Law of large numbers and central limit theorem for randomly forced PDE's

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Abstract

We consider a class of dissipative PDE's perturbed by an external random force. Under the condition that the distribution of perturbation is sufficiently non-degenerate, a strong law of large numbers (SLLN) and a central limit theorem (CLT) for solutions are established and the corresponding rates of convergence are estimated. It is also shown that the estimates obtained are close to being optimal. The proofs are based on the property of exponential mixing for the problem in question and some abstract SLLN and CLT for mixing-type Markov processes.

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Contents

0 Introduction

This paper deals with a class of randomly forced PDE's arising in mathematical physics. To be precise, we confine ourselves in this introduction to the 2D Navier–Stokes system perturbed by an external force white in time and smooth in the space variables:

$$\dot{u} - \Delta u + (u, \nabla)u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0, \quad x \in D. \tag{0.1}$$

Here $D \subset \mathbb{R}^2$ is a bounded domain with C^1 boundary ∂D, $u = (u_1, u_2)$ is the velocity field of the fluid, p is the pressure, and η is a random force. Equation (0.1) is supplemented with Dirichlet boundary condition

$$u\big|_{\partial D} = 0. \tag{0.2}$$

Excluding the pressure, we can write the problem (0.1), (0.2) as an evolution equation in the space H of divergence-free vector fields $u \in L^2(D, \mathbb{R}^2)$ whose normal component vanishes at ∂D (see [38]):

$$\dot{u} + Lu + B(u, u) = \eta(t). \tag{0.3}$$

Here L is the Stokes operator and B is a bilinear form resulting from the nonlinear term in (0.1). We assume that the right-hand side η, for which we retained the same notation as in the original equation, is a random process of the form

$$\eta(t) = \sum_{j=1}^{\infty} b_j \dot{\beta}_j(t) e_j, \tag{0.4}$$

where $b_j \geq 0$ are some constants such that $\sum_j b_j^2 < \infty$, $\{e_j\}$ is a complete set of normalised eigenfunctions of L, and $\{\beta_j\}$ is a sequence of independent standard Brownian motions. Assuming that

$$b_j \neq 0 \quad \text{for} \quad j = 1, \dots, N, \tag{0.5}$$

where $N \geq 1$ is sufficiently large, we obtain some estimates for the rate of convergence in the strong law of large numbers (SLLN) and central limit theorem (CLT) for solutions of Eq. (0.3). To this end, we establish some abstract versions of SLLN and CLT and then apply them to the problem in question. Before giving more detailed formulations, we discuss some earlier results in this direction and explain the main difficulties.

Exponential mixing for SDE's in $\mathbb{R}^n$. Let us consider the equation

$$\dot u = F(u) + \dot w, \quad u(t) \in \mathbb{R}^n, \tag{0.6}$$

where $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and w is a standard Brownian motion in $\mathbb{R}^n$. Assume that the function F satisfies the condition

$$\langle F(u), u\rangle \le -c|u|^2 + C \quad \text{for } u \in \mathbb{R}^n,$$

where C and c are positive constants, $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^n$, and $|\cdot|$ is the corresponding norm. In this case, it is not difficult to show that for any $v \in \mathbb{R}^n$ Eq. (0.6) has a unique solution $u(t)$, $t \ge 0$, adapted to the filtration of w and satisfying the initial condition

$$u(0) = v. \tag{0.7}$$

The large-time asymptotics of solutions of the problem (0.6), (0.7) was studied by many authors. First results in this domain were obtained in the papers [25, 41, 26, 14]. It was shown that the family of Markov processes associated with (0.6), (0.7) has a unique stationary measure μ. Moreover, the Markov family is mixing in the sense that, for any Borel subset $\Gamma \subset \mathbb{R}^n$ and any $v \in \mathbb{R}^n$,

$$P_t(v, \Gamma) \to \mu(\Gamma) \quad \text{as } t \to +\infty, \tag{0.8}$$

where $P_t(v,\Gamma)$ denotes the transition function.

These results were further developed in a number of works. In particular, it was shown in [39, 31] that the rate of convergence in (0.8) is exponential uniformly in all Borel subsets $\Gamma \subset \mathbb{R}^n$. In other word, for any $v \in \mathbb{R}^n$ we have

$$\|P_t(v,\Gamma) - \mu\|_{\mathrm{var}} \le C_v e^{-\gamma t} \quad \text{for } t \ge 0. \tag{0.9}$$

Here C_v and γ are positive constants, and for any probability measures μ_1, μ_2 on $\mathbb{R}^n$ we set

$$\|\mu_1 - \mu_2\|_{\mathrm{var}} = \sup_\Gamma |\mu_1(\Gamma) - \mu_2(\Gamma)|,$$

where the supremum is taken over all Borel subsets $\Gamma \subset \mathbb{R}^n$. Furthermore, if $\alpha_v(t)$ denotes the strong mixing coefficient for the solution with initial condition (0.7), then

$$\alpha_v(t) \le C_v e^{-\gamma t} \quad \text{for } t \ge 0. \tag{0.10}$$

SLLN and CLT for Markov processes with strong mixing. Inequalities (0.9) and (0.10) provide substantial information on the distribution of solutions and can be used for studying the time and ensemble averages of various functionals of solutions. For instance, it is a straightforward consequence of (0.9) that, for any $v \in \mathbb{R}^n$ and any bounded measurable function $f: \mathbb{R}^n \to \mathbb{R}$ satisfying the inequality $|f| \le 1$, we have

$$\bigl|\mathbb{E}\, f(u(t,v)) - (f,\mu)\bigr| \le C_v e^{-\gamma t}, \quad t \ge 0, \tag{0.11}$$

where $u(t,v)$ denotes the solution of (0.6), (0.7) and (f,μ) is the mean value of f with respect to μ. Furthermore, it is well known that Markov processes with strong mixing properties satisfy SLLN and CLT (see [31, 32, 15, 1]). Combining (0.9) and (0.10) with some general results of this type, one can prove the following two assertions:

SLLN: For any $\varepsilon > 0$, $v \in \mathbb{R}^n$, and $f \in L^\infty(\mathbb{R}^n)$, there is an almost surely finite random constant $C > 0$ such that

$$\left| \frac{1}{t} \int_0^t f(u(s,v))\, ds - (f,\mu) \right| \le C t^{-\frac{1}{2}+\varepsilon} \quad \text{for } t \ge 1. \tag{0.12}$$

CLT: For any $f \in L^\infty(\mathbb{R}^n)$ there is a constant $\sigma_f \ge 0$ such that

$$\frac{1}{\sqrt{t}} \int_0^t f(u(s,v))\, ds - \sqrt{t}(f,\mu) \to \mathcal{N}(0,\sigma_f) \quad \text{as } t \to +\infty, \tag{0.13}$$

where $\mathcal{N}(0,\sigma)$ denotes the one-dimensional centred Gaussian distribution with variance σ, and the convergence holds in the sense of distribution. Moreover, the rate of convergence of the corresponding distribution functions is $t^{-\frac{1}{2}}$.

We emphasize that it is important in the above CLT that the strong mixing coefficient $\alpha_v(t)$ decays sufficiently fast (see [32] for more details).

Exponential mixing, SLLN and CLT for randomly forced PDE's. The first result on ergodicity for randomly forced Navier–Stokes equations was obtained by Flandoli and Maslowski [10]. Assuming that the random perturbation is sufficiently irregular, they established the uniqueness of stationary measure and convergence to it in the total variation norm. Their result was refined by Ferrario [9]. Mattingly [28] considered the case in which the forcing is smooth and the viscosity is sufficiently large.

The first result on uniqueness in the case of smooth right-hand side and any positive viscosity was established by Kuksin and the author [18]. We studied a large class of randomly forced PDE's (including the 2D Navier–Stokes system and complex Ginzburg-Landau equation) perturbed by a discrete forcing. Assuming that the perturbation is sufficiently non-degenerate, we proved the uniqueness of stationary measure and convergence to it of other solutions in the weak* topology. E, Mattingly, Sinai [7] and Bricmont, Kupiainen, Lefevere [4] studied later the Navier–Stokes system in the case when the space variables x belong to the 2D torus and the right-hand side is white noise in time and trigonometric polynomial in x. They showed that there is a unique stationary measure. Moreover, it was proved by Bricmont et al. [4] that the above model possesses a property of exponential mixing. Eckmann and Hairer [8] used an infinite-dimensional version of the Malliavin calculus to study the problem of ergodicity for the real Ginzburg–Landau equation perturbed by a rough degenerate forcing.

Another approach for studying the problem of ergodicity for randomly forced PDE's was suggested in [19, 20, 29, 27, 12, 21]. It is based on the classical idea of coupling and enables one to improve the above-mentioned results. Using the coupling approach, Mattingly [29] gave a different proof of exponential mixing for the 2D Navier–Stokes equation on the torus and found an explicit dependence of the constants on the initial data, Masmoudi, Young [27] and Kuksin et al. [19, 20, 16] proved exponential convergence to the stationary measure for a class of parabolic PDE's perturbed by a discrete forcing, and Hairer [12] established similar results for some models in which the forcing does not act directly on all determining modes. The 2D Navier–Stokes system was further studied by Kuksin and the author [21, 22, 36] in the case of bounded domain and infinite-dimensional perturbation. We refer the reader to [5, 11, 17, 30, 37] for a more detailed account of the results obtained in this domain.

We note that inequalities (0.9) and (0.10), in general, do not hold for systems with infinite-dimensional phase space, even if the difference between two trajectories goes to zero exponentially fast (see Example 1.3). However, it was shown by Kuksin [17] that the above results combined with a coupling argument imply a SLLN and a CLT. As was mentioned above, the aim of this article is to estimate the corresponding rates of convergence. We emphasize that known abstract versions of SLLN and CLT for different classes of dependent random variables do not apply to our problem, since they require that the strong mixing coefficient decay sufficiently fast (for instance, see [1, 32]).

The following theorem is a simplified version of the main results of this paper.

MAIN THEOREM. *Suppose that the non-degeneracy condition* (0.5) *is satisfied for a sufficiently large N. Then for any uniformly Lipschitz bounded functional $f : H \to \mathbb{R}$ and any solution $u(t)$ of Eq.* (0.3) *with deterministic initial condition the following statements hold.*

Strong law of large numbers: *For any $\varepsilon > 0$ there is an a.s. finite random constant $T \geq 1$ such that*

$$\left|\frac{1}{t}\int_0^t f(u(s))\,ds - (f,\mu)\right| \leq \mathrm{const}\, t^{-\frac{1}{2}+\varepsilon} \quad \text{for } t \geq T. \tag{0.14}$$

Central limit theorem: *If $(f,\mu) = 0$, then there is a constant $\sigma \geq 0$ depending only on f such that, for any $\varepsilon > 0$, we have*

$$\sup_{z\in\mathbb{R}}\left(\theta_\sigma(z)\left|\mathbb{P}\left\{\frac{1}{\sqrt{t}}\int_0^t f(u(s))\,ds \leq z\right\} - \Phi_\sigma(z)\right|\right) \leq \mathrm{const}\, t^{-\frac{1}{4}+\varepsilon} \quad \text{for } t \geq 1, \tag{0.15}$$

where $\theta_\sigma \equiv 1$ for $\sigma > 0$, $\theta_0(z) = 1 \wedge |z|$, and $\Phi_\sigma(z)$ is the centred Gaussian distribution function with variance σ.

We note that (0.14) and (0.15) remain valid for a large class of Hölder continuous functionals on H with polynomial growth at infinity. Moreover, if we consider Eq. (0.1) on a 2D torus, similar results hold for functionals defined on a Sobolev space H^s with an arbitrarily large s, provided that the right-hand side is sufficiently smooth. We shall not give a precise formulation and a proof of this assertion, since they repeat almost literally the case of Dirichlet boundary condition.

Let us also note that the rates of convergence in the Main Theorem are close to being optimal. Indeed, one cannot take $\varepsilon = 0$ in (0.14), and therefore our SLLN is sharp in the power scale. The rate of convergence in CLT for dependent random variables is $t^{-\frac{1}{2}}$, provided that the strong mixing coefficient decays sufficiently fast (see [1]). If this condition is not satisfied, then the convergence to the limiting distribution holds, in general, with a rate slower than $t^{-\frac{1}{4}}$, and it is widely believed that the threshold $t^{-\frac{1}{4}}$ is critical (see [33, 23, 13]). Moreover, counterexamples show that the rate $t^{-\frac{1}{4}}$ cannot be achieved in the case of martingales (see [2, 13]).

Let us briefly describe the structure of the paper. In Section 1, the main results are presented. We consider the 2D Navier–Stokes system (0.1), (0.2), (0.4), as well as a class of dissipative PDE's perturbed by a random force of the form

$$\eta(t,x) = \sum_{k=1}^{\infty} \eta_k(x)\delta(t-k), \tag{0.16}$$

where $\delta(t)$ is the Dirac measure concentrated at zero and $\{\eta_k\}$ is a sequence of independent identically distributed (i.i.d.) random variables in an appropriate functional space. In Section 2, we establish an SLLN and a CLT for mixing-type Markov processes. Section 3 is devoted to the proof of the main results of this paper. In the Appendix, we have compiled some auxiliary assertions.

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Notation

Let H be a real Hilbert space with norm $|\cdot|$ and let $\alpha \in (0,1]$ be a constant. We shall use the following notation:
$B_H(R)$ is the closed ball in H of radius $R > 0$ centred at zero;
$\mathcal{B}(H)$ is the Borel σ-algebra in H;
$\mathcal{P}(H)$ is the family of probability measures on $(H, \mathcal{B}(H))$;
$C(H)$ is the space of continuous functionals $f\colon H \to \mathbb{R}$;
$C_b(H)$ is the space of bounded functionals $f \in C(H)$ endowed with the norm $\|f\|_\infty := \sup_{u\in H} |f(u)|$.
$\mathcal{W}$ is the space of increasing continuous functions $w(r) > 0$ defined for $r \geq 0$. The elements of $\mathcal{W}$ will be called *weight functions*. In particular, we use the functions $v_\delta(r) = e^{\delta r^2}$ and $w_p(r) = (1+r)^p$.
For the next two definitions, we fix an arbitrary weight function $w \in \mathcal{W}$.
$C(H,w)$ is the space of continuous functionals $f \in C(H)$ such that

$$|f|_w := \sup_{u\in H} \frac{|f(u)|}{w(|u|)} < \infty.$$

$C^\alpha(H,w)$ is the space of continuous functionals $f \in C(H)$ for which the following norm is finite:

$$|f|_{w,\alpha} := |f|_w + \sup_{u\neq v} \frac{|f(u)-f(v)|}{|u-v|^\alpha\big(w(|u|)+w(|v|)\big)}.$$

If $f: H \to \mathbb{R}$ is a $\mathcal{B}(H)$-measurable functional and $\mu \in \mathcal{P}(H)$, then we denote by (f,μ) the integral of f over H with respect to μ.
C_i, $i = 1,2,\dots$, stand for unessential positive constants.

1 Main results

1.1 Dissipative PDE's perturbed by random kicks

Let H be a real Hilbert space with norm $|\cdot|$ and orthonormal base $\{e_j\}$ and let $S: H \to H$ be a continuous operator such that $S(0) = 0$. We consider a discrete-time random dynamical system (RDS) in H,

$$u_k = S(u_{k-1}) + \eta_k, \tag{1.1}$$

where $k \geq 1$ and $\{\eta_k\}$ is a sequence of i.i.d. random variables in H. As was explained in [18, 19], a large class of dissipative PDE's perturbed by a random force of the form (0.16) reduces to the RDS (1.1), and in this case S is the time-one shift along trajectories of the unperturbed equation. We assume that the operator S satisfies the following three conditions introduced in [18, 19]:

(A) For any $R > r > 0$ there are positive constants $a = a(R,r) < 1$ and $C = C(R)$ and an integer $n_0 = n_0(R,r) \geq 1$ such that

$$|S(u_1) - S(u_2)| \leq C(R)|u_1 - u_2| \quad \text{for} \quad u_1, u_2 \in B_H(R),$$
$$|S^n(u)| \leq \max\{a|u|, r\} \quad \text{for} \quad u \in B_H(R), \quad n \geq n_0.$$

(B) For any compact set $\mathcal{K} \subset H$ and any bounded set $B \subset H$ there is a constant $R > 0$ such that the sets $\mathcal{A}_k(\mathcal{K}, B)$ defined recursively by the formulas $\mathcal{A}_0(\mathcal{K}, B) = B$ and $\mathcal{A}_k(\mathcal{K}, B) = S(\mathcal{A}_{k-1}(\mathcal{K}, B)) + \mathcal{K}$ are contained in the ball $B_H(R)$ for all $k \geq 0$.

(C) For any $R > 0$ there is an integer $N \geq 1$ such that

$$\big|\mathsf{Q}_N(S(u_1) - S(u_2))\big| \leq \tfrac{1}{2}|u_1 - u_2| \quad \text{for} \quad u_1, u_2 \in B_H(R),$$

where Q_N is the orthogonal projection onto the closed subspace spanned by $\{e_j, j \geq N+1\}$.

We note that the above conditions are satisfied for the resolving operators of the 2D Navier–Stokes system and the complex Ginzburg–Landau equation.

As for the random kicks η_k, we assume that they are i.i.d. random variables in H of the form

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j,$$

where $b_j \geq 0$ are some constants such that

$$B_0 := \sum_{j=1}^{\infty} b_j^2 < \infty, \tag{1.2}$$

and ξ_{jk} are independent scalar random variables satisfying the following condition:

(D) For any $j \geq 1$, the random variables ξ_{jk} have the same distribution $\pi_j(dr)$, which is absolutely continuous with respect to the Lebesgue measure. Moreover, the corresponding density $p_j(r)$ is a function of bounded total variation, is supported by the interval $[-1, 1]$, and satisfies the condition $\int_{|r|\leq\varepsilon} p_j(r)\, dr > 0$ for any $\varepsilon > 0$.

Let $(u_k, \mathbb{P}_u)$ be the family of Markov chains that is associated with the RDS (1.1) and is parametrised by the initial condition $u \in H$. We denote by $P_k(u, \Gamma)$ the corresponding transition function and by $\mathfrak{P}_k$ and $\mathfrak{P}_k^*$ the Markov operators generated by P_k:

$$\mathfrak{P}_k : C_b(H) \to C_b(H), \qquad \mathfrak{P}_k f(u) = \int_H P_k(u, dv) f(v),$$

$$\mathfrak{P}_k^* : \mathcal{P}(H) \to \mathcal{P}(H), \qquad \mathfrak{P}_k^* \mu(\Gamma) = \int_H P_k(u, \Gamma) \mu(du).$$

Recall that a measure $\mu \in \mathcal{P}(H)$ is said to be *stationary* for the family $(u_k, \mathbb{P}_u)$ if $\mathfrak{P}_1^* \mu = \mu$.

It was proved in [19, 20, 27, 16] that if Hypotheses (A)–(D) are fulfilled together with the non-degeneracy condition (0.5), where $N \geq 1$ is sufficiently large, then the RDS (1.1) has a unique stationary measure μ, which is exponentially mixing in the following sense: for any $\alpha \in (0, 1]$ and $w \in \mathcal{W}$ there is a constant $\beta > 0$ and an increasing function $C(r)$, $r \geq 0$, such that [1]

$$\left|\mathfrak{P}_k f(u) - (f, \mu)\right| \leq C(|u|)\, |f|_{w,\alpha}\, e^{-\beta k}, \quad k \geq 0, \tag{1.3}$$

where $u \in H$ and $f \in C^\alpha(H, w)$ are arbitrary. (See Notation in the Introduction for the definition of the space $C^\alpha(H, w)$.)

The following theorem establishes an SLLN for the family $(u_k, \mathbb{P}_u)$ with an estimate of the rate of convergence.

[1] In [19, 20, 27, 16], inequality (1.3) is proved for uniformly Lipschitz bounded functionals on H. However, the proofs given there remain valid for any functional that is uniformly Hölder continuous on bounded subsets of H.

Theorem 1.1. *Suppose that Hypotheses* (A) – (D) *and the non-degeneracy condition* (0.5) *are satisfied. Then for any* $\alpha \in (0,1]$ *and* $w \in \mathcal{W}$ *there is a constant* $D > 0$ *such that, for any* $f \in C^\alpha(H,w)$ *and* $\varepsilon \in (0,\frac{1}{2})$*, the following statements hold:*

(i) *There is a random integer* $K_\varepsilon(\omega) \geq 1$ *depending on* f *and* ε *such that*

$$\left| k^{-1} \sum_{l=0}^{k-1} f(u_l) - (f,\mu) \right| \leq D\, |f|_w k^{-\frac{1}{2}+\varepsilon} \quad \textit{for} \quad k \geq K_\varepsilon(\omega).$$

(ii) *For any* $u \in H$*, the random integer* K_ε *is* $\mathbb{P}_u$*-a.s. finite. Moreover, for any* $m \geq 1$ *there is a constant* p_m *and an increasing function* $C_m(r)$*,* $r \geq 0$*, such that*

$$\mathbb{E}_u K_\varepsilon^m \leq C_m(|u|)\, |f|_{w,\alpha}^{p_m}.$$

We now turn to the CLT. For any function $f \in C^\alpha(H,w)$ satisfying the condition $(f,\mu) = 0$, we set

$$g(u) = \sum_{l=0}^{\infty} \mathfrak{P}_k f(u), \quad u \in H.$$

The fact that $g(u)$ is well defined follows from inequality (1.3). We introduce a non-negative constant σ_f such that

$$\sigma_f^2 = 2(gf,\mu) - (f^2,\mu). \tag{1.4}$$

The following relation, which can easily be verified with the help of (1.3) and the Markov property, shows that the right-hand side of (1.4) is indeed non-negative:

$$\sigma_f^2 = \lim_{k\to\infty} \mathbb{E}_\mu \left(\frac{1}{\sqrt{k}} \sum_{l=0}^{k-1} f(u_l) \right)^2, \tag{1.5}$$

where $\mathbb{E}_\mu$ is the expectation corresponding to the stationary measure:

$$\mathbb{P}_\mu(\Gamma) = \int_H \mathbb{P}_u(\Gamma)\mu(du), \quad \Gamma \in \mathcal{B}(H). \tag{1.6}$$

For any $\sigma > 0$, we denote by $\Phi_\sigma(r)$ the one-dimensional centred Gaussian distribution with variance σ:

$$\Phi_\sigma(r) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{r} e^{-s^2/2\sigma^2}\, ds.$$

Finally, for $\sigma = 0$, we set

$$\Phi_0(r) = \begin{cases} 1, & r \geq 0, \\ 0, & r < 0. \end{cases}$$

Theorem 1.2. *Suppose that Hypotheses* (A) – (D) *and the non-degeneracy condition* (0.5) *are satisfied. Then for any $\alpha \in (0,1]$ and $w \in \mathcal{W}$ the following statements hold:*

(i) *For any $\bar\sigma > 0$ and $\varepsilon \in (0, \frac{1}{4})$ there is a function $h_{\bar\sigma,\varepsilon}(r_1, r_2) \geq 0$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and increasing in both arguments such that, for any functional $f \in C^\alpha(H, w)$ satisfying the conditions $\sigma_f \geq \bar\sigma$ and $(f, \mu) = 0$, we have*

$$\sup_{z\in\mathbb{R}} \Big|\mathbb{P}_u\Big\{k^{-\frac{1}{2}} \sum_{l=0}^{k-1} f(u_l) \leq z\Big\} - \Phi_{\sigma_f}(z)\Big| \leq h_{\bar\sigma,\varepsilon}(|u|, |f|_{w,\alpha}) k^{-\frac{1}{4}+\varepsilon},$$

where $k \geq 1$ and $u \in H$.

(ii) *There is a function $h(r_1, r_2) \geq 0$ defined for $\mathbb{R}_+ \times \mathbb{R}_+$ and increasing in both arguments such that, for any functional $f \in C^\alpha(H, w)$ satisfying the conditions $\sigma_f = 0$ and $(f, \mu) = 0$, we have*

$$\sup_{z\in\mathbb{R}} \Big((|z| \wedge 1) \Big|\mathbb{P}_u\Big\{k^{-\frac{1}{2}} \sum_{l=0}^{k-1} f(u_l) \leq z\Big\} - \Phi_0(z)\Big|\Big) \leq h(|u|, |f|_{w,\alpha})\, k^{-\frac{1}{4}},$$

where $k \geq 1$ and $u \in H$.

We emphasize that Theorems 1.1 and 1.2 are valid for any family of Markov chains with bounded trajectories that is uniformly mixing in the sense of inequality (1.3), and the function $C(r)$ entering (1.3) may grow at infinity arbitrarily fast.

Before turning to the case of the NS system perturbed by a white noise force, we consider an example showing that even if the operator $S \colon H \to H$ is a uniform contraction, convergence to the stationary measure does not hold, in general, in the total variation norm, and the strong mixing coefficient does not decay to zero.

Example 1.3. Let H be the space of real-valued sequences $\boldsymbol{u} = \{u_j\}$ such that

$$\|\boldsymbol{u}\|^2 := \sum_{j=1}^{\infty} u_j^2 < \infty.$$

We consider the following RDS in H:

$$\boldsymbol{u}^k = S(\boldsymbol{u}^{k-1}) + \boldsymbol{\eta}^k. \tag{1.7}$$

Here $S : H \to H$ is an operator of the form $S(\boldsymbol{u}) = (0, \varphi(u_1), \varphi(u_2), \dots)$, where $\varphi \in C^1(\mathbb{R}, \mathbb{R})$, and $\boldsymbol{\eta}^k = (\xi_k, 0, 0, \dots)$, where $\{\xi_k\}$ is a sequence i.i.d. random variables in $\mathbb{R}$ whose distribution has a smooth density $\rho(x)$ with respect to the Lebesgue measure. In what follows, we assume that ρ has a compact support.

Let us take $\varphi(u) = \varepsilon\,\chi(u)u$, where $\varepsilon > 0$ is a small parameter, $\chi \in C^\infty(\mathbb{R})$, $0 \le \chi \le 1$, $\chi(u) = 1$ for $|u| \le 1$ and $\chi(u) = 0$ for $|u| \ge 2$. It is matter of direct verification to show that if $2\varepsilon(1 + 2\sup|\chi'|) \le 1$, then

$$\|S(\boldsymbol{u}) - S(\boldsymbol{v})\| \le \tfrac{1}{2}\,\|\boldsymbol{u} - \boldsymbol{v}\| \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in H.$$

Thus, the RDS (1.7) has a unique stationary measure μ, which is exponentially mixing in the sense that

$$\big|\mathfrak{P}_k f(\boldsymbol{u}) - (f, \mu)\big| \le C\,\big(1 + \|\boldsymbol{u}\|\big)\,e^{-\beta k}, \quad k \ge 0, \quad \boldsymbol{u} \in H,$$

where $\mathfrak{P}_k : C_b(H) \to C_b(H)$ denotes the Markov semigroup associated with (1.7), $f : H \to \mathbb{R}$ is an arbitrary uniformly Lipschitz functional with constant ≤ 1, and C and β are positive constants not depending on f and $\boldsymbol{u}$. We claim that the following two assertions hold:

(i) For any initial point $\boldsymbol{v} \in H$ such that $|v_1| \le 1$, we have

$$\|P_k(\boldsymbol{v}, \cdot) - \mu\|_{\mathrm{var}} = 1 \quad \text{for all } k \ge 0. \tag{1.8}$$

(ii) Suppose that the support of ρ contains the interval $[-2, 2]$. Then there is a constant $c > 0$ such that for any stationary trajectory $\boldsymbol{u}^k$, $k \ge 0$, we have

$$\alpha_k \ge c \quad \text{for all } k \ge 0, \tag{1.9}$$

where α_k is the strong mixing coefficient of $\{\boldsymbol{u}^k\}$.

Proof of (i). Let us note that, for any initial point $\boldsymbol{v} \in H$, the trajectory of (1.7) starting from $\boldsymbol{v}$ has the form

$$\boldsymbol{v}^k = (\xi_k, \varphi_1(\xi_{k-1}), \dots, \varphi_{k-1}(\xi_1), \varphi_k(v_1), \varphi_k(v_2), \dots), \tag{1.10}$$

where $\varphi_m = \varphi\circ\cdots\circ\varphi$ (m times). It follows that the unique stationary measure μ coincides with the distribution of the random variable $(\xi_1, \varphi_1(\xi_2), \varphi_2(\xi_3), \dots)$.

Setting $\Gamma_k = \{\boldsymbol{u} \in H : u_{k+1} = \varphi_k(v_1)\}$, we see from (1.10) that

$$P_k(\boldsymbol{v}, \Gamma_k) = 1 \quad \text{for any } k \ge 0. \tag{1.11}$$

On the other hand, the definition of φ implies that, for any $z \ne 0$, the set $\{y \in \mathbb{R} : \varphi_k(y) = z\}$ consists of at most two points. Since $\varphi_k(v_1) \ne 0$ for $|v_1| \le 1$ and the distribution of ξ_{k+1} has a density, we conclude that

$$\mu(\Gamma_k) = \mathbb{P}\big\{\varphi_k(\xi_{k+1}) = \varphi_k(v_1)\big\} = 0 \quad \text{for any } k \ge 0. \tag{1.12}$$

Relations (1.11) and (1.12) imply (1.8). □

Proof of (ii). It is shown in [1] that the strong mixing coefficient can be represented in the form

$$\alpha_k = \frac{1}{2}\sup_f \int_H \big|\mathfrak{P}_k f(\boldsymbol{v}) - (f, \mu)\big|\,\mu(d\boldsymbol{v}), \tag{1.13}$$

where the supremum is taken over all measurable functions $f : H \to \mathbb{R}$ such that $0 \le f \le 1$. Let $g_k(\boldsymbol{v}) = 0$ for $v_{k+1} = 0$ and $g_k(\boldsymbol{v}) = 1$ otherwise. In this case, relation (1.10) implies that

$$\mathfrak{P}_k g_k(\boldsymbol{v}) = \mathbb{E}\, g_k(\boldsymbol{v}^k) = I_{\mathbb{R}^*}(\varphi_k(v_1)) = I_\Gamma(v_1),$$

where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\Gamma = \{x \in \mathbb{R} : 0 < |x| < 2\}$. Combining this with (1.13), we see that

$$\alpha_k \ge \frac{1}{2} \int_H \bigl|\mathfrak{P}_k g_k(\boldsymbol{v}) - (g_k, \mu)\bigr|\, \mu(d\boldsymbol{v}) = \frac{1}{2} \int_{\mathbb{R}} \bigl|I_\Gamma(x) - (g_k, \mu)\bigr| \rho(x)\, dx.$$

Since $\operatorname{supp} \rho \supset [-2, 2]$, the integral over $\mathbb{R}$ is separated from zero uniformly in $k \ge 0$. □

1.2 Navier–Stokes system perturbed by white noise

We now turn to the problem (0.3), (0.4), where the right-hand side η satisfies the conditions formulated in the Introduction. Let H be the space of divergence-free vector fields $u \in L^2(D, \mathbb{R}^2)$ such that $(u, \nu)\big|_{\partial D} = 0$, where ν is the unit normal to ∂D (see [38] for details). It is well known (see [40, 6]) that the problem (0.3), (0.4) generates a family of Markov processes $(u_t, \mathbb{P}_u)$ in the space H. We denote by $P_t(u, \Gamma) = \mathbb{P}_u\{u_t \in \Gamma\}$ the corresponding transition function and by $\mathfrak{P}_t$ and $\mathfrak{P}_t^*$ the Markov operators associated with $P_t(u, \Gamma)$. It was shown in [7, 4, 29, 21] that if

$$B_1 := \sum_j \alpha_j b_j^2 < \infty,$$

where α_j are the eigenvalues of the Stokes operator, and the non-degeneracy condition (0.5) holds with a sufficiently large $N \ge 1$, then the family $(u_t, \mathbb{P}_u)$ has a unique stationary measure $\mu \in \mathcal{P}(H)$, and any other solution converges to it exponentially fast in the weak* topology. Moreover, according to Proposition 3.2 in [22], there is $d > 0$ such that, for any positive constants $\alpha \le 1$ and $\delta \le d$ and any functional $f \in C^\alpha(H, v_\delta)$ (where $v_\delta(r) = e^{\delta r^2}$), we have

$$\bigl|\mathfrak{P}_t f(u) - (f, \mu)\bigr| \le C |f|_{v_\delta, \alpha} v_\delta(|u|)\, e^{-\beta t}, \quad t \ge 0, \tag{1.14}$$

where $u \in H$, and β and C are positive constants depending only on u and f.

Let $w \in \mathcal{W}$ be a weight function such that

$$\lim_{r \to +\infty} w(r)\, e^{\delta r^2} = 0 \quad \text{for any } \delta > 0. \tag{1.15}$$

The following result on SLLN is an analog of Theorem 1.1 for the case of the problem (0.3), (0.4).

Theorem 1.4. *Let $d > 0$ be the constant mentioned above and let $\alpha \in (0, 1]$. Suppose that* (1.15) *holds and that the non-degeneracy condition* (0.5) *is satisfied with sufficiently large $N \ge 1$. Then there is a constant $D > 0$ such that the following statements hold.*

(i) *For any $f \in C^\alpha(H,w)$ and $\varepsilon \in (0,\frac{1}{2})$ there is a random variable $T_\varepsilon(\omega) \geq 1$ such that*

$$\left| t^{-1} \int_0^t f(u_s)\,ds - (f,\mu) \right| \leq D\,|f|_w t^{-\frac{1}{2}+\varepsilon} \quad \text{for} \quad t \geq T_\varepsilon(\omega). \tag{1.16}$$

(ii) *For any $u \in H$, the random variable T_ε is $\mathbb{P}_u$-a.s. finite. Moreover, for any $m \geq 1$ there are constants $p_m \geq 1$ and $C_m > 0$ such that*

$$\mathbb{E}_u T_\varepsilon^m \leq C_m \left(|f|_{v_\delta,\alpha}^{p_m} + 1 \right) e^{d|u|^2}. \tag{1.17}$$

The above theorem concerns functionals $f : H \to \mathbb{R}$ that grow at infinity slower than $e^{\delta|u|^2}$ for any $\delta > 0$. A similar result takes place for functionals $f \in C^\alpha(H, v_\delta)$ with sufficiently small $\delta > 0$. In this case, however, we can only claim that (1.16) and (1.17) hold for some fixed constants $\varepsilon \in (0,\frac{1}{2})$ and $m > 0$ depending on δ. We shall not give a precise formulation.

We now discuss the CLT for the problem (0.3), (0.4). Let us fix an arbitrary constant $p > 0$ and set $w_p(r) = (1+r)^p$. For any $f \in C^\alpha(H, w_p)$ satisfying the condition $(f,\mu) = 0$ we introduce the function

$$g(u) = \int_0^\infty \mathfrak{P}_s f(u)\,ds, \quad u \in H. \tag{1.18}$$

Inequality (1.14) implies that g is well defined. Furthermore, we introduce a non-negative constant σ_f such that (cf. (1.4) and (1.5))

$$\sigma_f^2 = 2(gf,\mu) = \lim_{t\to+\infty} \mathbb{E}_\mu \left(t^{-\frac{1}{2}} \int_0^t f(u_s)\,ds \right)^2. \tag{1.19}$$

Theorem 1.5. *Suppose that the non-degeneracy condition* (0.5) *is satisfied with sufficiently large $N \geq 1$. Then for any $\alpha \in (0,1]$ and $p > 0$ the following statements hold:*

(i) *For any $\bar\sigma > 0$ and $\varepsilon \in (0,\frac{1}{4})$ there is a function $h_{\bar\sigma,\varepsilon}(r_1,r_2) \geq 0$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and increasing in both arguments such that, for any functional $f \in C^\alpha(H,w_p)$ satisfying the conditions $\sigma_f \geq \bar\sigma$ and $(f,\mu) = 0$, we have*

$$\sup_{z\in\mathbb{R}} \left| \mathbb{P}_u \left\{ t^{-\frac{1}{2}} \int_0^t f(u_s)\,ds \leq z \right\} - \Phi_{\sigma_f}(z) \right| \leq h_{\bar\sigma}(|u|, |f|_{w_p,\alpha})\, t^{-\frac{1}{4}+\varepsilon}, \tag{1.20}$$

where $t \geq 1$ and $u \in H$.

(ii) *There is a function $h(r_1,r_2) \geq 0$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and increasing in both arguments such that, for any functional $f \in C^\alpha(H,w_p)$ satisfying the conditions $\sigma_f = 0$ and $(f,\mu) = 0$, we have*

$$\sup_{z\in\mathbb{R}} \left((|z| \wedge 1) \left| \mathbb{P}_u \left\{ t^{-\frac{1}{2}} \int_0^t f(u_s)\,ds \leq z \right\} - \Phi_0(z) \right| \right) \leq h(|u|, |f|_{w_p,\alpha})\, t^{-\frac{1}{4}}, \tag{1.21}$$

where $t \geq 1$ and $u \in H$.

2 LLN and CLT for mixing-type Markov processes

In this section, we establish some versions of SLLN and CLT (with rates of convergence) for Markov processes possessing a mixing property. They are used in the next section to prove Theorems 1.4 and 1.5.

2.1 Strong law of large numbers

Let H be a real Hilbert space with norm $|\cdot|$, let $(u_t, \mathbb{P}_u)$ be a family of H-valued Markov processes, and let $\mathcal{F}_t$ be the σ-algebra generated by u_s, $0 \le s \le t$. We shall assume that, for any $u \in H$ and $\mathbb{P}_u$-a.e. $\omega \in \Omega$, the trajectory $u_t(\omega)$, $t \ge 0$, is continuous. (In what follows, a family of Markov processes satisfying this additional condition is said to be *continuous*.) Let $P_t(u, \Gamma) = \mathbb{P}_u\{u_t \in \Gamma\}$ be the transition function for $(u_t, \mathbb{P}_u)$ and let $\mathfrak{P}_t$ and $\mathfrak{P}_t^*$ be the corresponding Markov semi-groups (see Section 1.1).

Let us fix a constant $\alpha \in (0, 1]$ and a weight function $w \in \mathcal{W}$. For any $f \in C^\alpha(H, w)$, we set

$$S_t(f) = \int_0^t f(u_s)\, ds, \quad s_t(f) = t^{-1} S_t(f). \tag{2.1}$$

Suppose that the Markov family in question satisfies the following two conditions.

Condition 2.1. The family $(u_t, \mathbb{P}_u)$ has a unique stationary measure $\mu \in \mathcal{P}(H)$, and the mean value (f, μ) of any functional $f \in C^\alpha(H, w)$ is finite. Moreover, there is a constant $p > 1/2$ and a continuous function $\varkappa(r)$, $r \ge 0$, that do not depend on f such that

$$\mathbb{E}_u \left| s_t(f) - (f, \mu) \right|^{2p} \le \varkappa(|u|) |f|_{w,\alpha}^{2p} t^{-p} \quad \text{for } t \ge 1,\, u \in H. \tag{2.2}$$

Condition 2.2. There are constants $q \in (0, \frac{1}{2})$ and $s > 0$, a random time $M(\omega) \ge 1$, and a continuous function $\tau(r)$, $r \ge 0$, such that, for any $u \in H$, we have

$$\mathbb{P}_u\bigl\{|u_t(\omega)| \le w^{-1}(t^q) \text{ for } t \ge M\bigr\} = 1, \tag{2.3}$$

$$\mathbb{E}_u M^s \le \tau(|u|), \tag{2.4}$$

where $w^{-1}(r)$ is the inverse function of $w(r)$.

The following theorem shows that Conditions 2.1 and 2.2 imply an SLLN with an estimate for the rate of convergence.

Theorem 2.3. *Let $(u_t, \mathbb{P}_u)$ be a family of continuous Markov processes in H that satisfies Conditions 2.1 and 2.2 with some $\alpha \in (0, 1]$, $w \in \mathcal{W}$, $p > 1/2$, $q < 1/2$, and $s > 0$. Then there is a constant $D > 0$ not depending on the these parameters such that the following statements hold.*

(i) *For any $f \in C^\alpha(H,w)$ and $\nu \in (0,2p-1)$ there is a random time $T(\omega) \geq 1$ such that*

$$\left|t^{-1}S_t(f)-(f,\mu)\right| \leq D\,|f|_w t^{-\frac{1}{2}+r_\nu} \quad \text{for} \quad t \geq T(\omega), \tag{2.5}$$

where $r_\nu = q \vee \left(\frac{1+\nu}{4p}\right)$.

(ii) *For any $u \in H$, the random time T is $\mathbb{P}_u$-a.s. finite. Moreover, for any $\ell \leq s$ satisfying the inequality $\ell < \nu/2$, we have*

$$\mathbb{E}_u T^\ell \leq \tfrac{2p}{\nu-2\ell}\,|f|_{w,\alpha}^{2p}\,\varkappa(|u|)+\tau(|u|). \tag{2.6}$$

We note that if $w \in \mathcal{W}$ is bounded (and hence any functional in $C^\alpha(H,w)$ is bounded), then Condition 2.2 can be omitted. In this case, we should take $q=0$ and $\tau \equiv 0$ in the formulation of the theorem; see [35] for a particular case of this result.

Proof of Theorem 2.3. Let us fix a constant $\nu \in (0,2p-1)$ and an arbitrary function $f \in C^\alpha(H,w)$. There is no loss of generality in assuming that $|f|_w \leq 1$ and $(f,\mu)=0$; the proof in the general case is similar.

Step 1. Let us set

$$t_n = n^2, \quad \delta = 1-\tfrac{1+\nu}{2p}$$

and consider the events

$$G_n = \left\{\omega \in \Omega : |s_{t_n}(f)| > n^{-\delta}\right\}, \quad n \geq 1.$$

Using (2.2) and the Chebyshev inequality, for any $u \in H$ and $n \geq 1$ we derive

$$\mathbb{P}_u(G_n) \leq n^{2p\delta}\mathbb{E}_u|s_{t_n}(f)|^{2p} \leq \varkappa(|u|)|f|_{w,\alpha}^{2p}\,n^{-1-\nu}. \tag{2.7}$$

Let us define $m(\omega)$ as the smallest integer $m \geq 0$ such that

$$|s_{t_n}(f)| \leq n^{-\delta} = t_n^{-\frac{\delta}{2}} \quad \text{for} \quad n \geq m+1. \tag{2.8}$$

Inequality (2.7) and the Borel–Cantelli lemma imply that, for any $u \in H$, the random integer $m(\omega)$ is $\mathbb{P}_u$-a.s. finite. Moreover, it follows from the definition that, if $m(\omega) \geq 1$, then

$$|s_{t_m}(f)| > m^{-\delta}. \tag{2.9}$$

We now estimate $|s_t(f)|$ for $t_n < t < t_{n+1}$, $n \geq m$. To this end, we note that

$$|s_t(f)-s_{t_{n+1}}(f)| \leq \left(t^{-1}-t_{n+1}^{-1}\right)|S_{t_{n+1}}(f)| + t^{-1}|S_t(f)-S_{t_{n+1}}(f)|. \tag{2.10}$$

In view of (2.8), we have

$$\left(t^{-1}-t_{n+1}^{-1}\right)|S_{t_{n+1}}(f)| \leq \tfrac{t_{n+1}-t}{t}\,|s_{t_{n+1}}(f)| \leq \tfrac{t_{n+1}-t_n}{t_n}\,t_{n+1}^{-\frac{\delta}{2}}. \tag{2.11}$$

Furthermore, it follows from (2.3) that, for $t \geq M(\omega)$,

$$
\begin{aligned}
t^{-1}|S_t(f)-S_{t_{n+1}}(f)| &\leq t^{-1}\int_t^{t_{n+1}} |f(u_s)|\,ds \leq t^{-1}\int_t^{t_{n+1}} w\big(|u_s|\big)\,ds \\
&\leq t^{-1}\int_t^{t_{n+1}} s^q ds \leq \frac{t_{n+1}^{q+1}-t_n^{q+1}}{t_n}. \qquad (2.12)
\end{aligned}
$$

Since $t_n = n^2$, there is $C>0$ such that for any $r\in[1,3/2]$ we have

$$
\frac{t_{n+1}^r - t_n^r}{t_n} \leq C\,(n+1)^{2r-3} = C\,t_{n+1}^{r-\frac{3}{2}}.
$$

Combining this inequality with (2.10) – (2.12) and (2.8), for $t_n < t < t_{n+1}$, $n \geq m(\omega)$, $t \geq M(\omega)$, we obtain

$$
\begin{aligned}
|s_t(f)| &\leq |s_t(f) - s_{t_{n+1}}(f)| + |s_{t_{n+1}}(f)| \\
&\leq C t_{n+1}^{-\frac{1+\delta}{2}} + C\,t_{n+1}^{q-\frac{1}{2}} + t_{n+1}^{-\frac{\delta}{2}} \leq D\,t^{-\frac{1}{2}+r_\nu}.
\end{aligned}
$$

Thus, inequality (2.5) holds with $T(\omega) = M(\omega)\vee m(\omega)^2$.

Step 2. We now prove (2.6). To this end, let us note that, for $0<\ell<\nu/2$, we have

$$
\begin{aligned}
\mathbb{E}_u m^{2\ell} &= \sum_{n=1}^\infty \mathbb{P}_u\{m=n\} n^{2\ell} \leq \sum_{n=1}^\infty \mathbb{P}_u(G_n) n^{2\ell} \\
&\leq \varkappa(|u|)\,|f|_{w,\alpha}^{2p} \sum_{n=1}^\infty n^{-1-\nu+2\ell} \leq \tfrac{2p}{\nu-2\ell}\,\varkappa(|u|)\,|f|_{w,\alpha}^{2p}, \qquad (2.13)
\end{aligned}
$$

where we used inequalities (2.7), (2.9) and the definition of $m(\omega)$ and G_n. Furthermore, if $\ell \leq s$, then (2.4) implies that

$$
\mathbb{E}_u M^\ell \leq \mathbb{E}_u M^s \leq \tau(|u|). \qquad (2.14)
$$

Now note that

$$
\mathbb{E}_u T^\ell \leq \mathbb{E}_u m^{2\ell} + \mathbb{E}_u M^\ell.
$$

Hence, using (2.13) and (2.14) to estimate the right-hand side of the above inequality, we obtain (2.6). The proof of Theorem 2.3 is complete. □

Theorem 2.3 implies the following corollary, in which the rate of convergence is arbitrarily close to $\frac{1}{2}$.

Corollary 2.4. *Let $(u_t,\mathbb{P}_u)$ be a family of continuous Markov processes in H that satisfies Conditions 2.1 and 2.2 with some fixed $\alpha\in(0,1]$ and $w\in\mathcal{W}$, arbitrary $p>1/2$ and $q<1/2$, and $s=q^{-1}$. We denote by $\varkappa_p$, τ_q, and M_q the corresponding functions and random time in Conditions 2.1 and 2.2. Then the following statements hold.*

(i) *For any $f \in C^\alpha(H,w)$ and $\varepsilon \in (0,\frac{1}{2})$ there is a random time $T_\varepsilon(\omega) \geq 1$ such that*

$$\left|t^{-1}S_t(f) - (f,\mu)\right| \leq D\,|f|_w t^{-\frac{1}{2}+\varepsilon} \quad \text{for} \quad t \geq T_\varepsilon(\omega), \tag{2.15}$$

where $D > 0$ is the constant constructed in Theorem 2.3.

(ii) *For any $u \in H$, the random variable T_ε is $\mathbb{P}_u$-a.s. finite. Moreover, for any $m \geq 1$ there is a constant $p_m \geq 1$ such that*

$$\mathbb{E}_u T_\varepsilon^m \leq p_m |f|_{w,\alpha}^{p_m} \varkappa_{p_m}(|u|) + \tau_m(|u|). \tag{2.16}$$

Proof. We fix arbitrary $\varepsilon \in (0,\frac{1}{2})$ and $f \in C^\alpha(H,w)$ and, for any $\omega \in \Omega$, denote by $T_\varepsilon \geq 0$ the smallest constant for which (2.15) is fulfilled. The definition of T_ε implies that assertion (i) holds. To prove (ii), we choose an integer $m \geq 3$ such that $\varepsilon \leq m^{-1}$ and apply Theorem 2.3 with

$$p = \frac{m(m+1)}{2}, \quad q = \frac{1}{m}, \quad s = m, \quad \nu = 2m+1.$$

Denoting by $T(\omega,m)$ the corresponding random variable and recalling the definition of $T_\varepsilon(\omega)$, we see that $T_\varepsilon(\omega) \leq T(\omega,m)$ for all $\omega \in \Omega$. In view of inequality (2.6) with $\ell = m$, we have

$$\mathbb{E}_u T(\cdot,m)^m \leq p_m\,|f|_{w,\alpha}^{p_m} \varkappa_{p_m}(|u|) + \tau_m(|u|),$$

where $p_m = m(m+1)$. This completes the proof of the corollary. ☐

In what follows, we shall need a sufficient condition for the validity of inequality (2.2). To this end, we introduce the following definition.

Definition 2.5. We shall say that the family $(u_t,\mathbb{P}_u)$ is *uniformly mixing for the class $C^\alpha(H,w)$* if it has a unique stationary measure $\mu \in \mathcal{P}(H)$ and there are non-negative continuous functions $\rho(r)$, $r \geq 0$, and $\gamma(t)$, $t \geq 0$, such that

$$\bar\gamma := \int_0^\infty \gamma(t)\,dt < \infty, \tag{2.17}$$

and, for any $f \in C^\alpha(H,w)$ and $u \in H$, we have[2]

$$\left|\mathfrak{P}_t f(u) - (f,\mu)\right| \leq \gamma(t)\rho(|u|)|f|_{w,\alpha}, \quad t \geq 0. \tag{2.18}$$

Note that, taking $t = 0$ in (2.18), we derive

$$|(f,\mu)| \leq C\,|f|_{w,\alpha}, \quad |f(u)| \leq C\,|f|_{w,\alpha}\big(\rho(|u|)+1\big), \tag{2.19}$$

where $C > 0$ is a constant not depending on f and u. The proposition below shows that uniform mixing combined with an additional assumption on the function ρ implies Condition 2.1.

[2] In particular, we assume that $(f,\mu) < \infty$ for any $f \in C^\alpha(H,w)$.

Proposition 2.6. *Let $(u_t, \mathbb{P}_u)$ be a family of continuous Markov processes that is uniformly mixing for a class $C^\alpha(H,w)$. Suppose that the function ρ in Definition 2.5 satisfies the inequality*

$$\mathbb{E}_u \rho^{2p}(|u_t|) \le \sigma(|u|) \quad \text{for all } t \ge 0,\ u \in H, \tag{2.20}$$

where $p \ge 1$ is an integer and $\sigma(r)$, $r \ge 0$, is a continuous function. Then Condition 2.1 holds with the above p and the function

$$\varkappa(r) = \bigl(2p(2p-1)\bar{\gamma}\gamma(0)\bigr)^p \sigma(r).$$

Proof. Let us fix an arbitrary functional $f \in C^\alpha(H,w)$. Without loss of generality, we assume that $|f|_w \le 1$ and $(f,\mu) = 0$ and set

$$I_p(t) = \sup_{0 \le r \le t} \mathbb{E}_u \bigl|S_r(f)\bigr|^{2p}.$$

We have

$$\begin{aligned}
\mathbb{E}_u \bigl|S_r(f)\bigr|^{2p} &= \mathbb{E}_u \int_{[0,r]^{2p}} f(u_{r_1}) \cdots f(u_{r_{2p}})\, dr_1 \cdots d_{r_{2p}} \\
&= (2p)!\, \mathbb{E}_u \int_{\Delta_p(r)} \bigl(f(u_{r_1}) \cdots f(u_{r_{2p}})\bigr)\, dr_1 \cdots d_{r_{2p}} \\
&= (2p)!\, \mathbb{E}_u \int_{\Delta_p(r)} f(u_{r_1}) \cdots f(u_{r_{2p-2}}) g(r_{2p-1}, r_{2p})\, dr_1 \cdots d_{r_{2p}},
\end{aligned} \tag{2.21}$$

where we set

$$\begin{aligned}
\Delta_p(r) &= \bigl\{(r_1, \dots, r_{2p}) \in \mathbb{R}^{2p} : 0 \le r_1 \le \cdots \le r_{2p} \le r\bigr\}, \\
g(s_1, s_2) &= f(u_{s_1}) \mathbb{E}_u\bigl(f(u_{s_2}) \,|\, \mathcal{F}_{s_1}\bigr), \quad s_1 \le s_2.
\end{aligned}$$

The integral on the right-hand side of (2.21) can be represented as

$$\begin{aligned}
&\int_{\Delta_1(r)} g(r_{2p-1}, r_{2p}) \left\{ \int_{\Delta_{p-1}(r_{2p-1})} f(u_{r_1}) \cdots f(u_{r_{2p-2}})\, dr_1 \cdots d_{r_{2p-2}} \right\} dr_{2p-1} dr_{2p} \\
&\qquad = \frac{1}{(2p-2)!} \int_{\Delta_1(r)} g(r_{2p-1}, r_{2p}) \bigl|S_{r_{2p-1}}(f)\bigr|^{2(p-1)} dr_{2p-1} dr_{2p},
\end{aligned}$$

where the domain of integration $\Delta_1(r)$ is taken for the variables r_{2p-1} and r_{2p}. Substituting this expression into (2.21) and applying the Hölder inequality, we obtain

$$\mathbb{E}_u \bigl|S_r(f)\bigr|^{2p} = C_p \int_{\Delta_1(r)} \bigl(\mathbb{E}_u \bigl|S_{r_{2p-1}}(f)\bigr|^{2p}\bigr)^{\frac{p-1}{p}} \bigl(\mathbb{E}_u \left|g(r_{2p-1}, r_{2p})\right|^p\bigr)^{\frac{1}{p}} dr_{2p-1} dr_{2p},$$

where $C_p = 2p(2p-1)$. Taking the supremum over $r \in [0,t]$, we see that

$$I_p(t) \le C_p\big(I_p(t)\big)^{\frac{p-1}{p}} \int_{\Delta_1(t)} \big(\mathbb{E}_u\,|g(s_1,s_2)|^p\big)^{\frac{1}{p}} ds_1 ds_2.$$

Thus, we have arrived at the inequality

$$I_p(t) \le \left(C_p \int_{\Delta_1(t)} \big(\mathbb{E}_u\,|g(s_1,s_2)|^p\big)^{\frac{1}{p}} ds_1 ds_2\right)^p. \tag{2.22}$$

We now estimate the function $g(s_1,s_2)$. It follows from the Markov property and inequality (2.18) that

$$\big|\mathbb{E}_u\big(f(u_{s_2}\,|\,\mathcal{F}_{s_1}\big)\big| = \big|\mathfrak{P}_{s_2-s_1} f(u_{s_1})\big| \le \gamma(s_2-s_1)\rho(|u_{s_1}|)|f|_{w,\alpha}. \tag{2.23}$$

Using inequality (2.18) with $t=0$, we obtain

$$\begin{aligned} |g(s_1,s_2)| &\le \gamma(s_2-s_1)\,|f(u_{s_1})|\,\rho(|u_{s_1}|)\,|f|_{w,\alpha} \\ &\le \gamma(0)\gamma(s_2-s_1)\,\rho^2(|u_{s_1}|)\,|f|^2_{w,\alpha}. \end{aligned}$$

It follows from (2.20) that

$$\begin{aligned} \big(\mathbb{E}_u\,|g(s_1,s_2)|^p\big)^{\frac{1}{p}} &\le \gamma(0)\gamma(s_2-s_1)\,|f|^2_{w,\alpha}\,\big(\mathbb{E}_u \rho^{2p}(|u_{s_1}|)\big)^{\frac{1}{p}} \\ &\le \gamma(0)\gamma(s_2-s_1)\,|f|^2_{w,\alpha}\sigma(|u|)^{\frac{1}{p}}. \end{aligned}$$

Substituting this inequality into (2.22) and taking into account (2.17), we arrive at the required estimate (2.2). □

Remark 2.7. It is clear that Proposition 2.6 remains valid for discrete-time Markov processes. In what follows, we shall refer to it for both continuous and discrete time.

2.2 Central limit theorem

In this subsection, we show that the rate of convergence in CLT for uniformly mixing Markov processes can be expressed in terms of the conditional variance for an associated martingale. To formulate the corresponding result, we shall need some notation.

Let $(u_t, \mathbb{P}_u)$ be a family of continuous Markov processes that is uniformly mixing in the sense of Definition 2.5. As before, we denote by $\mathcal{F}_t$ the filtration generated by the Markov family. Let us fix $u \in H$ and an arbitrary function $f \in C^\alpha(H,w)$ such that $(f,\mu)=0$ and set

$$M_t = \int_0^\infty \big\{\mathbb{E}_u\big(f(u_s)\,|\,\mathcal{F}_t\big) - \mathbb{E}_u\big(f(u_s)\,|\,\mathcal{F}_0\big)\big\} ds. \tag{2.24}$$

We claim that, for any $u \in H$, the random variables M_t, $t \geq 0$, are $\mathbb{P}_u$-a.s. finite and form a zero-mean martingale. Indeed, using (2.23) we derive

$$\int_0^\infty \bigl|\mathbb{E}_u\bigl(f(u_s)\,|\,\mathcal{F}_t\bigr)\bigr|\,ds \leq \int_0^t \bigl|f(u_s)\bigr|\,ds + \bar{\gamma}\,|f|_{w,\alpha}\rho(|u_t|), \quad t \geq 0,$$

where $\bar{\gamma}$ is the constant in (2.17). Thus, integral (2.24) converges absolutely for $\mathbb{P}_u$-a.e. ω. The fact that M_t is a zero-mean martingale can easily be established by taking in (2.24) the (conditional) expectation.

For any integer $k \geq 1$, we define a conditional variance for M_k by the formula

$$V_k^2 = \sum_{l=1}^k \mathbb{E}_u\bigl((M_l - M_{l-1})^2\,|\,\mathcal{F}_{l-1}\bigr).$$

Given a random variable ζ and a non-negative constant σ, we set

$$\Delta_\sigma(\zeta, z) = \begin{cases} F_\zeta(z) - \Phi_\sigma(z), & \sigma > 0, \\ (|z| \wedge 1)\bigl(F_\zeta(z) - \Phi_0(z)\bigr), & \sigma = 0, \end{cases}$$

where $F_\zeta(z)$ is the distribution function of ζ.

The following theorem reduces the CLT for uniformly mixing Markov families to an LLN for the conditional variance (cf. [15, Theorem VIII.3.22]).

Theorem 2.8. *Let $(u_t, \mathbb{P}_u)$ be a family of Markov processes that is uniformly mixing in the sense of Definition 2.5. Suppose that there is a constant $a > 0$ and a continuous function $\varkappa(r)$, $r \geq 0$, such that*

$$\mathbb{E}_u\Bigl(\sup_{t\in[k,k+1]} \exp\{\rho^a(|u_t|)\}\Bigr) \leq \varkappa(|u|) \quad \textit{for} \quad k \geq 0, \quad u \in H. \tag{2.25}$$

Then the following statements hold:

- **(i)** *For any $\bar{\sigma} > 0$ and $\varepsilon \in (0, \frac{1}{4})$ there is a non-negative continuous function $h_{\bar{\sigma}}(r_1, r_2)$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and increasing in both arguments such that, for any $\sigma \geq \bar{\sigma}$, $q > 0$, and $f \in C^\alpha(H, w)$ with $(f, \mu) = 0$, we have*

$$\sup_{z\in\mathbb{R}}\bigl|\Delta_\sigma\bigl(t^{-\frac{1}{2}}S_t(f), z\bigr)\bigr| \leq t^{-\frac{1}{4}+\varepsilon}\,h_{\bar{\sigma},\varepsilon}\bigl(|u|, |f|_{w,\alpha}\bigr) + \\ + \sigma^{-4q}\hat{t}^{q(1-4\varepsilon)}\,\mathbb{E}_u\bigl|\hat{t}^{-1}V_{\hat{t}}^2 - \sigma^2\bigr|^{2q}, \tag{2.26}$$

  where $t \geq 1$ and $\hat{t}$ is the integer part of t.

- **(ii)** *There is a non-negative continuous function $h(r_1, r_2)$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and increasing in both arguments such that, for any function $f \in C^\alpha(H, w)$ with $(f, \mu) = 0$ and any $t \geq 1$, we have*

$$\sup_{z\in\mathbb{R}}\bigl|\Delta_0\bigl(t^{-\frac{1}{2}}S_t(f), z\bigr)\bigr| \leq t^{-\frac{1}{4}}\,h\bigl(|u|, |f|_{w,\alpha}\bigr) + \hat{t}^{-\frac{1}{2}}\left(\mathbb{E}_u V_{\hat{t}}^2\right)^{\frac{1}{2}}. \tag{2.27}$$

Proof. We first describe the idea of the proof. Let us fix an arbitrary functional $f \in C^\alpha(H,w)$ and, following a well-known argument (cf. [24, 15]), represent $S_t = S_t(f)$ in the form

$$\begin{aligned} S_t &= M_{\hat t} + \int_{\hat t}^t f(u_s)\,ds - \int_{\hat t}^\infty \mathbb{E}_u\bigl(f(u_s)\,|\,\mathcal{F}_{\hat t}\bigr)\,ds + \int_0^\infty \mathbb{E}_u\bigl(f(u_s)\,|\,\mathcal{F}_0\bigr)\,ds \\ &= M_{\hat t} + \zeta_t - g(u_{\hat t}) + g(u_0), \end{aligned} \tag{2.28}$$

where $\hat t$ is the integer part of t, M_t is defined by (2.24), and

$$\zeta_t := \int_{\hat t}^t f(u_s)\,ds, \tag{2.29}$$

$$g(u) := \int_0^\infty \mathbb{E}_u f(u_s)\,ds = \int_0^\infty \mathfrak{P}_s f(u)\,ds. \tag{2.30}$$

It follows from (2.18) and (2.17) that the function $g(u)$ is well defined and satisfies the inequality

$$|g(u)| \le \bar\gamma\,\rho(|u|)|f|_{w,\alpha}, \quad u \in H. \tag{2.31}$$

Combining (2.28), (2.31), and (2.25), it is not difficult to show that

$$\bigl|t^{-\frac12}S_t - \hat t^{-\frac12}M_{\hat t}\bigr| \le \mathrm{const}\,t^{-\frac14} \quad \text{for } t \gg 1.$$

Therefore, to establish (2.26) and (2.27), it suffices to estimate the rate of convergence of $\hat t^{-\frac12}M_{\hat t}$ to a Gaussian distribution in terms of the conditional variance $V_{\hat t}$. This will be done by applying a result from [13].

The accurate proof is divided into three steps.

Step 1. Let us show that it suffices to establish inequalities (2.26) and (2.27) with $t^{-\frac12}S_t$ replaced by $\hat t^{-\frac12}M_{\hat t}$. We shall need the following simple lemma, whose proof is given in the Appendix (see Section 4.1).

Lemma 2.9. *Let ξ and η be real-valued random variables defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Then for any $\sigma \ge 0$ and $\varepsilon > 0$ we have*

$$\sup_{z\in\mathbb{R}}\bigl|\Delta_\sigma(\xi,z)\bigr| \le \sup_{z\in\mathbb{R}}\bigl|\Delta_\sigma(\eta,z)\bigr| + \mathbb{P}\{|\xi-\eta|>\varepsilon\} + c_\sigma\varepsilon, \tag{2.32}$$

where $c_\sigma = (\sigma\sqrt{2\pi})^{-1}$ for $\sigma > 0$ and $c_0 = 2$.

We wish to apply Lemma 2.9 with $\xi = t^{-\frac12}S_t$ and $\eta = \hat t^{-\frac12}M_{\hat t}$. To this end, we first estimate $\mathbb{E}_u|\xi-\eta|$. It follows from (2.28) that

$$\bigl|t^{-\frac12}S_t - \hat t^{-\frac12}M_{\hat t}\bigr| \le \bigl(t\,\hat t^{\frac12}\bigr)^{-1}|S_t| + \hat t^{-\frac12}\bigl|\zeta_t - g(u_{\hat t}) + g(u_0)\bigr|. \tag{2.33}$$

Inequality (2.25) implies that the conditions of Proposition 2.6 are satisfied for any integer $p \ge 1$. In particular, taking $p = 1$, we obtain

$$\mathbb{E}_u|S_t| \le \bigl(\mathbb{E}_u S_t^2\bigr)^{\frac12} \le C_1 t^{\frac12}|f|_{w,\alpha}\varkappa^{\frac12}(|u|). \tag{2.34}$$

Using (2.17), (2.18) and the first inequality in (2.19), we derive

$$\begin{aligned}\mathbb{E}_u|\zeta_t| &\le \int_{\hat t}^{t} \big|\mathbb{E}_u|f(u_s)| - (|f|,\mu)\big|\,ds + (|f|,\mu)\\ &\le \bar\gamma\,\rho(|u|)|f|_{w,\alpha} + (|f|,\mu) \le C_2|f|_{w,\alpha}\rho(|u|). \end{aligned}\tag{2.35}$$

Furthermore, inequalities (2.31) and (2.25) imply that

$$\mathbb{E}_u\big|g(u_{\hat t}) - g(u_0)\big| \le \bar\gamma\,|f|_{w,\alpha}\mathbb{E}_u\big(\rho(|u_0|)+\rho(|u_{\hat t}|)\big) \le C_3\bar\gamma\,|f|_{w,\alpha}\varkappa(|u|). \tag{2.36}$$

Combining (2.33) – (2.36), we see that

$$\mathbb{E}_u\big|t^{-\frac12}S_t - \hat t^{-\frac12}M_{\hat t}\big| \le t^{-\frac12}d_1(|u|)|f|_{w,\alpha}, \quad t\ge 1.$$

Here and henceforth, we denote by $d_i(r)$, $r\ge 0$, non-negative continuous functions. Applying now the Chebyshev inequality and using (2.32) with $\varepsilon = t^{-\frac14}$, we see that

$$\sup_{z\in\mathbb{R}}\big|\Delta_\sigma(t^{-1}S_t,z)\big| \le \sup_{z\in\mathbb{R}}\big|\Delta_\sigma(\hat t^{-1}M_{\hat t},z)\big| + t^{-\frac14}d_2(|u|)\,\big(|f|_{w,\alpha}+1\big).$$

This implies the assertion formulated in the beginning of Step 1.

Step 2. We now prove (2.26) with $t^{-\frac12}S_t$ replaced by $\hat t^{-\frac12}M_{\hat t}$. We shall need the following proposition, which can be obtained by a slight modification of the proof of Theorem 3.7 in [13] (see Section 4.2 of the Appendix).

Proposition 2.10. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let M_k, $0\le k\le n$, be a zero-mean martingale with respect to σ-algebras $\mathcal{F}_k$. Suppose that, for some positive constants β and B, we have*

$$\mathbb{E}\,\exp\big(|M_k - M_{k-1}|^\beta\big) \le B, \quad 1\le k\le n. \tag{2.37}$$

Then for any $\bar\sigma>0$ and $\varepsilon\in(0,\frac14)$ there is a constant $A_\varepsilon(\bar\sigma)>0$ depending on β and B such that, for any $q>0$, we have

$$\sup_{z\in\mathbb{R}}\big|\Delta_\sigma(n^{-\frac12}M_n,z)\big| \le A_\varepsilon(\bar\sigma)n^{-\frac14+\varepsilon} + \sigma^{-4q}n^{q(1-4\varepsilon)}\,\mathbb{E}\,\big|n^{-1}V_n^2-\sigma^2\big|^{2q}, \tag{2.38}$$

where $\sigma\ge\bar\sigma$ is an arbitrary constant and V_n^2 is the conditional variance:

$$V_n^2 = \sum_{k=1}^{n}\mathbb{E}\big((M_k-M_{k-1})^2\,|\,\mathcal{F}_{k-1}\big).$$

Let us fix $t\ge 1$ and $\varepsilon\in(0,\frac14)$. We wish to apply Proposition 2.10 with $n=\hat t$ to the zero-mean martingale M_k defined by (2.24). To this end, we shall show that condition (2.37) is satisfied with $\beta=\frac{a}{2}$ (see (2.25)) and some constant $B=B(|f|_{w,\alpha})$. This will imply the required inequality (2.26).

It follows from (2.24) and the Markov property that

$$M_k - M_{k-1} = \int_{k-1}^{k} f(u_s)\, ds + g(u_k) - g(u_{k-1}). \tag{2.39}$$

Therefore, using the second inequality in (2.19) and (2.31), we obtain

$$|M_k - M_{k-1}| \le C_5 |f|_{w,\alpha} \Big(1 + \sup_{s\in[k-1,k]} \rho(|u_s|)\Big),$$

whence it follows that

$$\exp\big(|M_k - M_{k-1}|^{\frac{a}{2}}\big) \le C_6 \exp\big(C_6 |f|_{w,\alpha}^a\big) \sup_{s\in[k-1,k]} \exp\{\rho^a(|u_s|)\},$$

Inequality (2.37) follows now from (2.25).

Step 3. To prove inequality (2.27) with $t^{-\frac{1}{2}} S_t$ replaced by $\hat{t}^{-\frac{1}{2}} M_{\hat{t}}$, it suffices to show that

$$\sup_{z\in\mathbb{R}} \big|\Delta_0(k^{-\frac{1}{2}} M_k, z)\big| \le \big(k^{-1} \mathbb{E}_u V_k^2\big)^{\frac{1}{2}}, \quad k \ge 1. \tag{2.40}$$

It is a matter of direct verification to show that $\mathbb{E}_u M_k^2 = \mathbb{E}_u V_k^2$. Therefore, by the Chebyshev inequality, for any $z > 0$ we have

$$\mathbb{P}_u\big\{|k^{-\frac{1}{2}} M_k| \ge z\big\} \le z^{-1} k^{-\frac{1}{2}}\, \mathbb{E}_u |M_k| \le z^{-1} k^{-\frac{1}{2}} \left(\mathbb{E}_u V_k^2\right)^{\frac{1}{2}}.$$

To prove (2.40), it remains to note that

$$\big|\Delta_0(k^{-\frac{1}{2}} M_k, z)\big| \le \big(|z| \wedge 1\big) \mathbb{P}_u\big\{|k^{-\frac{1}{2}} M_k| \ge |z|\big\}.$$

The proof of Theorem 2.8 is complete. □

3 Proof of the main results

We shall confine ourselves to the case of the Navier–Stokes system perturbed by the random force (0.4). The proof of the results on the discrete-time RDS (1.1) is similar and technically much simpler.

3.1 Proof of Theorem 1.4

We wish to apply Corollary 2.4. To this end, we shall show that Conditions 2.1 and 2.2 are satisfied for any $p > 1/2$, $q < 1/2$ and $s = q^{-1}$, and that we can take

$$\varkappa_p(r) = a_p e^{dr^2}, \quad \tau_q(r) = b_q e^{dr^2}, \tag{3.1}$$

where a_p and b_q are some positive constants. Once this claim is established, Corollary 2.4 will imply all required assertions.

Step 1: Checking Condition 2.1. Let us fix a constant $\alpha \in (0,1]$ and a weight function $w \in \mathcal{W}$ satisfying (1.15). As was mentioned in the beginning of Section 1.2, the Markov family $(u_t, \mathbb{P}_u)$ associated to the problem (0.3), (0.4) has a unique stationary measure $\mu \in \mathcal{P}(H)$, and there are positive constants d, C, and β such that (1.14) holds for any $u \in H$ and $f \in C^\alpha(H, v_\delta)$ with $\delta \in (0, d]$. Furthermore, it is shown in [3, 34] that [3]

$$\mathbb{E}_u e^{d|u_t|^2} \le C_1\, e^{d|u|^2}, \quad t \ge 0, \quad u \in H. \tag{3.2}$$

Here and henceforth, we denote by C_i positive constants not depending on u and t.

What has been said implies that the family $(u_t, \mathbb{P}_u)$ is uniformly mixing for the class $C^\alpha(H, w)$ in the sense of Definition 2.5. Indeed, it follows from (1.15) that the space $C^\alpha(H, w)$ is continuously embedded into $C^\alpha(H, v_\delta)$ for any $\delta > 0$. Hence, by (1.14), for every $\delta > 0$ there is a constant $A_\delta > 0$ such that for any functional $f \in C^\alpha(H, w)$ inequality (2.18) holds with

$$\gamma(t) = A_\delta e^{-\beta t}, \quad \rho(r) = v_\delta(r). \tag{3.3}$$

It is clear that (2.17) also holds.

We now fix an arbitrary integer $p \ge 1$ and note that the conditions of Proposition 2.6 are satisfied. Indeed, taking $\delta = \frac{d}{2p}$ in (3.3) and using (3.2), we see that

$$\mathbb{E}_u \rho^{2p}(|u_t|) = E_u e^{d|u_t|^2} \le C\, e^{d|u|^2} \quad \text{for } t \ge 0.$$

Thus, Proposition 2.6 applies, and we conclude that Condition 2.1 is satisfied for any $p > 1/2$ with the function $\varkappa_p(r)$ given in (3.1).

Step 2: Checking Condition 2.2. We shall show that Condition 2.2 is satisfied for any positive constants s and q. Inequality (1.15) implies that, for any $\delta > 0$, we have

$$w(r) \le C_\delta e^{\delta r^2} \quad \text{for all } r \ge 0,$$

where $C_\delta \ge 1$ is a constant depending on δ. It follows that

$$w^{-1}(r)^2 \ge \delta^{-1} \log(r/C_\delta) \ge (2\delta)^{-1} \log r \quad \text{for } r \ge w(0) \vee C_\delta^2.$$

Hence, the required assertion will be established if for any $q > 0$ and $s > 0$ we find a constant $\delta > 0$ such that, for any $u \in H$, the random variable

$$M(\omega) = \min\{T \ge 0 : |u_t(\omega)|^2 \le \tfrac{q}{2\delta} \log t \text{ for } t \ge T\}$$

is $\mathbb{P}_u$-a.s. finite, and

$$\mathbb{E}_u M^s \le b_{q,s} e^{d|u|^2}. \tag{3.4}$$

To this end, let us set

$$U_k = \sup_{t \in [k, k+1]} |u_t|^2, \tag{3.5}$$

[3] We can assume that the constant d in inequalities (1.14) and (3.2) is the same.

and show that (cf. (3.2))

$$\mathbb{E}_u e^{\nu U_k} \le C_2 e^{2\nu|u|^2}, \quad k \ge 0, \quad u \in H, \tag{3.6}$$

where $\nu \le \frac{d}{2}$ is a positive constant not depending on u and t. Indeed, it follows from the Markov property that

$$\mathbb{E}_u e^{\nu U_k} = \mathbb{E}_u \big\{ \mathbb{E}_u \big(e^{\nu U_k} \mid \mathcal{F}_k \big) \big\} = \mathbb{E}_u \big(\mathbb{E}_{u_k} e^{\nu U_0} \big). \tag{3.7}$$

Now note that, applying the Hölder inequality, we can show that (3.2) holds with d replaced by any $d' \in [0, d]$. Combining inequalities (3.7) and (3.2) (with $d = 2\nu$), we see that it suffices to prove (3.6) for $k = 0$.

As is shown in [21] (see Lemma 2.3 with $T = 1$), there is a constant $c \in (0, d]$ such that

$$\mathbb{P}_u \big\{ U_0 - B_0 - |u|^2 \ge z \big\} \le e^{-cz}, \quad z \in \mathbb{R}, \tag{3.8}$$

where B_0 is defined in (1.2). Since $U_0 \ge 0$, it follows that

$$\begin{aligned} \mathbb{E}_u e^{\nu U_0} &= \int_0^\infty \mathbb{P}_u \big\{ e^{\nu U_0} > z \big\}\, dz = 1 + \int_1^\infty \mathbb{P}_u \big\{ U_0 > \nu^{-1} \log z \big\}\, dz \\ &\le 1 + e^{c(B_0 + |u|^2)} \int_1^\infty \exp\Big\{ -\frac{c \log z}{\nu} \Big\} dz. \end{aligned} \tag{3.9}$$

Thus, if $\nu = \frac{c}{2}$, then the right-hand side of (3.9) is equal to $1 + e^{2\nu(B_0 + |u|^2)}$. This completes the proof of (3.6).

We now prove that the random constant M is $\mathbb{P}_u$-a.s. finite and that (3.4) holds. Let us fix an arbitrary $q > 0$ and define a random integer by the formula

$$K(\omega) = \big\{ m \ge 0 : U_k \le \tfrac{q}{2\delta} \log k \text{ for } k \ge m + 1 \big\}.$$

Combining the Chebyshev inequality with (3.6), for $2\delta < q\nu$ we derive

$$\sum_{k=1}^\infty \mathbb{P}_u \Big\{ U_k > \frac{q}{2\delta} \log k \Big\} \le \sum_{k=1}^\infty k^{-\frac{q\nu}{2\delta}} \mathbb{E}_u e^{\nu U_k} \le C_3 e^{2\nu|u|^2}.$$

Hence, by the Borel–Cantelli lemma, we have $\mathbb{P}_u\{K < \infty\} = 1$ for any $u \in H$. Moreover, if $\delta > 0$ is so small that $2\delta(s+1) < q\nu$, then

$$\begin{aligned} \mathbb{E}_u K^s &= \sum_{k=1}^\infty k \mathbb{P}_u \{ K = k \} \le \sum_{k=1}^\infty k \mathbb{P}_u \Big\{ U_k > \frac{q}{2\delta} \log k \Big\} \\ &\le \sum_{k=1}^\infty k^{s - \frac{q\nu}{2\delta}} \mathbb{E}_u e^{\nu U_k} \le C_4 e^{2\nu|u|^2}. \end{aligned}$$

It remains to note that $K \ge M$ for all $\omega \in \Omega$, and therefore the above inequality implies (3.4). The proof of Theorem 1.4 is complete.

3.2 Proof of Theorem 1.5

We first describe the scheme of the proof. The Markov family $(u_t, \mathbb{P}_u)$ associated with the NS system (0.3), (0.4) is uniformly mixing for the class $C^\alpha(H, w_p)$ for any $\alpha \in (0,1]$ and $p \geq 1$. Moreover, using (3.3), it is not difficult to prove that (2.25) is also valid. Therefore, by Theorem 2.8, inequalities (2.26) and (2.27) hold. To obtain the required assertion, we show that

$$V_k^2 = \sum_{l=0}^{k-1} \varphi(u_l), \tag{3.10}$$

where $\varphi \in C^\gamma(H, v_\delta)$ for some sufficiently small $\gamma \in (0,1]$ and $\delta > 0$. This will imply that $k^{-1}V_k^2$ converges (in an appropriate sense) to the mean value (φ, μ) as $k \to \infty$. It turns out that $(\varphi, \mu) = \sigma_f^2$. Therefore, if $\sigma_f > 0$, then taking $\sigma = \sigma_f$ in (2.26) and using Proposition 2.6, we prove assertion (i) of the theorem. If $\sigma_f = 0$, then a similar argument enables one to establish assertion (ii).

The accurate proof is divided into several steps.

Step 1. We first show that the family $(u_t, \mathbb{P}_u)$ is uniformly mixing and that inequality (2.25) holds. To this end, we shall need the following lemma.

Lemma 3.1. *Under the conditions of Theorem 1.5, for any $\alpha \in (0,1]$ and $p > 0$ there are positive constants C, β, and m such that*

$$\left|\mathfrak{P}_t f(u) - (f, \mu)\right| \leq C\, e^{-\beta t} |f|_{w_p,\alpha} (1 + |u|)^m, \quad t \geq 0, \tag{3.11}$$

where $w_p(r) = (1+r)^p$ and $f \in C^\alpha(H, w_p)$ is an arbitrary functional.

Inequality (3.11) is an analogue of (1.14) for Hölder continuous functionals with polynomial growth at infinity. To prove (3.11), it suffices to repeat the scheme used in [22, Section 3] for deriving (1.14); we shall not dwell on it.

We now show that (2.25) holds for some $a > 0$. Indeed, in view of (3.11), we can take $\rho(r) = (1+r)^m$. Therefore, if $a = \frac{1}{m}$, then

$$\sup_{t \in [k,k+1]} \exp\bigl(\rho^a(|u_t|)\bigr) \leq \exp\bigl(1 + U_k^{\frac{1}{2}}\bigr), \quad k \geq 0, \tag{3.12}$$

where the random variable U_k is defined by (3.5). Hence, using the Markov property, we obtain (cf. (3.7))

$$\mathbb{E}_u\Bigl(\sup_{t \in [k,k+1]} \exp\{\rho^a(|u_t|)\}\Bigr) \leq \mathbb{E}_u \exp\bigl(1 + U_k^{\frac{1}{2}}\bigr) = \mathbb{E}_u\bigl(\mathbb{E}_{u_k} \exp\bigl(1 + U_0^{\frac{1}{2}}\bigr)\bigr). \tag{3.13}$$

Furthermore, it follows from (3.6) that

$$\mathbb{E}_u \exp\bigl(1 + U_0^{\frac{1}{2}}\bigr) \leq C_5 e^{d|u|^2}.$$

Combining this estimate with (3.13), we derive

$$\mathbb{E}_u\Bigl(\sup_{t \in [k,k+1]} \exp\{\rho^a(|u_t|)\}\Bigr) \leq C_6 \mathbb{E}_u e^{d|u_k|^2}, \quad k \geq 0, \quad u \in H. \tag{3.14}$$

Using (3.2) to estimate the right-hand side of (3.14), we obtain the required inequality (2.25), in which $\varkappa(r) = e^{dr^2}$. Thus, the conditions of Theorem 2.8 are fulfilled, and statements (i) and (ii) take place.

Step 2. Our next goal is to estimate the expectations on the right-hand sides of (2.26) and (2.27). To this end, we first establish (3.10). Namely, we shall show that there is a functional $\varphi\colon H \to \mathbb{R}$ belonging $C^\gamma(H, v_\delta)$ for any $\delta > 0$ and a sufficiently small $\gamma = \gamma(\delta, \alpha, p) \in (0, 1]$ such that, for any $u \in H$, relation (3.10) holds $\mathbb{P}_u$-almost surely.

Let us recall that (see (2.39), (2.29), and (2.30))

$$M_l - M_{l-1} = \zeta_l^- + g(u_l) - g(u_{l-1}), \quad l = 1, \dots, k,$$

where $\zeta_l^- = \int_{l-1}^l f(u_s)\,ds$. Therefore, by the Markov property, for any $u \in H$ we have

$$\mathbb{E}_u\big((M_l - M_{l-1})^2 \,|\, \mathcal{F}_{l-1}\big) = \varphi(u_{l-1}) \quad \mathbb{P}_u\text{-a.s.}, \tag{3.15}$$

where we set

$$\varphi(u) = \mathbb{E}_u M_1^2 = \mathbb{E}_u\big(\zeta_1^- + g(u_1) - g(u_0)\big)^2. \tag{3.16}$$

Thus, relation (3.10) holds $\mathbb{P}_u$-a.s., and it remains to show that $\varphi \in C^\gamma(H, v_\delta)$.

We shall need the following lemma, whose proof is given in the Appendix (see Section 4.3).

Lemma 3.2. *Under the conditions of Theorem 1.5, the Markov semigroup $\mathfrak{P}_t$ associated with the problem* (0.3), (0.4) *possesses the following properties:*

(i) *For any $\alpha \in (0, 1]$, $p > 0$, and $\delta > 0$ there is $\gamma > 0$ such that the operator*

$$\mathfrak{P}_t : C^\alpha(H, w_p) \to C^\gamma(H, v_\delta), \quad t \geq 0, \tag{3.17}$$

is continuous, and its norm is uniformly bounded on any interval $[0, T]$. Moreover, if $f \in C^\alpha(H, w_p)$ and $(f, \mu) = 0$, then the function $g(u)$ defined by (1.18) *belongs to $C^\gamma(H, v_\delta)$, and its norm can be estimated by* $\mathrm{const}\,|f|_{w_p,\alpha}$.

(ii) *For any $\alpha \in (0, 1]$ and sufficiently small $\nu > 0$ there are $\gamma > 0$ and $\delta > 0$ such that the operator*

$$\mathfrak{P}_t : C^\alpha(H, v_\nu) \to C^\gamma(H, v_\delta), \quad t \geq 0, \tag{3.18}$$

is continuous, and its norm is uniformly bounded on any interval $[0, T]$. Moreover, the constants γ and δ can be chosen in such a way that $\delta \to 0$ as $\nu \to 0$.

It follows from relation (3.16) that

$$\varphi(u) = \mathbb{E}_u(\zeta_1^-)^2 + \mathbb{E}_u g^2(u_1) + \mathbb{E}_u g^2(u_0) + \\ + 2\,\mathbb{E}_u\big(\zeta_1^- g(u_1)\big) - 2\,\mathbb{E}_u\big(\zeta_1^- g(u_0)\big) - 2\,\mathbb{E}_u\big(g(u_1)g(u_0)\big). \tag{3.19}$$

By Lemma 3.2, for any $\delta > 0$ there is $\gamma \in (0,1]$ such that the functionals

$$\mathbb{E}_u g^2(u_0) = g^2(u), \qquad \mathbb{E}_u g^2(u_1) = \mathfrak{P}_1 g^2(u),$$
$$\mathbb{E}_u\bigl(g(u_0)g(u_1)\bigr) = g(u)\mathfrak{P}_1 g(u), \qquad \mathbb{E}_u\bigl(\zeta_1^- g(u_0)\bigr) = g(u)\int_0^1 \mathfrak{P}_s f(u)\,ds$$

belong to the space $C^\gamma(H, v_\delta)$. Furthermore, using the Markov property, we write

$$\begin{aligned}\mathbb{E}_u\left(\int_0^1 f(u_s)\,ds\right)^2 &= \int_0^1\int_0^1 \mathbb{E}_u\left(f(u_s)f(u_t)\right)dsdt\\ &= 2\int_0^1\int_0^t \mathfrak{P}_s\bigl(f\mathfrak{P}_{t-s}f\bigr)(u)\,dsdt,\\ \mathbb{E}_u\left(g(u_1)\int_0^1 f(u_s)\,ds\right) &= \int_0^1 \mathfrak{P}_s\bigl(f\mathfrak{P}_{1-s}g\bigr)(u)\,ds.\end{aligned}$$

Applying again Lemma 3.2, we see that these two functions are also elements of $C^\gamma(H, v_\delta)$ for any $\delta > 0$ and sufficiently small $\gamma > 0$. What has been said implies that for any $\delta > 0$ there is a constant $\gamma > 0$ and a non-negative increasing function $d_\delta(r)$ defined for $r \ge 0$ such that

$$|\varphi|_{v_\delta,\gamma} \le d_\delta\bigl(|f|_{w_p,\alpha}\bigr). \tag{3.20}$$

Step 3. We now estimate the second terms in the right-hand sides of (2.26) and (2.27). To this end, we shall use Proposition 2.6.

Let us fix an arbitrary $\varepsilon > 0$ and choose an integer $q \ge 1$ and a constant $\delta > 0$ such that

$$16q\varepsilon > 1, \quad 2q\delta \le d, \tag{3.21}$$

where $d > 0$ is so small that (1.14) holds for $\delta \in (0, d]$ and inequality (3.2) is also valid. As was shown in Step 3, we can find $\gamma \in (0,1]$ such that $\varphi \in C^\gamma(H, v_\delta)$.

We claim that the conditions of the discrete analogue of Proposition 2.6 are satisfied with $\alpha = \gamma$, $w = v_\delta$ and $p = q$. Indeed, the second inequality in (3.21) implies that $\delta \le d$. Therefore, by (1.14), the family $(u_t, \mathbb{P}_u)$ is uniformly mixing for the class $C^\gamma(H, v_\delta)$ in the sense of Definition 2.5, and we can take $\rho(r) = v_\delta(r)$. Furthermore, it follows from (3.2) that (2.20) holds with $\sigma(r) = e^{dr^2}$. Hence, by Proposition 2.6 (see Remark 2.7), for any $u \in H$ and $k \ge 1$, we have

$$\begin{aligned}\mathbb{E}_u\bigl|k^{-1}V_k^2 - (\varphi,\mu)\bigr|^{2q} &\le C_7\,|\varphi|_{v_\delta,\varepsilon}^{2q} e^{d|u|^2}\,k^{-q}\\ &\le C_7\,d_\delta\bigl(|f|_{w_p,\alpha}\bigr)^{2q} e^{d|u|^2}\,k^{-q},\end{aligned} \tag{3.22}$$

where $C_7 > 0$ does not depend on u and k, and we used inequality (3.20).

Suppose now that we have established the relation

$$(\varphi,\mu) = \sigma_f^2, \tag{3.23}$$

where σ_f is defined in (1.19). If $\sigma_f \geq \bar{\sigma}$, then setting $k = \hat{t}$, substituting (3.22) and (3.23) into (2.26), and taking into account the first inequality in (3.21), we obtain (1.20). Similarly, if $\sigma_f = 0$, then applying the Cauchy inequality to estimate the second term on the right-hand side of (2.27) and using inequality (3.22) in which $(\varphi, \mu) = 0$, $k = \hat{t}$, and $q = 1$, we get (1.21). Thus, Theorem 1.5 will be established if we prove (3.23).

Step 4. Let us recall that $\mathbb{E}_\mu$ stands for the mean value corresponding to the stationary measure (see (1.6)). In view of (3.16), we have

$$(\varphi, \mu) = \mathbb{E}_\mu\big(\zeta_1^- + g(u_1) - g(u_0)\big)^2. \tag{3.24}$$

On multiplying out the brackets, we represent the right-hand side of (3.24) as a sum of six terms (cf. relation (3.19) with $\mathbb{E}_u$ replaced by $\mathbb{E}_\mu$). Let us calculate each of them. Using the Markov property and the stationarity of μ, we derive

$$\begin{aligned}\mathbb{E}_\mu\big(\zeta_1^-\big)^2 &= \int_0^1\int_0^1 \mathbb{E}_\mu\big(f(u_s)f(u_t)\big)\,dsdt = 2\int_0^1\int_0^t \mathbb{E}_\mu\big(f(u_s)\mathbb{E}_\mu(f(u_t)\,|\,\mathcal{F}_s)\big)\,dsdt\\ &= 2\int_0^1\int_0^t \mathbb{E}_\mu\big(f(u_s)\mathfrak{P}_{t-s}f(u_s)\big)\,dsdt = 2\int_0^1\int_0^t (f\mathfrak{P}_{t-s}f,\mu)\,dsdt\\ &= 2\int_0^1 (1-s)(f\mathfrak{P}_s f,\mu)\,ds.\end{aligned} \tag{3.25}$$

Similar arguments combined with the relation

$$\mathfrak{P}_s g = \int_0^\infty \mathfrak{P}_{s+t} f\,dt = g - \int_0^s \mathfrak{P}_t f\,dt$$

enable one to show that

$$\begin{aligned}\mathbb{E}_\mu g^2(u_0) &= \mathbb{E}_\mu g^2(u_1) = (g^2,\mu),\\ \mathbb{E}_\mu\big(g(u_0)g(u_1)\big) &= (g\mathfrak{P}_1 g,\mu) = (g^2,\mu) - \int_0^1 (g\mathfrak{P}_s f,\mu)\,ds,\\ \mathbb{E}_\mu\big(\zeta_1^- g(u_0)\big) &= \int_0^1 (g\mathfrak{P}_s f,\mu)\,ds,\\ \mathbb{E}_\mu\big(\zeta_1^- g(u_1)\big) &= \int_0^1 (f\mathfrak{P}_s g,\mu)\,ds = (fg,\mu) - \int_0^1 (1-s)(f\mathfrak{P}_s f,\mu)\,ds.\end{aligned}$$

Substituting these relations together with (3.25) into (3.24), we derive

$$(\varphi,\mu) = 2(fg,\mu).$$

Recalling the definition of σ_f (see (1.19)), we see that this relation coincides with (3.23). The proof of Theorem 1.5 is complete.

4 Appendix

4.1 Proof of Lemma 2.9

We confine ourselves to the case $\sigma > 0$, since the proof for $\sigma = 0$ is similar. For any $\varepsilon > 0$ and $z \in \mathbb{R}$, we have

$$F_\xi(z) \le \mathbb{P}\{\xi \le z, |\xi - \eta| \le \varepsilon\} + \mathbb{P}\{|\xi - \eta| > \varepsilon\} \le F_\eta(z+\varepsilon) + \mathbb{P}\{|\xi - \eta| > \varepsilon\}.$$

It follows that

$$\begin{aligned}\Delta_\sigma(\xi, z) &\le \Delta_\sigma(\eta, z+\varepsilon) + \Phi_\sigma(z+\varepsilon) - \Phi_\sigma(z) + \mathbb{P}\{|\xi - \eta| > \varepsilon\} \\ &\le \Delta_\sigma(\eta, z+\varepsilon) + \frac{\varepsilon}{\sigma\sqrt{2\pi}} + \mathbb{P}\{|\xi - \eta| > \varepsilon\}. \qquad (4.1)\end{aligned}$$

Interchanging the roles of ξ and η and replacing z by $z - \varepsilon$, we obtain

$$\Delta_\sigma(\xi, z) \ge \Delta_\sigma(\eta, z-\varepsilon) - \frac{\varepsilon}{\sigma\sqrt{2\pi}} - \mathbb{P}\{|\xi - \eta| > \varepsilon\}. \qquad (4.2)$$

Combining (4.1) and (4.2), we derive (2.32).

4.2 Proof of Proposition 2.10

Let us fix an arbitrary $\varepsilon > 0$. As is shown in [13] (see inequality (3.74) with $\Delta = n^{-\frac{1}{2}+2\varepsilon}$ and $p = \frac{1-4\varepsilon}{8\varepsilon}$),

$$\sup_{z\in\mathbb{R}} \big|\Delta_1(n^{-\frac{1}{2}} M_n, z)\big| \le A_\varepsilon\, n^{-\frac{1}{4}+\varepsilon} + \mathbb{P}\big\{|n^{-1}V_n^2 - 1| > n^{-\frac{1}{2}+2\varepsilon}\big\}, \qquad (4.3)$$

where $A_\varepsilon > 0$ is a constant depending only on B and β. Applying the Chebyshev inequality to the second term on the right-hand side of (4.3), we obtain (2.38) with $\sigma = 1$.

To prove (2.38) for an arbitrary $\sigma > 0$, let us note that

$$\Delta_\sigma(n^{-\frac{1}{2}} M_n, z) = \Delta_1(n^{-\frac{1}{2}} M_n(\sigma), \sigma^{-1} z),$$

where we set $M_n(\sigma) = M_n/\sigma$. It follows from the estimate

$$|M_n(\sigma)|^{\frac{\beta}{2}} \le |M_n|^\beta + \frac{1}{4\sigma^\beta}$$

that the zero-mean martingale $M_n(\sigma)$ satisfies inequality (2.37) with β and B replaced by $\frac{\beta}{2}$ and $B_\sigma := B\exp(\frac{1}{4\sigma^\beta})$, respectively.

Let us fix $\bar\sigma > 0$. The constants B_σ are uniformly bounded for $\sigma \ge \bar\sigma$. Hence, by inequality (2.38) with $\sigma = 1$, there is $A_\varepsilon(\bar\sigma) > 0$ such that

$$\begin{aligned}\sup_{z\in\mathbb{R}} \big|\Delta_\sigma(n^{-\frac{1}{2}} M_n, z)\big| &= \sup_{z\in\mathbb{R}} \big|\Delta_1(n^{-\frac{1}{2}} M_n(\sigma), \sigma^{-1} z)\big| \\ &\le A_\varepsilon(\bar\sigma) n^{-\frac{1}{4}+\varepsilon} + n^{q(1-4\varepsilon)}\mathbb{E}\left|n^{-1}V_n^2(\sigma) - 1\right|^{2q},\end{aligned}$$

where $V_n^2(\sigma)$ is the conditional variance for $M_n(\sigma)$. It remains to note that $V_n^2(\sigma) = \sigma^{-2} V_n^2$. The proof is complete.

4.3 Proof of Lemma 3.2

We shall confine ourselves to the proof of (i), since assertion (ii) can be established using similar ideas.

Let us fix arbitrary constants $\alpha \in (0,1]$, $p > 0$, $\delta > 0$ and a functional $f \in C^\alpha(H, w_p)$ with norm $|f|_{w_p,\alpha} \le 1$. The continuity of operator (3.17) and uniform boundedness of its norm will be established if we show that

$$\big|\mathfrak{P}_t f(u)\big| \le C\, e^{\delta|u|^2}, \tag{4.4}$$

$$\big|\mathfrak{P}_t f(u) - \mathfrak{P}_t f(v)\big| \le C\,|u-v|^\gamma\, e^{bt+\delta(|u|^2+|v|^2)}, \tag{4.5}$$

where $u, v \in H$, $t \ge 0$, and γ, b, C are some positive constants depending only on α, p, and δ. Inequality (4.4) follows immediately from (3.2). To prove (4.5), let us denote by u_t and v_t the solutions of the problem (0.3), (0.4) that correspond to the initial functions u and v, respectively. Then the difference $u_t - v_t$ satisfies the inequality (see [38])

$$|u_t - v_t| \le |u-v|\, \exp\Big(C_1 \int_0^t \|u_s\|^2 ds\Big), \quad t \ge 0, \tag{4.6}$$

where $\|\cdot\|$ denotes the H^1 norm. Now note that, for any $\gamma \in (0,\alpha]$, we have

$$\begin{aligned} \big|f(u_t) - f(v_t)\big| &\le |u_t - v_t|^\alpha \big(w_p(|u_t|) + w_p(|v_t|)\big) \\ &\le C_2 |u_t - v_t|^\gamma \big(1 + |u_t| + |v_t|\big)^{p+1}. \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7), we derive

$$\begin{aligned} \big|\mathfrak{P}_t f(u) - \mathfrak{P}_t f(v)\big| &\le \mathbb{E}\,\big|f(u_t) - f(v_t)\big| \\ &\le C_3\,|u-v|^\gamma\, \mathbb{E}\exp\Big(\gamma C_1 \int_0^t \|u_s\|^2 ds + \gamma(|u_t|^2 + |v_t|^2)\Big). \end{aligned} \tag{4.8}$$

It follows from Lemma 2.3 in [21] that

$$\mathbb{P}_u\Big\{|u_t|^2 + \int_0^t \|u_s\|^2 ds - B_0 t - |u|^2 \ge z\Big\} \le e^{-cz} \quad \text{for all } t, z \in \mathbb{R},$$

where $c > 0$ does not depend on t, z and $u \in H$. Therefore, if $\gamma > 0$ is sufficiently small, then the expectation on the right-hand side of (4.8) does not exceed $e^{bt+\delta(|u|^2+|v|^2)}$ (cf. (3.9)). This completes the proof of (4.5).

We now fix a functional $f \in C^\alpha(H, w_p)$ such that $|f|_{w_p,\alpha} \le 1$ and $(f, \mu) = 0$ and consider the function $g(u)$. Inequalities (2.31) and (3.11) imply that

$$|g(u)| \le C_4\big(1 + |u|\big)^m, \quad u \in H. \tag{4.9}$$

Furthermore, it follows from (3.11) and (4.5) that, for any $u, v \in H$ and $T > 0$, we have

$$\begin{aligned} \big|g(u) - g(v)\big| &\le \int_0^T \big|\mathfrak{P}_t f(u) - \mathfrak{P}_t f(v)\big|\, dt + \int_T^\infty \big(|\mathfrak{P}_t f(u)| + |\mathfrak{P}_t f(v)|\big)\, dt \\ &\le C_5\,|u-v|^\gamma\, e^{cT+\delta(|u|^2+|v|^2)} + C_6 e^{-\beta T}(1 + |u| + |v|)^m. \end{aligned} \tag{4.10}$$

Minimizing the right-hand side of (4.10) with respect to T, we obtain

$$\big|g(u) - g(v)\big| \le C_7\, |u - v|^{\hat{\gamma}} e^{\delta(|u|^2 + |v|^2)}.$$

where $\hat{\gamma} = \frac{\beta\gamma}{\beta + c}$. This completes the proof of assertion (i).

References

[1] E. Bolthausen, The Berry-Esseén theorem for strongly mixing Harris recurrent Markov chains, *Z. Wahrsch. Verw. Gebiete* **60** (1982), no. 3, 283–289.

[2] E. Bolthausen, Exact convergence rates in some martingale central limit theorems, *Ann. Probab.* **10** (1982), no. 3, 672–688.

[3] J. Bricmont, A. Kupiainen, and R. Lefevere, Probabilistic estimates for the two-dimensional Navier–Stokes equations, *J. Statist. Phys.* **100** (2000), no. 3-4, 743–756.

[4] J. Bricmont, A. Kupiainen, and R. Lefevere, Exponential mixing for the 2D stochastic Navier–Stokes dynamics, *Comm. Math. Phys.* **230** (2002), no. 1, 87–132.

[5] J. Bricmont, Ergodicity and mixing for stochastic partial differential equations, *Proceedings of the International Congress of Mathematicians*, Vol. I (Beijing, 2002), 567–585, Higher Ed. Press, Beijing, 2002.

[6] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite–Dimensional Systems*, Cambridge University Press, Cambridge, 1996.

[7] W. E, J. C. Mattingly, and Ya. G. Sinai, Gibbsian dynamics and ergodicity for the stochastically forced Navier–Stokes equation, *Comm. Math. Phys.* **224** (2001), no. 1, 83–106.

[8] J.-P. Eckmann and M. Hairer, Uniqueness of the invariant measure for a stochastic PDE driven by degenerate noise, *Comm. Math. Phys.* **219** (2001), no. 3, 523–565.

[9] B. Ferrario, Ergodic results for stochastic Navier-Stokes equation, *Stochastics Stochastics Rep.* **60** (1997), no. 3–4, 271–288.

[10] F. Flandoli and B. Maslowski, Ergodicity of the 2-D Navier-Stokes equation under random perturbations, *Comm. Math. Phys.* **172** (1995), no. 1, 119–141.

[11] G. Gallavotti, *Foundations of Fluid Dynamics*, Springer-Verlag, Berlin, 2002.

[12] M. Hairer, Exponential mixing properties of stochastic PDE's through asymptotic coupling, *Probab. Theory Related Fields* **124** (2002), no. 3, 345–380.

[13] P. Hall and C. C. Heyde, *Martingale Limit Theory and Its Application*, Academic Press, New York–London, 1980.

[14] R. Z. Hasminskiĭ, Ergodic properties of recurrent diffusion processes and stabilization of the solution of the Cauchy problem for parabolic equations, *Teor. Verojatnost. i Primenen.* **5** (1960), 196–214.

[15] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin, 1987.

[16] S. Kuksin, On exponential convergence to a stationary measure for nonlinear PDE's, perturbed by random kick-forces, and the turbulence-limit, The M. I. Vishik Moscow PDE seminar, AMS Translations, 2002.

[17] S. Kuksin, Ergodic theorems for 2D statistical hydrodynamics, *Rev. Math. Physics* **14** (2002), no. 6, 585–600.

[18] S. Kuksin and A. Shirikyan, Stochastic dissipative PDE's and Gibbs measures, Comm. Math. Phys. **213** (2000), no. 2, 291–330.

[19] S. Kuksin and A. Shirikyan, A coupling approach to randomly forced nonlinear PDE's. I, *Comm. Math. Phys.* **221** (2001), no. 2, 351–366.

[20] S. Kuksin, A. Piatnitski, and A. Shirikyan, A coupling approach to randomly forced nonlinear PDE's. II, *Comm. Math. Phys.* **230** (2002), no. 1, 81–85.

[21] S. Kuksin and A. Shirikyan, Coupling approach to white-forced nonlinear PDE's, *J. Math. Pures Appl.* **81** (2002), no. 6, 567–602.

[22] S. Kuksin and A. Shirikyan, Some limiting properties of randomly forced 2D Navier–Stokes equations, *Proc. Roy. Soc. Edinburgh Sect. A* **133** (2003), no. 4, 875–891.

[23] D. Landers and L. Rogge, On the rate of convergence in the central limit theorem for Markov-chains, *Z. Wahrsch. Verw. Gebiete* **35** (1976), no. 1, 57–63.

[24] R. S. Liptser and A. N. Shiryayev, *Theory of Martingales*, Kluwer, Dordrecht, 1989.

[25] G. Maruyama and H. Tanaka, Some properties of one-dimensional diffusion processes, *Mem. Fac. Sci. Kyusyu Univ. Ser. A. Math.* **11** (1957), 117–141.

[26] G. Maruyama and H. Tanaka, Ergodic property of N-dimensional recurrent Markov processes, *Mem. Fac. Sci. Kyushu Univ. Ser. A* **13** (1959), 157–172.

[27] N. Masmoudi and L.-S. Young, Ergodic theory of infinite dimensional systems with applications to dissipative parabolic PDEs, *Comm. Math. Phys.* **227** (2002), no. 3, 461–481.

[28] J. C. Mattingly, Ergodicity of 2D Navier-Stokes equations with random forcing and large viscosity, *Comm. Math. Phys.* **206** (1999), no. 2, 273–288.

[29] J. C. Mattingly, Exponential convergence for the stochastically forced Navier–Stokes equations and other partially dissipative dynamics, *Comm. Math. Phys.* **230** (2002), no. 3, 421–462.

[30] J. C. Mattingly, On recent progress for the stochastic Navier Stokes equations, Journes "quations aux Drives Partielles", Exp. No. XI, 52 pp., Univ. Nantes, Nantes, 2003.

[31] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, Springer-Verlag London, London, 1993.

[32] E. Rio, *Théorie Asymptotique des Processus Aléatoires Faiblement Dépendants*, Springer, Berlin–Heidelberg, 2000.

[33] S. Sawyer, Rates of convergence for some functionals in probability, *Ann. Math. Statist.* **43** (1972), 273–284.

[34] A. Shirikyan, Analyticity of solutions for randomly forced two-dimensional Navier-Stokes equations, *Russian Math. Surveys* **57** (2002), no. 4, 785–799.

[35] A. Shirikyan, A version of the law of large numbers and applications, In: *Probabilistic Methods in Fluids*, Proceedings of the Swansea Workshop held on 14 – 19 April 2002, 263–272, I. M. Davies et al (Edt.), World Scientific, New Jersey, 2003.

[36] A. Shirikyan, Exponential mixing for 2D Navier–Stokes equations perturbed by an unbounded noise, *J. Math. Fluid Mech.* **6** (2004), no. 2, 169–193.

[37] A. Shirikyan, Some mathematical problems of statistical hydrodynamics, In: *Proceedings of the International Congress of Mathematical Physics*, Lisbon, 2003, to appear.

[38] R. Temam, *Navier–Stokes Equations. Theory and Numerical Analysis*, North-Holland, Amsterdam–New York–Oxford, 1977.

[39] A. Yu. Veretennikov, Estimates of the mixing rate for stochastic equations, *Theory Probab. Appl.* **32** (1987), no. 2, 273–281.

[40] M. I. Vishik and A. V. Fursikov, *Mathematical Problems in Statistical Hydromechanics*, Kluwer, Dordrecht, 1988.

[41] H. Watanabe and M. Hisao, Ergodic property of recurrent diffusion processes. *J. Math. Soc. Japan* **10** (1958), 272–286.