and proof of Theorem 2.3

We would like to show that uniform limit conditions of the type in (2.1) are somehow natural. First of all, a condition of this kind is necessary to avoid, for instance, periodic one-dimensional solutions. To motivate the decay property we assume, even if we do not need the following result in the present paper, we think it is convenient to point out that global solutions of (1.5) attaining a uniform limit always possess an exponential decay:

Proposition 9.1. *Let $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ be a weak solution of* (1.5) *such that*

$$\int_{\mathbb{R}^N} |\nabla u|^p\, dx + \int_{\mathbb{R}^N} h_0(u)\, dx < +\infty \tag{9.1}$$

Assume h_0' nondecreasing in $[-1, -1+\theta^*]$ and suppose $|u| < 1$ and

$$u < -1 + \theta^* \qquad \text{for any} \quad x \qquad \text{such that} \quad x_N \le K\,.$$

Then, if we set $\beta(x) := -1 + k_1\, e^{k_2 x_N}$, we find positive constants k_1 and k_2, such that

$$u(x) \le \beta(x) \qquad\qquad \text{for any} \quad x \quad \text{such that} \quad x_N \le K\,.$$

Proof. Let us start with the following easy calculations

$$\begin{aligned}\Delta_p\beta &= (p-1)k_2^{2(p-1)}(k_1\, e^{k_2 x_N})^{p-1}\\ &= (p-1)k_2^{2(p-1)}(1+\beta)^{p-1}\frac{h_0'(\beta)}{h_0'(\beta)}\\ &\le h_0'(\beta)\end{aligned} \tag{9.2}$$

for k_2 sufficiently small.
Moreover, we take k_1 such that $\beta(x) = -1 + \theta^*$ on $\{x_N = K\}$, hence

$$u < \beta \quad \text{on} \quad \{x_N = K\} \tag{9.3}$$

By (1.5) and by (9.2) we get

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u - |\nabla\beta|^{p-2}\nabla\beta)\cdot\nabla\varphi\, dx \le \int_{\mathbb{R}^N}(h_0'(\beta) - h_0'(u))\varphi\, dx \tag{9.4}$$

for any nonnegative $\varphi \in C_0^\infty(\mathbb{R}^N)$.
We consider now w defined as follows

$$w := \begin{cases}(u-\beta)^+ & \text{if} \quad x_N \le K\\ 0 & \text{if} \quad x_N > K\end{cases} \tag{9.5}$$

Since $u < \beta$ on $\{x_N = K\}$ we have that $\operatorname{supp} w \subset \{x_N < K\}$. Moreover, in light of (1.2) and (9.1), we get that $1 + u \in L^p(\mathbb{R}^N)$ and $|\nabla u| \in L^p(\mathbb{R}^N)$, and thus $w \in W^{1,p}(\mathbb{R}^N)$. Therefore, there exists a sequence of nonnegative functions $\varphi_n \in C_0^\infty(\mathbb{R}^N)$ converging to w in $W^{1,p}(\mathbb{R}^N)$. We may therefore take φ_n as test function in (9.4). Moreover, since $(h_0'(\beta) - h_0'(u)) \in L^{\frac{p}{p-1}}(\mathbb{R}^N)$ and $(|\nabla u|^{p-1} - |\nabla\beta|^{p-1}) \in L^{\frac{p}{p-1}}(\mathbb{R}^N)$, by Hölder's inequality and letting $n \longrightarrow +\infty$, we deduce that

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u - |\nabla\beta|^{p-2}\nabla\beta)\nabla w \le \int_{\mathbb{R}^N}(h_0'(\beta) - h_0'(u))w\, dx \tag{9.6}$$

Therefore, by construction,

$$\begin{aligned}&\int_{\{u\ge\beta\}\cap\{x_N<K\}} (|\nabla u|^{p-2}\nabla u - |\nabla\beta|^{p-2}\nabla\beta)\cdot(\nabla u - \nabla\beta)\, dx\\ \le\ &\int_{\{u\ge\beta\}\cap\{x_N<K\}} (h_0'(\beta) - h_0'(u))(u-\beta)\, dx\,.\end{aligned}$$

Since for $x_N < K$ we have that u and β take values in the domain where h_0' is non-decreasing, we get that $(h_0'(\beta) - h_0'(u))(u-\beta) \le 0$ so that,

$$\int_{\{u\ge\beta\}\cap\{x_N<K\}} (|\nabla u|^{p-2}\nabla u - |\nabla\beta|^{p-2}\nabla\beta)(\nabla u - \nabla\beta)\, dx \le 0$$

By standard estimates for the p-Laplace operator (see, for instance Lemma 2.1 in [6]), we get

$$\int_{\{u\geq\beta\}\cap\{x_N<K\}}(|\nabla u|+|\nabla\beta|)^{p-2}|\nabla(u-\beta)^2|\,dx=0\,.$$

Since, by the construction of β and the regularity results in [8] and [19],

$$0<|\nabla u|+|\nabla\beta|\leq\text{ const}\,,$$

we infer from the above estimate that $\nabla(u-\beta)=0$ in $\{u\geq\beta\}\cap\{x_N<K\}$, and therefore $u-\beta=c$ there, for some $c\geq 0$. Then, if, by contradiction, $\{u>\beta\}\cap\{x_N<K\}$ were non-empty, c is actually strictly positive, and a contradiction arises from (9.3). □

We now show that Class A minimizers of $\mathcal{F}$ fulfill (2.1), provided h_0 is convex near ± 1.
Here, it is crucial the use of some density estimates for minimizers of $\mathcal{F}$, proved in [15]. In our setting, we may state this result as follows:

Theorem 9.2. *Let u be a Class A minimizer of $\mathcal{F}$. Suppose that the hypothesis (1.2) and (1.3) are fulfilled. Assume also that h_0' is monotone increasing in $(-1,-1+\theta^*)\cup(1-\theta^*,1)$.*

Fix $\epsilon\in(0,1)$ and suppose $x\in\{-1+\epsilon<u<1-\epsilon\}$. Then, there exist positive $r_0(\epsilon)$ and $c(\epsilon)$ such that

$$\mathfrak{L}(B_r(x)\cap\{u>-1+\epsilon\})\geq c(\epsilon)r^N \quad \textit{and} \quad \mathfrak{L}(B_r(x)\cap\{u<1-\epsilon\})\geq c(\epsilon)r^N \tag{9.7}$$

for any $r\geq r_0(\epsilon)$, being $\mathfrak{L}$ the N-dimensional Lebesgue measure.

Corollary 9.3. *If h_0' is monotone increasing in $(-1,-1+\theta^*)\cup(1-\theta^*,1)$, then any Class A minimizer of $\mathcal{F}$ satisfies (2.1).*

Proof. Suppose $\{u=0\}\cap\{|x-(\omega\cdot x)\omega|\leq l\}\subset\{\omega\cdot x\geq -l/10\}$ and let $x=(x',x_N)\in\mathbb{R}^{N-1}\times\mathbb{R}$ satisfying $\omega\cdot x<-3l/5$ and $|x-(\omega\cdot x)\omega|\leq l/2$. Fixed $\epsilon>0$ assume by contradiction $u(x)>-1+\epsilon$
By Theorem 9.2, with l sufficiently large, so that $l/2\geq r_0(\epsilon)$, we get that $B_{l/2}(x)$ contains points where $u=0$. This is a clear contradiction with the assumption on the set $u=0$ since $B_{l/2}(x)\subset\{|x-(\omega\cdot x)\omega|\leq l\}$ and $\{\omega\cdot x<-l/10\}$ for any $x\in B_{l/2}(x)$.
This contradiction shows that $u(x)\leq -1+\epsilon$ for l large such that $l/2\geq r_0(\epsilon)$. The second part of (2.1) follows with the same arguments. □

The proof of Theorem 2.3 is now completed via Corollary 9.3.

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[6]Preliminary versions of [14] and [15] are available on-line at
`http://www.math.utexas.edu/mp_arc`