CLUSTER EXPANSIONS OF SMALL CONTOURS IN ABSTRACT PIROGOV – SINAI MODELS

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Abstract. We develop the Pirogov – Sinai theory for "abstract Pirogov – Sinai models" where configuration is already represented as a system of matching compatible contours. The Hamiltonian is then expressed in terms of contour energies and an additional external field, the latter acting not on single spins but more generally on their "clusters", in the space ("colored" by various "local ground states") outside of contours. This general model is formulated in such a way that i) its general structure does not change when cluster expansion of its "smallest possible" contours resp. contour systems is applied; ii) it includes, after suitable preparation, many interesting lattice spin models, notably the reformulation of the Ising Kac ferromagnet studied in the paper [BZK]. In fact the result of the latter paper – a model with expanded restricted ensembles living outside of suitably defined contours – is the main testing example for the usefulness of this general method, further elaborating and simplifying the method of the paper [ZR].

0. Introduction

This paper arose in connection with the paper [BZK]. The latter paper deals with cluster expansions of long range Kac – Ising models, and develops a parallel approach to that of [LMP]. Originally, the present paper was considered just as "Part II" of [BZK]. However, the subject of this new paper is different and more general. Also, its method is to a large extent complementary to [BZK]. In this new paper, there is even no explicit relation to models with (Kac) long range interactions. Formally, there is no overlap between these two papers except that we repeat here for convenience the final steps of [BZK]. Still, Kac – Ising models are the main new application where the presented form of P.S. theory seems to be useful.

Shortly speaking, we take the result of the previous paper namely the cluster expansion of the "restricted ensembles" (of low density perturbations of a "constant color" regime), under suitably defined contours as a basic reference point here. Then we develop the Pirogov–Sinai

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theory of a generalized version (1.11), (1.23) called here an "abstract Pirogov–Sinai model", of such an "elaborated" (by [BZK]) Kac–Ising model.

Our basic starting concept here is the model (1.11) with "matching compatible" contours and with an "external cluster field". As it was just said it describes, among other situations, the elaboration of the long range Kac Ising model from [BZK]. Now, it is our main aim to develop a suitable technology dealing with such an "abstract Pirogov–Sinai" model.

The basic ideas of this technology are not new, but we believe that some simplifications and improvements (compared to paper [ZR]) appear in this paper. In general, we believe that the method now converges to what was its aim from the very beginning – becoming a standard and very general *method of low temperature cluster expansions*, for quite a general class of lattice spin models with no symmetry in their Hamiltonian.

The assumptions imposed on our starting, "abstract Pirogov Sinai model" have now a very simple logical structure: In addition to the Peierls condition imposed on energies of *contours* of our abstract model we assume also a validity of suitably formulated condition (1.13) on quick decay of the "cluster field"; the latter being a concept greatly generalizing the notion of an external "magnetic" field acting on single spins.

It is instructive to imagine that our "cluster field" is actually a cluster expansion of some *restricted ensemble* of some previous model, in the regions outside contours. Depending on the concrete model this can be either a simple expansion of high temperature type, of some restricted ensemble of "almost ground configurations" (e.g. in the continuous spin models) or something more complicated like the situation analyzed in [BZK]. Or, alternatively it can be also that the considered cluster field is actually a cluster expansion of some low temperature polymer model of "small" contours.

By small contours we will mean the "well behaved" ones, not exhibiting the characteristic difficulties which large contours cause in nonsymmetric situations of P.S. theory. Finally, one should keep in mind the most general possibility that the considered "starting" cluster field actually appeared as a result of a *combination* of both such expansion procedures.

So, the "preparation" of some original (spin) model to an abstract Pirogov Sinai model of the type (1.11) should be considered as a task which is *complementary* to the task of the presented paper. The presented paper could be read independently of the paper [BZK] and its subject is in principle considerably more general. However, the result of [BZK] still gives a most relevant "starting model" for which the technology of this new paper is developed. It is, at the moment, the main example which can be treated by our new version of P.S. theory but which can*not* by treated by other versions of that theory appearing in the literature.

While we expect that in future, there will be other examples (than the output of [BZK]) where our method turns out to be useful, the situation of [BZK] still gives a main motivation for the presented paper. In fact, many of the typical applications of the P.S. theory treated in the existing literature already fit the more special scheme (1.1) presented below. We convert it first (and this was already done in [BZK]) into a more general scheme (1.11). The latter situation *is* more general and we expect it to be useful in further applications that do not allow formulation of a type (1.1). For example, many models of continuous spins, also of Kac type, seem to offer nice reformulations of the type (1.11), but not of the type (1.1).

Also conceptionally, the scheme (1.11) is more adequate than (1.1) because the presence of the "cluster field" in (1.23) already suggests also the basic *method*, of successive "partial expansions", how to treat such a model. Thus, our basic method is an analytical one, and with such an approach the P.S. theory becomes part of analysis rather than probability theory. In fact the main body of the theory does not change at all if also *complex* Hamiltonians are considered. One has to be only careful to express things in terms of partition functions, rather than with probabilities.

A probabilistically inclined reader would expect, in the paper on Pirogov–Sinai theory, some more explicit statements concerning the probabilities of events like that a spin having in a given site a specified value, inside a finite volume and under a given boundary condition. Strictly speaking, answers to such problems are going beyond the basic technique developed here – which is formulated solely *in the language of contours* (of some original spin model).

The translation from a spin model to a contour one is a model dependent task, and so some questions concerning the original model in principle cannot be (at least immediately) formulated in terms of abstract contours. This is not to say that such questions do not deserve a more detailed treatment, which is unfortunately mostly omitted here – though the important preparatory work enabling to solve these questions is developed in detail.

Concerning the probabilistically flavored questions in general, we minimize *intentionally* their discussion here. It would not be "stylish" to make a (false!) impression that truly probabilistic arguments should play any important role in the development of a theory which should deal, in mostly unchanged form, also with complex Hamiltonians. The idea that contours should be viewed as an energetical barriers of course suggests that contours will have a small probability (in ensembles with "stable" boundary condition), and surely the latter statement helps to visualize the structure of the infinite volume Gibbs state(s) of the model. However, the reader will realize that there is surprisingly little of probability theory (and classical theory of Gibbs states in particular) in the Pirogov Sinai theory. The technology is mostly developed in finite volumes, and for complex Hamiltonians in principle .

1. Setting of the problem

As we already noticed above, we give two possible settings. While the second one (1.11) will be the actual reference point for our further work, the first one comes as an example reminding from where we actually reached, in the last section of [BZK], our second setting. To get (1.11) from (1.1), one has to take cluster expansion of all the polymers P from (1.1)):

A starting example.

Let Λ be a ν–dimensional finite torus, $\nu \geq 2$. Assume that we have some family $\{q \in Q\}$ of "reference colors" whose elements are typically interpreted as spins q for which some "original Hamiltonian" (which was later converted to a form (1.1)) had its "almost ground value"[1] at the constant configuration $\{x_i = q, i \in \Lambda\}$. Assume that some "contours" denoted by symbols $\mathbf{\Gamma}$ and also some "polymers" denoted by symbols P are already defined in the considered model. See below for more details on these objects.

Normally, if we relate the general picture (1.1) to a corresponding original spin model, then contour $\mathbf{\Gamma}$ is defined as a triple $\mathbf{\Gamma} = (G \equiv \underline{\mathbf{\Gamma}}, x_{G'}, I)$ where $\underline{\mathbf{\Gamma}} \subset \Lambda$ is the "support" of a contour $\mathbf{\Gamma}$, $x_{G'}$ a spin configuration on some $G' \subset G$ (often we have simply $G' \equiv G$ but imagine also the possibility when we "factorize over spins inside $\mathbf{\Gamma}$"; then G' is the boundary of G) and I is some additional information (often empty one) formulating possibly some other restrictions on allowed objects $\mathbf{\Gamma}$ resp. some conditions on their compatibility with other objects. While polymers used in such a scheme (1.1) will be always "monochromatic" – in the sense that a fixed color $q \in Q$ will be always assigned to them, contours will be in general "polychromatic" objects assigning possibly *different* colors (usually these are just the boundary values of spins x_G) to the connected components of their supports.

Let some (in)compatibility relation denoted by c (resp. ι) between all these objects be given and also some translation invariant (incompatibility may cause a violation of this property) weights $w_{\mathbf{\Gamma}}$ and w_P. The relation ι will be considered as some abstract relation but typically the compatibility of contours and polymers is defined in terms of disconnectedness resp. disjointness of the "supports" $\underline{\mathbf{\Gamma}}$ resp. $\underline{P}$ of $\mathbf{\Gamma}$ resp. P. For contours one also requires "matching compatibility" of the colours determined by them. In particular, in [BZK] the supports of contours are required to be disconnected while compatibility of a polymer P with an other object P' or $\mathbf{\Gamma}$ means that the support of P does not intersect its support.

In many previous situations of Pirogov - Sinai theory there was no need to distinguish monochromatic contours and polymers; then polymers disappear from the setting (1.1). This is actually the situation which was called an "abstract P.S. model" before by the author of this article. However, in [BZK] and in other applications too, monochromatic contours are objects of rather a different nature ("connected sets & configuration on them) than polymers (which are considered as graphs there; their "gas" representing the restricted ensemble outside contours).

The partition function on a torus Λ will be then defined as

$$Z(\Lambda) = \sum_{\{\mathbf{\Gamma}_1, \ldots \mathbf{\Gamma}_m, P_1, \ldots, P_k\}} \prod_{i=1}^{m} w_{\mathbf{\Gamma}_i} \prod_{q \in Q} \exp(-e_q \, |\Lambda_q|) \prod_{j=1}^{k} w_{P_i} \tag{1.1}$$

where the sum is over all compatible collections of contours and polymers. Such collections will be simply called *configurations* of our model, sometimes denoted also by symbol x_Λ.

The set Λ_q in (1.1) denotes the collection of all "q–correct" points of the configuration $\{\mathbf{\Gamma}_i\} \cup \{P_j\}$ i.e. of all points outside of contours having a prescribed color q. Then we have the decomposition $\Lambda = \cup_{q \in Q} \Lambda_q \cup \cup_i \underline{\mathbf{\Gamma}}_i$ where $\underline{\mathbf{\Gamma}}$ denotes the support of $\mathbf{\Gamma}$. This is nonambiguous for nonempty systems $\{\mathbf{\Gamma}_i\}$only; an important additional remark is that we consider no single "empty" configuration $\{\mathbf{\Gamma}_i\}$ but, instead, a whole collection (marked by

[1] "Almost ground state" should be understood as a "ground state for a suitably (slightly) changed Hamiltonian". However, this is meant just for an intuition, we will not have to make precise such a notion.

$q \in Q$) of configurations interpreted as "colored emptiness" $x_\Lambda^q = \{P_j^q\}$ (emptiness filled only by polymers) i.e. configurations with $\{\mathbf{\Gamma}_i\} = \emptyset$ and with $\Lambda^q \equiv \Lambda$, $\Lambda^{q'} = \emptyset$ for other $q' \neq q$.

Note. It is instructive to write first the "ground" terms $e^{-e_q|\Lambda_q|}$ and only then the polymer activities w_P to emphasize that while contours $\mathbf{\Gamma}$ may change their external color – we will often write $\mathbf{\Gamma} = \mathbf{\Gamma}^q$ if the latter is equal to q – to another color, say $\tilde{q}$, in the interior components (usually denoted by $\operatorname{int}\mathbf{\Gamma} = \operatorname{int}_{\tilde{q}}\mathbf{\Gamma}$) of the complement of the support $\underline{\mathbf{\Gamma}}$. In contrary to contours, polymers will be always "monochromatic" objects living in some Λ_q and having a constant color q. Their interpretation is is fact less "physical" than that of contours, and it will be also necessary to assume that their activities w_P are not necessarily positive, while for contours their activities will be always[2] of the type $w_{\mathbf{\Gamma}} = e^{-E(\mathbf{\Gamma})}$ where $E(\mathbf{\Gamma})$ is a positive (and large) *energy* of the contour $\mathbf{\Gamma}$.

The notion of an interior of a contour on a (usually *finite*) torus clearly has to be more specified. Let us call a contour $\mathbf{\Gamma}$ a nonastronomical one if its size is considerably smaller than the size of our "universe" Λ say

$$\operatorname{diam}\underline{\mathbf{\Gamma}} \ < \ 1/2 \ \operatorname{diam}\Lambda. \tag{1.2}$$

Let us make a technically convenient agreement that *only* configurations with *no* astronomically large contours will be considered in what follows. In particular, partition function $Z(\Lambda)$ will be always taken only over them. Few simple comments on how to adapt our study also to the "full" partition function taken also over configurations (of contours and polymers) with no "nonastronomical" constraint (such a constraint turns out to be of extremely small relevance in large tori) will be given at the end of the paper.

More generally, we will consider also partitition function with some limitation $\mathcal{C}$, which has to be fulfilled for all considered configurations for which we sum in (1.1). Typically the constraint $\mathcal{C}$ will be just a fixed collection of contours or (less naturally) polymers. The partition function $Z_{\mathcal{C}}$ will be then defined as

$$Z_{\mathcal{C}}(\Lambda) = \sum_{\{\mathbf{\Gamma}_1,\dots\mathbf{\Gamma}_m,P_1,\dots,P_k\}} \prod_{i=1}^{m} w_{\mathbf{\Gamma}_i} \prod_{q\in Q} \exp(-e_q\,|\Lambda_q|) \prod_{j=1}^{k} w_{P_j}. \tag{1.3}$$

Here, the sum is taken over all compatible collections of contours and polymers fulfilling the limitation $\mathcal{C}$. For $\mathcal{C} = \emptyset$, this is just the partition function $Z(\Lambda) \equiv Z_\emptyset(\Lambda)$ above.

In the following we will often normalize the "external field" $\{e_q, q \in Q\}$ such that

$$e = \min_{q\in Q} e_q = 0. \tag{1.4}$$

The exception is the case, studied only at the end of the article, when the dependence of e_q on parameters of the model will be discussed; then (1.4) would be inconvenient.

[2] At least in this paper, where we do not develop a systematic treatment of *complex* Hamiltonians.

The Peierls condition for contours and polymers.

The weights $w_{\mathbf{\Gamma}}$ and w_P will be assumed to satisfy the following bounds: Weights $w_{\mathbf{\Gamma}}$ of contours are given as $e^{-E(\mathbf{\Gamma})}$ where $E(\mathbf{\Gamma})$ is the "energy" of $\mathbf{\Gamma}$ satisfying the Peierls condition

$$E(\mathbf{\Gamma}) \geq \tau \, |\underline{\mathbf{\Gamma}}| \tag{1.5}$$

where $\tau \gg 1$. The constant τ has the order of inverse temperature in typical applications. Below we will rewrite this bound in another, more suitable "exponential" form (1.14).

Analogous condition will be assumed for the polymers, but in such a case it will be even more advisable to write it in the exponential form because w_P may be negative, even in models with real Hamiltonian H (in contrary to the weights $w_{\mathbf{\Gamma}}$). In fact, our main example [BZK] requires polymers to be geometrical objects of either a very simple geometric structure ("thin" graphs, typically with only a few bonds, called "constellations" in [BZK]) or much more "dense" objects, with somehow more complicated geometric nature ("galaxies") similar to contours. It is best to treat them in a unified way as graphs with weights given as

$$|w_P| \leq \prod_{b\in P} |w_b| \qquad \text{where} \qquad \sum_{b\ni i} |w_b| \leq e^{-\tilde{\tau}} \tag{1.6}$$

with another $\tilde{\tau} \gg 1$. The constants τ and $\tilde{\tau}$ introduced here do not correspond literally to the constants denoted by the same letters in [BZK]. Later we will rewrite the bound (1.6) in another way, based on the following standard tree summation bound (see e.g. [BZK]):

Lemma. *Let some tree weights w_t be given satisfying the following estimates:*

$$|w_t| \ \leq \ e^{-\zeta|\underline{t}|} \prod_{b\in t} (e^{w_b} - 1) \qquad \textit{where} \qquad \sum_{b\ni i} |w_b| \leq \zeta \tag{1.7}$$

and where the product is over all bonds b of the tree t. Then the sum $\sum_{t:r(t)=i} |w_t|$ over trees having the "root" i (and at least one bond) can be bounded as follows

$$\sum_{t:r(t)=i} |w_t| \leq 1 - e^{-\zeta} \ \leq \zeta. \tag{1.8}$$

Now, with the help of such a lemma one can rewrite (see again [BZK] for more details) the bound (1.6) in the following way, and this will be our actual starting assumption later:

Reformulated assumption (1.6) on polymer weights w_P.

With the help of the above lemma, taking $\zeta = e^{-\tilde{\tau}}$ one can obviously rewrite the bound (1.6) in the following way, with suitable small (see below for more details) $\tilde{p}$

$$\sum_{P:\underline{P}\ni i} |w_P| \, e^{p|\underline{P}|} \ \leq \ \tilde{p} \ \ll \ 1 \ \ll \ p \qquad \text{where} \qquad p = \tilde{\tau} - e^{-\tilde{\tau}}. \tag{1.9}$$

Note. The constant p in bounds of the type (1.9) is often in the literature [ES],[KP] etc. denoted by symbol a. However, we choose the letter p for polymers P and analogously, below, g for contours $\mathbf{\Gamma}$. Later we rewrite also (1.5) in a similar way as (1.9) but in that case we will have to strengthen the requirements on the corresponding constant g.

So, this is our basic example. The important idea now is to apply *cluster expansion* both to polymers and (later) also to contours of such a model. There is, however, a crucial difference between the "monochromatic" polymers on one side – whose expansion should cause no problems under assumptions of the type (1.9) – and on the other side the contours having in general "polychromatic" boundaries inducing different colors q in the rest of the space. The latter is the feature which is very characteristic for the applications of P.S. theory and distinguishes them from the standard polymer low temperature expansions.

Nevertheless, the "zeroth" step of our expansion namely the expansion of all polymers causes no problems and it is really advisable to do it. It envisages our basic future strategy, which can be characterized as an attempt to expand, in a sequence of steps, also all *contours* of the model. The polychromatic character of contours will then require an application of additional tools developed to deal with such a situation, having no analogy in the theory of polymer models. These tools, namely the basic notion of a contour functional, are rather characteristic for the method called the P.S. theory.

So, imagine that while fixing a family $\{\mathbf{\Gamma}_i\}$ of contours of a configuration in (1.1) we expand, by standard cluster expansion (Mayer expansion) method (discussed later in some detail, but see also [BZE] and [NOZ]) the abstract polymer model "living" outside of the collection $\{\mathbf{\Gamma}_i\}$. Then we can reformulate (1.1) in the following way, and this will be an actual starting point of our paper; the situation (1.1) being now considered just as an example leading, by expansion, to the following more general setting:

Our new, reformulated "zeroth level" model.

Assume that some objects called contours $\mathbf{\Gamma}$ are defined having the properties described in (1.1). Assume that their weights $w_{\mathbf{\Gamma}}$ satisfy the bound (1.5). Usually, the weights $w_{\mathbf{\Gamma}}$ will be translation invariant quantities given as $w_{\mathbf{\Gamma}} = e^{-E(\mathbf{\Gamma})}$; then the quantity

$$H(\{\mathbf{\Gamma}_i\}) \equiv \sum_i E(\mathbf{\Gamma}_i) + \sum_q e_q|\Lambda^q| \tag{1.10}$$

could be called the "energy" of the (factorized) "configuration" $\{\mathbf{\Gamma}_i\}$. Now assume further that some "cluster field" (see below for more details) w_T is given, with translation invariant (for T not touching Λ^c) values. Consider the partition function on a torus Λ

$$Z(\Lambda) = \sum_{\{\mathbf{\Gamma}_1,\dots\mathbf{\Gamma}_m\}} \prod_{i=1}^m w_{\mathbf{\Gamma}_i} \prod_q \exp(-e_q|\Lambda_q|)\ \exp(\sum_{T_q \sqsubset \Lambda^q} w_{T^q}) \tag{1.11}$$

where the symbol $T \sqsubset M$ means that the cluster T is incompatible with the set M (i.e. the support of T either is a subset of M or at least "touches" M. See below. More precisely the incompatibility of T with M will be usually meant in the sense that $M \cap \underline{T} \neq \emptyset$. Later we will write the term in the exponents approximately as $(e_q - s_q)|\Lambda_q| = f_q|\Lambda_q|$ where f_q resp. s_q is the "free energy" of $\{\mathbf{\Gamma}_i\}$ resp. of polymers compatible with it.

Again, it will be necessary to consider also conditioned partition functions, for any limitation (required presence of a family of contours) $\mathcal{C}$; we have $Z_\emptyset(\Lambda) \equiv Z(\Lambda)$

$$Z_{\mathcal{C}}(\Lambda) = \sum_{\{\mathbf{\Gamma}_1,\ldots\mathbf{\Gamma}_m\}} \prod_{i=1}^{m} w_{\mathbf{\Gamma}_i} \prod_q \exp(-e_q|\Lambda_q|)\ \exp(\sum_{T_q \sqsubset \tilde{\Lambda}^q} w_{T^q}) \tag{1.12}$$

where the summation is over all configurations $\{\mathbf{\Gamma}_i\}$ satisfying the limitation $\mathcal{C}$.

Note. A typical example of a limitation $\mathcal{C}$ will be a required presence of some contour(s), in questions like "what happens inside of a given contour $\mathbf{\Gamma}$". Or, on the contrary, we can require an *absence* of contours at some points. For example, we can choose a "very distant" point of Λ called "infinity" and we may require that such a point i) does not belong to interior or support of any contour ii) has a prescribed color q. This is the boundary condition "in infinity" fixing this "remote point" in the regime q outside of all contours. It may seem surprising but even such a relatively weak condition will create, in the thermodynamical limit and for a "stable" q, a measure which is very close to the extremal Gibbs state of a type q. To attain the extremal q–th Gibs state precisely one has to consider *multipoint* set Λ_∞ instead of a single point "∞" and then to take $|\Lambda_\infty| \to \infty$ during the thermodynamical limit $\Lambda \to \mathbb{Z}^\nu$, while holding the values of spins on Λ_∞ equal to q and kept all contours outside Λ_∞. No special requirement on the "enough quick" growth of $|\Lambda_\infty|$ will be necessary.

Assumptions on the cluster weights w_T: expanded version of (1.9).

We require a quick decay of the cluster weights w_{T^q}, namely[3]

$$\sum_{T \ni x_i} |w_T|\ e^{c|\underline{T}|} \ \leq\ \epsilon \ll 1 \ll\ c \tag{1.13}$$

where the constant c will be close to previous p for a large p i.e. typically of the order of τ.

See the references [KP], [BZE], [BZK] and also the forthcoming text for more details concerning how (1.9) implies such a bound (1.13) – with a still large c which is actually only slightly smaller, something like $c = p - \tilde{p}$, than p . Clusters T are then just "indecomposable collections" of polymers P where decomposability of a collection $\mathcal{P}$ means that $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ where all elements of $\mathcal{P}_1$ are compatible with all elements of $\mathcal{P}_2$.

Reformulated Peierls condition for contour weights.

The Peierls bound (1.5) will be now written in a similar style as (1.9) namely

$$\sum_{\mathbf{\Gamma}:\underline{\mathbf{\Gamma}} \ni i} w_{\mathbf{\Gamma}}\ e^{g|\underline{\mathbf{\Gamma}}|}\ \leq\ \omega\ \ll 1 \ll g \qquad \text{where} \qquad w_{\mathbf{\Gamma}} = e^{-E(\mathbf{\Gamma})}. \tag{1.14}$$

[3] We will need such a bound only for the sum with a fixed color q of $T = T^q$. Analogously, in (1.14) the sum will be over $\mathbf{\Gamma}$ with a fixed external color q. The ambiguity in formulations is unimportant for small $|Q|$.

A more detailed discussion of (1.14) will be given below. While polymers will be expanded "at once", (infinitely) many expansions steps will be needed in the expansion of contours. Later we will need a more explicit control also over the tails of (1.14): Let us formulate it as follows, with $g' \gg \varepsilon + \omega$ (this will be used only much later, at (5.5))

$$\sum_{\mathbf{\Gamma}:\underline{\mathbf{\Gamma}}\ni i, |\underline{\mathbf{\Gamma}}|\geq N} |w_{\mathbf{\Gamma}}|\ e^{g|\underline{\mathbf{\Gamma}}|} \ \leq\ \omega_N \quad \text{where} \quad \sum_N \omega_N = \omega \quad \text{and} \quad \omega_N \leq\ K\ e^{-g'|N|}. \tag{1.15}$$

On the goal of Pirogov Sinai theory.

The first and main goal is to find an expression for the free energy[4]

$$h = -\lim_{\Lambda\to\mathbb{Z}^\nu} \frac{\log Z(\Lambda)}{|\Lambda|} \tag{1.16}$$

of the model (1.11) by a converging cluster expansion series, and then to look how possible (non)analyticity of such a sum of cluster expansion terms corresponds to unicity of a Gibbs state resp. to a phase transition. The manifolds bordering the regions with nonanalytic h are precisely the coexistence manifolds and one of the main goals of the theory is to determine these manifolds. We will see that all this can be attained if we are able to keep the control of the decay of all the cluster expansion terms.

The above questions have, of course, a sense even for a *complex* Hamiltonian. However, an analyticity of all the cluster expansion terms (and their sum) can be established only in the uniqueness regions (and in a suitable, more narrow sense also on the coexistence manifolds). The cluster expansion terms of the P.S. theory are *not* analytical functions (of the parameters in the Hamiltonian) on the boundaries of regions where uniqueness of the Gibbs state is established i.e. on the boundaries where a phase transition takes place.

For real valued Hamiltonians of classical statistical mechanics one also wants to give a detailed information on the structure of the Gibbs states of a given Hamiltonian. For this, it is not enough to have only the expansion of the full partition function Z_Λ on a torus. One would like also to expand some special, $\mathcal{C}$ – conditioned partition functions introduced above.

Take a simplest and most basic probabilistic question (in the setting we have): "What is a probability that a contour $\mathbf{\Gamma}$ appears in the Gibbs ensemble with the partition function $Z(\Lambda)$" ? Or more generally: "What is a probability of the same event in the ensemble conditioned by the assumption of a required presence of some contour subsystem $\mathcal{C}$"? Then, by choosing Λ and $\mathcal{C}$ properly we can model any boundary condition on any finite subset of $\mathbb{Z}^\nu$ and then we can go to the thermodynamic limit. To conclude, our restriction to tori, with a suitable choice of a conditioning formulated in terms of a presence (absence) of some contour subsystem $\mathcal{C}$ is not at all restrictive, and all limit Gibbs states on $\mathbb{Z}^\nu$ can be attained as limits of suitably conditioned ensembles on tori.

Fixing a restrictive condition $\mathcal{C}$ has precisely the aim to allow a proper formulation of expansion results also for the probabilities of correspondingly conditioned events. Conditioned

[4]The quantity f^∞, our main output of the expansions constructed below, will be equal to h.

events are then interpereted as events in the ensemble with the partition function (1.12). In practise, the limitation $\mathcal{C}$ will often include a requirement of a "fixed color q at infinity"; then the most interesting case is that of q being the external color of $\mathbf{\Gamma}$.

Conditional probability in the ensemble with partition function (1.12) (with conditioning being formulated in terms of a limitation $\mathcal{C}$ which has to be fulfilled for any considered configuration) of the event "contour $\mathbf{\Gamma}$ occurs, as a subsystem of the family of all contours of a given configuration" is then given by formulas like (see more precisely later in (6.6))

$$P_\Lambda^{\mathcal{C}}(\mathbf{\Gamma}) = \frac{Z_{\mathcal{C}\cup\mathbf{\Gamma}}(\Lambda)}{Z_{\mathcal{C}}(\Lambda)} \quad = \quad \exp(-E(\mathbf{\Gamma}) + \sum_{T\iota(\mathbf{\Gamma}\cup\operatorname{int}\mathbf{\Gamma}\cup\mathcal{C})} (w_T - \tilde{w}_T)) \tag{1.17}$$

where in the numerator we add the limitation "$\mathbf{\Gamma}$ must appear" to the previous limitation $\mathcal{C}$. The second equation suggests that later we will take the expansion of both the numerator and the denominator of the fraction. Then the (small and quickly decaying!) expansion terms w_T resp. $\tilde{w}_T$ (in the expressions of partition functions correspoding to situations when $\mathbf{\Gamma}$ is present resp. absent) in each of the two (complicated!) series are the same and therefore cancel if the cluster T is outside $\underline{\mathbf{\Gamma}} \cup \operatorname{int}\mathbf{\Gamma}$.

For $T \subset \operatorname{int}\mathbf{\Gamma}$ the situation is *more delicate*, a cancellation (roughly) or even volume energy increase takes place only for special, "stable" external colors q of $\mathbf{\Gamma}$. (This will be the situation where extremal Gibbs state of a type q exists.) The weights w_T resp. $\tilde{w}_T$ for $T\iota\mathbf{\Gamma}$ do not cancel but their sum is unimportant, much smaller than the leading term $E(\mathbf{\Gamma})$. So, only for a stable q we really get a useful formula, with only those (quickly decaying) terms w_T resp. $\tilde{w}_T$ remaining which are indexed by clusters "touching" $\mathbf{\Gamma}$ (or $\mathcal{C}$). Then, for "stable" external colours of $\mathbf{\Gamma}$ we will get a very accurate evaluation of the probability (1.17).

Note. we emphasize that neither (1.1) nor (1.11) are spin models. (They reformulate and generalize such models.) Questions formulated in terms of behaviour of spins of an original spin model must be first translated into the language of objects used here in (1.11) i.e. into the language of contours, and clusters. Analogously, the answers have to be translated from the language of contours back to the spin model. Often, this is a straightforward task like in the above example but it can be a more tedious one for more complicated events. Anyway, this is "(spin) model dependent" work, not directly related to the aim of this article.

For example, the behaviour of blockspins can be eventually, of course, also described in the language of contours of the spin model but the expansion formulas for such a contour description will look more cumbersome than for the events described in terms of contours more directly. In this paper, we do not have place enough to discuss these transcriptions in detail. So we concentrate only on events (and conditionings) formulated directly in terms of a presence or absence of some contours. Moreover, when discussing formulas of the type (1.17) we will always start with the basic case $\mathcal{C} = \emptyset$.

Equivalent models, giving the same partition functions.

Let us always denote by $\mathcal{A}$ the collection of all "available" contours appearing in a given model of a type (1.11). Assume that we have two such models, and let $\mathcal{A}_1$ resp. $\mathcal{A}_2$ be the families of available contours used in the first resp. second model. In fact we will consider only the case when $\mathcal{A}_1 \subset \mathcal{A}_2$. Let $\mathcal{G} \subset \mathcal{A}_1 \cap \mathcal{A}_2$. Say that these two models are $\mathcal{G}$ – equivalent if for all limitations $\mathcal{C}$ expressed in terms of contours from $\mathcal{G}$ the partition functions $Z_{\mathcal{C}}(\Lambda)$ taken over all configurations (= compatible systems of contours from $\mathcal{A}_1$ resp. $\mathcal{A}_2$ satisfying the limitation $\mathcal{C}$) are the same for *all tori* Λ, in both models.

We always consider here, at the first instance, the case $\mathcal{G} = \emptyset$. However, for a more detailed analysis of Gibbs probabilities one has to choose a suitably richer family $\mathcal{G}$. If the two models are $\mathcal{G}$ equivalent for $\mathcal{G} = \mathcal{A}_1 \cap \mathcal{A}_2$ then we will simply call them *equivalent*.

Needless to say, two equivalent (even $\emptyset$ – equivalent!) models have the *same free energy*. Two equivalent models moreover have the same probabilities of events which are expressible in the language of contours from $\mathcal{A}_1 \cap \mathcal{A}_2$.

A generalized approach to the model (1.11) with (1.13) and (1.14).

It is in the very nature of the contour approach (= the P.S. theory) that any question in the model (1.11) – or in some original spin model, which was later rephrased by (1.11), possibly through (1.1)– is formulated in the language of contours. Translation, into the language of contours, is then needed for any question formulated directly in terms of spins of the original model. Possible complexity of a translation of such a question affects, of course, also the complexity of the translation, back to the language of spins, of the answer. Events in the original spin model which are not straightforwardly expressed in terms of contours must be then dealt with the help of the conditional expectations w.r. to various possible configurations of contours allowing a presence of the given event.

We already argued that it is advisable to minimize the usage of probabilistic concepts in all these investigations, giving emphasis to the expansion of partition functions. This allows also a later generalization to complex Hamiltonians.

When computing the partition function, the key point is to keep in mind the structure of considered configurations $\{\mathbf{\Gamma}_i\}$. Any such configuration induces a coloring of the rest $(\cup_i \underline{\mathbf{\Gamma}_i})^c$ of the torus Λ, and the interior components of $(\underline{\mathbf{\Gamma}})^c$ have in general a color different from that of the exterior of $\mathbf{\Gamma}$. This creates, together with the fact that generally we have *no symmetry* in the model, all the difficulties (and the interest) of Pirogov – Sinai theory. The compatibility of contours and clusters should be viewed as rather an abstract relation, its details being mostly irrelevant below.

It will be important to slightly generalize the model (1.11), keeping a careful eye on the validity of the Peierls conditions (1.14) and (1.13), with suitably large g and c. The generalization will have two main aspects:

1) a broader approach to considered collections of allowed contours $\mathbf{\Gamma}$, and their energies

2) a broader approach to considered collections of clusters T, and their weights

While 1) could have been tricky in some other situations (for other models, notably for those with continuous spins), here it will have always a very straightforward meaning. Namely, we have to acquire only the idea that also appropriately selected *subsets* of the original family $\mathcal{A}$ of all contours should be considered, and also that some collections of contour should be "glued" together later i.e. considered as a single entity.

On the other hand, a careful generalization of a notion 2) of a cluster will be absolutely necessary for the approach to Pirogov – Sinai theory we are developing here. This will be characterized by the desire to *iteratively expand the smallest contours* resp. contour systems of the model and to convert them into some *newly formed* (more complicated) cluster field w_T – such that a control over the decay of (both the old and new) cluster weights w_t is retained, even when using an infinite number of such expansion steps.

What it means "to expand" will be discussed below in detail, in particular see our Theorem 4 and its equation (3.17) which is really the basic technical step of our approach. At this point it suffices to mention a "principle of correspondence" stating that our newly defined operation of expansion of contours should be *identical* with the well understood operation of an expansion of *polymers* in the cases when i) expanded contours are "monochromatic" (such that neighboring values of spins q are everywhere the same along the boundary of $\underline{\mathbf{\Gamma}}$) resp. more generally in the cases where ii) there is a *symmetry* between the different "colors" of the model. The main aim of the whole construction is to construct a cluster expansion of the partition function $Z(\Lambda)$ and to derive an expansion formula for free energy from it.

Note on an abstract P.S. model with tree like contours. The clusters T used in (1.23) have essentially a tree structure. In fact one could treat not only these objects but also *contours* more systematically as graphs resp. trees carrying some additional information, in particular about the "color" from Q attached to the *connected components* of the complement of a support of a contour. The prescriptions on the colors of the contours should be "matching compatible", in the complement of the union of supports of these objects. One could reformulate the abstract Pirogov – Sinai theory for such a more general setting, with contours treated as graphs (trees) inducing a "'connected coloring" of the rest of the space. We expect that for some more complicated situations, like for models with infinite range "tail" interactions, this could actually the most adequate approach.

Auxiliary ordering on the collection $\mathcal{A}$ of all contours of the model.

The idea of the forthcoming expansions will be to expand the model successively, according to the size of contours or contours systems which have to be expanded. It is natural to consider the following partial ordering $\prec$ on the family of all contours $\mathcal{A}$, given as

$$\mathbf{\Gamma} \prec \mathbf{\Gamma}' \iff \underline{\mathbf{\Gamma}} \subset \operatorname{int} \mathbf{\Gamma}'. \tag{1.18}$$

Recall that our "universe" is a (possibly very large) torus Λ, and we restrict our interest only to ensembles of configurations (and we take the partition function $Z(\Lambda)$ over those ensembles only) having contours of "nonastronomical" size only, such that

$$\mathbf{\Gamma} \in \mathcal{A} \Longrightarrow \operatorname{diam} \underline{\mathbf{\Gamma}} \ < \ 1/2 \ \operatorname{diam} \Lambda. \tag{1.19}$$

This assumption on the the diameter of $\mathbf{\Gamma}$ clarifies the notion of interior components of $\underline{\mathbf{\Gamma}}^c$ on Λ. It is rather elementary to show that with Peierls condition, contribution of configurations containing some "astronomical" contour is extremely small, when $\operatorname{diam} \Lambda \to \infty$, in the ensemble with the full partition function $Z_{full}(\Lambda)$. Their relative contribution is actually of of order $e^{-C\tau \operatorname{diam}\Lambda}$. We omit a more detailed discussion here.

Notice that the ordering above is invariant w.r. to all translations in Λ.

"Expandable" contours in a model (1.23). Extremal contour systems.

Denote by $\mathcal{A} = \cup_q \mathcal{A}^q$, more precisely $\mathcal{A}(\Lambda) = \cup_q \mathcal{A}^q(\Lambda)$ the system of all contours of the model (1.23) in a given torus Λ. In the following we will select, for any $q \in Q$, a suitable subsystem $\mathcal{H}^q \subset \mathcal{A}^q$ (of "expandable, small" contours; see below). The subsystem $\mathcal{H}^q$ will be nonempty for *at least one* $q \in Q$. We will assume that its choice is always made such that

$$\mathbf{\Gamma}' \in \mathcal{H}^q,\ \mathbf{\Gamma} \prec \mathbf{\Gamma}' \quad \Longrightarrow \mathbf{\Gamma} \notin \ \cup_{q' \in Q} \ \mathcal{H}^{q'} \tag{1.20}$$

i.e. a nontrivial subordination relation $\prec$ cannot appear between the elements of $\mathcal{H} = \cup_q \mathcal{H}^q$. Such a condition is just an "organizational one", how to select properly the families $\mathcal{H}^q$.

Any compatible system $\mathcal{D} \subset \mathcal{A}$ having a single external contour $\mathbf{\Gamma} = \mathcal{D}_{ext} \in \mathcal{H}$ will be called an *extremal* contour system of the model. By (1.20), only contours from the family

$$\mathcal{R} \equiv \mathcal{A} \setminus \mathcal{H} \ = \cup_q \mathcal{R}^q \ ; \ \mathcal{R}^q \equiv \mathcal{A}^q \setminus \mathcal{H}^q \tag{1.21}$$

could be possibly placed compatibly into the interior of $\mathcal{D}_{ext}$. Extremal contour systems $\mathcal{D}$ will be later treated as smallest *expandable* objects of the model.

To have a concrete idea of the choice of $\mathcal{H}^q$ imagine the collection of all contours $\mathbf{\Gamma}^q$ whose diameter is the smallest possible, in the family $\mathcal{A}^q$. Assume for simplicity that $w_T \equiv 0$ in such a starting example. However, even then we have to assume that all e_q are roughly the same to make such a choice of a nonempty $\mathcal{H}^q$ a reasonable one for future expansions. Otherwise, one has to take $\mathcal{H}^q = \emptyset$ if e_q deviates from $\min_{q' \in Q} U(q')$ too much.

Note. So, for any contour from $\mathcal{H}^q$ we assume that inside of $\operatorname{int}\mathbf{\Gamma}^q$, no other "small" contours from some $\mathcal{H}^{q'}$ can reside, whatever the whole collection of interior contours inside $\mathbf{\Gamma}^q$ is. There may be a room, inside of $\operatorname{int}\mathbf{\Gamma}^q$, for "nonsmall" (slightly paradoxical name here!) contours from some $\mathcal{R}^{q'}$. This will be quite normal and characteristic, especially in the later expansion stages when the size of "small" contours from $\mathcal{H}^q$ will be already quite large. At the same time, some contours geometrically much smaller than $\mathbf{\Gamma}$ but from some $\mathcal{R}^{q'}$ (this can happen only for those q' for which $\mathcal{H}^{q'}$ is empty!) will persist in the model.

Typically we will be in a following situation. Assume that there is, for any nonempty $\mathcal{H}^q$, a common diametr N_q of all contours from $\mathcal{H}^q$. Assume that contours from corresponding families $\mathcal{R}^q$ have a size at least $R_q \geq N_q$. Write $N_k \equiv N_{q_k}$, $R_k \equiv R_{q_k}$, $\mathcal{H}^k \equiv \mathcal{H}^{q_k}$. Let us arrange the elements Q as $q_1, q_2, \ldots, q_n$ such that $R_1 < \cdots < R_m$.[5] So, we have a table

$$R_1 < \cdots < R_m \ \ ; \ \ \mathcal{H}^1 = \cdots = \mathcal{H}^m = \emptyset$$

[5]Intuitively, this can be often interpreted as increasing "stability" of q ("ability to form a $q-$ like Gibbs state in a large volume", see below) and in many (not all) examples this corresponds to $f_1 > \cdots > f_m$ (see (2.3)) especially in models which are small perturbations of symmetric ones.

$$\mathcal{H}^i \neq \emptyset \quad \text{for} \quad i > m \quad \text{and} \quad \mathbf{\Gamma} \in \mathcal{H}_i,\ \mathbf{\Gamma}' \prec \mathbf{\Gamma} \implies \mathbf{\Gamma}' \in \cup_{j=1}^n \mathcal{R}_j$$
$$N_{m+1} = \cdots = N_n = N \ > \ R_m \ \ ; \ i > m \implies R_i > N$$
$$f_{m+1} \doteq f_{m+2} \cdots \doteq f_n < \min(f_1, \ldots, f_m) \tag{1.22}$$

where N will be, in later expansion steps, possibly *much bigger* than R_m and either the relation $f_i \doteq f_j$ for $i, j > m$ will be valid with greatly increasing precision, see below, or some other $\mathcal{H}_i$ will become empty later – and the corresponding f_i will finally become distinctly bigger than those f_j for which $\mathcal{H}_j$ is still nonempty. In other words, "nonsmall" contours from $\mathcal{R}^l$, $l < m$ will *persist* all the later expansions procedures – erasing objects from $\mathcal{H}^l, l > m$ *much bigger* than the elements of the above $\mathcal{R}^l$! We note in advance that our expansion procedures will be organized such that they will erase, together with contours from $\mathcal{H}^l$, also all the objects (even those from $\mathcal{R}^l, l < k$) *inside* them.

Note. So, one should not take the adjective "small" too literally. The size of small contours will depend on q. For some special values of q, below called "stable", *any* contour $\mathbf{\Gamma}^q$ will become "small" at a suitable step of the construction – and will then be expanded in one of the forthcoming steps. The adjective big/small can be misleading in some cases. "Small" ($\equiv$ having an almost stable external color) and minimal should be read as "easy to expand". Also, the word "stable" could be rather misleading if we look at the dynamics of contours. Then the large ($\equiv$ having an instable external color in our terminology) contours are dynamically more stable than typical small ones (which evolve in time much more rapidly).

Notation. Denote by symbols $\mathbf{\Gamma}_\odot$, more precisely $\mathbf{\Gamma}^q_\odot$, the extremal contour systems of the type $(\mathbf{\Gamma}, \{\mathbf{\Gamma}_i\})$ where $\mathbf{\Gamma}$, the external contour of the system $\mathbf{\Gamma}_\odot$, belongs to $\mathcal{H}_q$ and $\mathbf{\Gamma}_i$ are inside of $\mathbf{\Gamma}$, belonging to some $\mathcal{R}_{q'}$. We will use the symbol $\mathcal{H}^q_\odot$ to denote the collection of such extremal contour systems containing precisely one external contour $\mathbf{\Gamma}^q \in \mathcal{H}^q$.

On the choice of (recursively defined) clusters.

Now we will discuss our general choice of the "clusters" i.e. indices marking the "cluster field" $\{w_T\}$ in our abstract Hamiltonian (1.23). We already argued that we need a general enough notion, such that "clusters of contours and other clusters are again clusters". While the set of considered contours in partially expanded models constructed below will always be a suitable *subset* of the family $\cup_q \mathcal{A}^q$ (of all contours in the starting level model (1.11)) the concept of a cluster has to be generalized, to become useful in situations where iterative partial expansions of the model are considered. Then, a recursive approach to the notion of a cluster is natural, also in the inductive study of the bounds for cluster weights.

Having established all these bounds one could simplify the structure of clusters, "factorize" them w.r. to their support, and treat them as connected sets or graphs. However, to do this ia adequate only when a summary information on these objects is required, when a detailed discussion of the convergence of expansion terms was already done.

For any $\mathcal{H}^q, q \in Q$ we will consider the family $\mathcal{H}^q_\odot \supset \mathcal{H}^q$ of all extremal contour *systems* $\mathbf{\Gamma}_\odot = (\mathbf{\Gamma}, \{\mathbf{\Gamma}_k\})$ having one external contour $\mathbf{\Gamma}$ from $\mathcal{H}^q$ and possibly some interior, "big and nonexpandable contours $\mathbf{\Gamma}_k$ taken from $\mathcal{R}^{q'}$, $q' \neq q$ (and bordering "holes" with a "more stable" color). Let further $\mathcal{T}_q, q \in Q$ be a family of "clusters" i.e. objects already defined before, denoted by symbols T_q or simply T. We assume that for these objects we already have the notion of a support, denoted by $\underline{T}$, which is defined as a connected graph in Λ. The forthcoming definition is a preparation for our main result (Theorem 4) stated below:

Newly formed $(\mathcal{H}^q_\odot, \mathcal{T}^q)$ clusters.

Say that a pair $\mathbb{C} = (\{\mathbf{\Gamma}_\odot{}^i\}, \{T_j\})$ where $\{\mathbf{\Gamma}_\odot{}^i\} \subset \mathcal{H}^q_\odot$ and $\{T_j\} \subset \mathcal{T}^q$ is a $(\mathcal{H}^q_\odot, \mathcal{T}^q)$ cluster if it is *connected* in the relation of incompatibility $\mathbf{\Gamma}_\odot \,\iota\, \widetilde{\mathbf{\Gamma}_\odot}$ resp. $T \,\iota\, \mathbf{\Gamma}_\odot$ between the elements of the system $\{\mathbf{\Gamma}_\odot{}^i\} \cup \{T_j\}$. By incompatibility we mean either the incompatibility between contour systems $\mathbf{\Gamma}_\odot$ and $\widetilde{\mathbf{\Gamma}_\odot}$ or incompatibility of a contour system with a cluster i.e. (here, in this article) the relation $\underline{\mathbf{\Gamma}_\odot} \cap \underline{T} \neq \emptyset$. Notice that with such a definition all the pairs $\{T, \tilde{T}\}$ of clusters are mutually compatible. Contour systems $\mathbf{\Gamma}_\odot{}^i$ will be called the "nuclei" of $\mathbb{C}$. A typical nontrivial cluster has one nucleus, and empty or nonempty collection of "legs" T_i.

We will now generalize the above notion also to all *multiindices* $\mathbb{T} = \{n_{\mathbf{\Gamma}_\odot}, n_T\}$ i.e. integer valued functions on the set of indices $\mathcal{H}^q_\odot \cup \mathcal{T}^q$ which have nonzero value precisely on elements of some $(\mathcal{H}^q_\odot, \mathcal{T}^q)$ cluster $\mathbb{C} = \{\mathbf{\Gamma}_\odot{}^i\} \cup \{T_j\}$. Then the collection $\mathbb{C}$ will be called the *trace* of the cluster $\mathbb{T}$, $\mathbb{C} = \mathrm{Tr}\,\mathbb{T}$ while $\cup_i \underline{\mathbf{\Gamma}_\odot{}^i} \cup_j \cup \underline{T}_j$ will be called the *support* of $\mathbb{T}$. (The latter notion will be only used for some resummation of cluster weights later, for deciding whether $\mathbb{T}$ is compatible with other objects it is too crude.)

Define now the relation of (in)compatibility $\mathbb{T} \,\iota\, \mathbf{\Gamma}_\odot$ resp. $\mathbb{T} c \mathbf{\Gamma}_\odot$ between the newly formed cluster $\mathbb{T} = (\{\mathbf{\Gamma}_\odot{}^i\}, \{T_j\})$ and a contour system $\mathbf{\Gamma}_\odot$ – according to whether the elements of the trace of $\mathbb{T}$ are all compatible (and this relation is defined by the induction assumption) with $\mathbf{\Gamma}_\odot$ or not. Any two clusters will be considered mutually compatible.

Notation of new clusters. Sometimes, we will have to distinguish between the notations of the "old clusters" from $\mathcal{T}^q$ and the newly formed $(\mathcal{H}^q_\odot, \mathcal{T}^q)$ clusters defined above. Let us use the notation $\mathbb{T}_q$ or simply $\mathbb{T}$ for the new objects $\{\mathbf{\Gamma}_\odot{}^i\} \cup \{T_j\}$ constructed above. More precisely we will write $\mathbb{T} \in \mathcal{T}(\mathcal{H}^q_\odot, \mathcal{T}^q)$ for the new clusters. The external color of $\mathbb{T} = \mathbb{T}_q$ is defined in accordance with the external color of all the considered contour systems $\{\mathbf{\Gamma}_\odot{}^i\}$ and $\{T_j\}$. If the system of contours in $\mathbb{T}$ is empty then the system $\{T_j\}$ of "legs" contains only *one* term and we use the old notation $\mathbb{T} = T$. When investigating the decay of w_T, the knowledge of the "history" how all these clusters were formed will be essential.

Given a volume $M \subset \Lambda$ we have to define what it means that a cluster T "touches" M resp."is inside of" M. These two properties will be denoted respectively as $T \sqsubset M$ and $T \subset M$. We will also say that T is compatible with M^c, $T c M^c$ in the latter case and that T is incompatible with M (T "touches" the set M; $T \sqsubset M$) in the former case.

It is best to treat compatibility as an abstract relation, such that compatibility of a new cluster $\mathbb{T}$ with M^c is always defined in terms of compatibility, with M^c, of all the "old" objects (contour systems $\mathbf{\Gamma}_\odot$ and "old clusters" T) forming it. Notice that here the compatibility of the original clusters T from (1.11) with M^c was defined in a little bit weaker sense (for technical reasons taken from [BZK]) than for contours; for T it meant just that $\underline{T} \cap M^c = \emptyset$.

The value of the cluster activity w_T will be in general influenced by the boundary of M, but for $T \subset M$ we will always assume that it is translation invariant.

Now we are ready to generalize our basic model (1.11):

An abstract Pirogov Sinai model with recursively defined "cluster field".

From now on, we will consider again a model of the type (1.11) with the partition function

$$Z(\Lambda) = \sum_{\{\mathbf{\Gamma}_1,\ldots\mathbf{\Gamma}_m\}} \prod_{i=1}^{m} \exp(-E(\mathbf{\Gamma}_i)) \prod_q \exp(-e_q|\Lambda_q|) \ \exp(\sum_{T_q \sqsubset \Lambda_q} w_{T_q}) \tag{1.23}$$

but with a more general notion of a cluster introduced above. Instead of saying that T_q is incompatible with Λ_q we are using also the above notation $T_q \sqsubset \Lambda_q$. We emphasize that we replace here the usual condition $T \subset \Lambda_q$ used in the theory of polymer models (and also in the theory of "partially expanded" polymer models, see [BZK]) by such a more general condition. Then the translation invariancy of w_T is assumed to hold only if T is *compatible* with Λ^q, $T \subset \Lambda^q$ while for clusters T^q touching $(\Lambda^q)^c$, no translation invariancy is assumed and w_T may depend on the "shape" of Λ^q around T, too. (We could assume some invariancy w.r. to the shifts of the pair $T, T \cap \Lambda^q$ but we will not do it.)

Consider also the more general, $\mathcal{C}$ – conditioned partition function

$$Z_{\mathcal{C}}(\Lambda) = \sum_{\{\mathbf{\Gamma}_1,\ldots\mathbf{\Gamma}_m\}} \prod_{i=1}^{m} \exp(-E(\mathbf{\Gamma}_i)) \prod_q \exp(-e_q|\Lambda_q|) \ \exp(\sum_{T_q \sqsubset \Lambda_q} w_{T_q}) \tag{1.24}$$

where $\mathcal{C}$ is some limitation on allowed configurations (in the sum (1.24)) formulated in the language of contours (taken from some given subsystem $\mathcal{S}$ of $\mathcal{A}$).

Note. Boundary condition "at infinity" on a torus Λ. Imagine e.g. the limitation $\mathcal{C} \equiv \mathcal{C}^q$ saying that external contours of any configuration x_Λ in the sum (1.24) have a fixed prescribed color q. Denote further by $\mathcal{S}_q$ the collection of all contours containing "infinity" in their interior. Then the limitations $\mathcal{C}_q$ & possibly also additional condition that "contours from $\mathcal{S}_q$ are all forbidden in x_Λ" give reasonable choices of what could be called the q–th boundary condition on Λ. The summation in (1.23) resp. in (1.24) is then over all compatible contour systems resp. over all compatible contour systems satisfying such a limitation.

Below we shall assume that the bounds (1.13) and (1.14), (1.15) for the cluster weights and for the contour weights are valid. This is a convenient way how the polymer and cluster bounds are formulated, and we refer to [KP], [DV], and especially to [BZK] (see also [NOZ]) for an additional information (and more reference to previous work) on these bounds. The constant g^{-1} can be interpreted as the correlation length of the contour gas (of contours $\mathbf{\Gamma}$, without the cluster field) and similarly the quantity c^{-1} is a correlation length of the "expanded gas" described by the cluster quantities w_T.

Rephrasement of the contour and cluster estimates (1.14) and (1.13).

Later, we will need to reinterpret the bounds (1.14) in view of the construction of contour functional $F(\mathbf{\Gamma})$, more specifically $F(\mathbf{\Gamma}_\odot)$. But first recall the bound (1.13)

$$\sum_{T \ni x_i} |w_T| \; e^{c|\underline{T}|} \; \leq \; \epsilon \ll \; 1 \ll \; c \tag{1.25}$$

which will determine our actual choice of the small constant ϵ in all the bounds below.

More interesting will be a reformulation of (1.14), motivated by the definition of a contour functional below. Rewrite it as follows, introducing a new auxiliary constant a ($\doteq g$)

$$a = g - \epsilon - \omega \tag{1.26}$$

where ϵ is from (1.13) and ω is the supremum of the l.h.s. of (1.14), written now as

$$\sum_{\underline{\mathbf{\Gamma}} \ni i} e^{-E(\mathbf{\Gamma})} \; e^{(a+\epsilon+\omega)|\underline{\mathbf{\Gamma}}|} \; \leq \; \omega. \tag{1.27}$$

More specifically, also the bound (1.15) can be rewritten as

$$\sum_{\underline{\mathbf{\Gamma}} \ni i, \, \mathrm{diam} \, \underline{\mathbf{\Gamma}} = N} e^{-E(\mathbf{\Gamma})} \; e^{(a+\epsilon+\omega)|\underline{\mathbf{\Gamma}}|} \; \leq \; \omega_N \; ; \qquad \sum_N \omega_N = \omega \tag{1.28}$$

where ω_N decay like $\omega_N \; \leq \; K \; \kappa^N, \; \kappa < 1$ and K is a suitable constant independent on a.

Recall that a is still of the order g (and τ) while ω is of the order $e^{-C\tau}$, so for a large τ we surely have a trivial inequality, see more precisely below in (2.10)

$$\omega < a \qquad \text{i.e.} \qquad 2\omega + \epsilon < g. \tag{1.29}$$

Note. While the cluster bound (1.25) is still just (1.13), the presence of another small constants ω and ϵ newly incorporated into (1.27) and later estimates (compare the formulations of Theorem 1 and Theorem 2 below, where another auxiliary, large enough, constant Ω will be also introduced) will be important and not trivial. While there is no necessity to introduce such new constants in polymer expansion problems (like in partially expanded polymer models of [BZE]) these constants will be essential later, in the context of general Pirogov Sinai theory dealing with matching compatible collections of "polychromatic" contours.

This is related to a later construction of a contour functional $F(\mathbf{\Gamma}_\odot)$, related to the energy $E(\mathbf{\Gamma}_\odot)$ as follows. See Theorem 1 and below it, for a detailed discussion on why we introduce a new, seemingly arbitrary term e^{Ω} here. One should keep in mind that the constants ω and ϵ are small, so we will still have roughly $E(\mathbf{\Gamma}_\odot)$ in the exponential on the r.h.s.

$$e^{-F(\mathbf{\Gamma}_\odot)} \; \leq \; e^{\Omega} \;\; e^{-E(\mathbf{\Gamma}_\odot)+(\omega+\epsilon)|\underline{\mathbf{\Gamma}_\odot}|}. \tag{1.30}$$

Definition of the contour functional $F(\mathbf{\Gamma}_\odot)$.

The quantity playing a fundamental role in developing of Pirogov Sinai theory is their "contour functional" $F(\mathbf{\Gamma}_\odot)$ – here considered as functional of contour systems $\mathbf{\Gamma}_\odot$ rather than single contours $\mathbf{\Gamma}$. This is really a key concept. It appeared first in the paper [PS]. We will use a variant (of [Z] and subsequent papers) of this quantity, technically more suitable for us. It could be also called the "free energy" of the contour system $\mathbf{\Gamma}_\odot$, or "work needed to install $\mathbf{\Gamma}_\odot$". Hence also the notation, $F(\mathbf{\Gamma}_\odot)$, in contrast to the *energy* $E(\mathbf{\Gamma}_\odot)$ of $\mathbf{\Gamma}_\odot$.

So, we will define such a quantity not only for single contours but more generally for special compatible *systems* $\mathbf{\Gamma}_\odot$ of contours. However, in contrary to previous versions of P.S. theory we will use this notion *only* for contours or contour systems which are "small"(see below) and "extremal" ones. Recall that latter means that no other contour from[6] $\cup_{\tilde q}\mathcal{H}^{\tilde q}$ can be placed inside $\cup_{\tilde q}\operatorname{int}_{\tilde q}\mathbf{\Gamma}_\odot$ compatibly with $\mathbf{\Gamma}_\odot$.

Let $\mathbf{\Gamma}_\odot$ be an extremal contour or a compatible system of contours, let $q = q_{\mathbf{\Gamma}_\odot}$ denote the external color of $\mathbf{\Gamma}_\odot$. Recall that one often writes $\mathbf{\Gamma}_\odot = \mathbf{\Gamma}^q_\odot$ in such a case. For any $\tilde q \in Q$ let $\operatorname{int}_{\tilde q}\mathbf{\Gamma}_\odot$ denote the union of components of $\operatorname{int}\mathbf{\Gamma}_\odot$ colored by $\tilde q$. Define

$$F(\mathbf{\Gamma}^q_\odot) = E(\mathbf{\Gamma}^q_\odot) + \sum_{\tilde q\neq q}\Big((e_{\tilde q} - e_q)|\operatorname{int}_{\tilde q}\mathbf{\Gamma}_\odot| + \sum_{T_q\subset \operatorname{int}_{\tilde q}\mathbf{\Gamma}_\odot} w_{T_q} - \sum_{T_{\tilde q}\subset \operatorname{int}_{\tilde q}\mathbf{\Gamma}_\odot} w_{T_{\tilde q}}\Big). \tag{2.1}$$

Note. Notice that we have sums over clusters of *different* colors T_q resp. $T_{\tilde q}$ here. It is important that the last two sums in (2.1) do *not* cancel. In fact they produce, like the terms $(e_{\tilde q} - e_q)|\operatorname{int}_{\tilde q}\mathbf{\Gamma}_\odot|$, a volume term possibly driving $F(\mathbf{\Gamma}_\odot)$ to negative values for large $\mathbf{\Gamma}_\odot$. While geometrically the clusters T_q and $T_{\tilde q}$ may be the same objects the value w_{T_q} depends substantially on the color q. Finally notice that the clusters $T \sqsubset \operatorname{int}\mathbf{\Gamma}_\odot$ not obeying $T \subset \operatorname{int}\mathbf{\Gamma}_\odot$ are ignored. Their values have only marginal effects. Cancellation of both $e_q - e_{\tilde q}$ and of the interior sums over T_q takes place typically *only* in the case when there is a symmetry between various q in the definition of the model. In such a case we obviously have

$$F(\mathbf{\Gamma}_\odot) \equiv E(\mathbf{\Gamma}_\odot) \tag{2.2}$$

and a simple Peierls argument can be used to estimate the probability of the event that $\mathbf{\Gamma}_\odot$ appears. If a Peierls condition holds we can also easily find cluster expansion formulas for these probabilities. It is a characteristic feature of P. S. theory that it develops a universal machinery applicable to the cases when such a cancellation does *not* take place.

It is worthwhile to give another expression of (2.1), clarifying better its meaning:

"Metastable free energies"of the model (1.23). Another expression of $F(\mathbf{\Gamma}_\odot)$.

[6] If q denotes the external color of $\mathbf{\Gamma}_\odot$ then we use the symbol $\tilde q$ to denote all the variables from Q.

Denote by f_q the "metastable free energy" of the model (1.23) (defined if we "remove all contours $\mathbf{\Gamma}^q$ from it"), more precisely the quantity (independent on a choice of $i \in \Lambda$)

$$f_q = e_q - \sum_{T_q \ni i} \frac{w_{T_q}}{|\underline{T}_q|} \;=\; e_q - \; s_q \tag{2.3}$$

where e_q is the mean energy of the constant configuration $x_\Lambda{}^q \equiv q$ and

$$s_q = \sum_{T_q \;\ni\; i} \frac{w_{T_q}}{|\underline{T}_q|} \;. \tag{2.4}$$

The quantity s_q can be often interpreted as the free energy of a restricted ensemble in some original model (1.1), which was first partially expanded to the form (1.11).

Theorem 0. *The contour functional (2.1) can be also expressed as follows, by using the quantities f_q: For any extremal system $\mathbf{\Gamma}_\odot = \mathbf{\Gamma}_\odot^q$ with an external color q we have*

$$F(\mathbf{\Gamma}_\odot) = E(\mathbf{\Gamma}_\odot) + \sum_{\tilde{q} \neq q} (f_{\tilde{q}} - f_q) |\operatorname{int}_{\tilde{q}} \mathbf{\Gamma}_\odot| + \Delta(\mathbf{\Gamma}_\odot) \tag{2.5}$$

where the additional "surface" term $\Delta(\mathbf{\Gamma}_\odot)$ is relatively very small, satisfying a bound

$$|\Delta(\mathbf{\Gamma}_\odot)| \leq \; (\omega + \epsilon) |\underline{\mathbf{\Gamma}_\odot}| \tag{2.6}$$

where ϵ and ω are taken from (1.27) and (1.25).

The quantity $\epsilon + \omega$ (we may understand now why we imposed it into (1.27)) bounds from above the "surface tension density" of the model. The latter gives a relatively small correction to the main term $E(\mathbf{\Gamma}_\odot)$ of (2.5). The role of the volume terms $(f_{\tilde{q}} - f_q)|\operatorname{int}_{\tilde{q}} \mathbf{\Gamma}_\odot^q|$ will be much more important for large enough systems $\mathbf{\Gamma}_\odot^q$ and will be investigated in detail below.

Note on a factorized contour functional. The following notion suggests another approach to the problem, closer in spirit to the original P.S. theory. Let $\mathbf{\Gamma}$ be a contour which is a frame of some extremal contour system $\mathbf{\Gamma}_\odot$. Define the "factorized" contour functional $F_f(\mathbf{\Gamma})$ by

$$\exp(-F_f(\mathbf{\Gamma})) = \sum_{\mathbf{\Gamma}_\odot} \exp(-F(\mathbf{\Gamma}_\odot)) \tag{2.7}$$

where the summation is over all extremal contour systems $\mathbf{\Gamma}_\odot$ having the same frame $\mathbf{\Gamma}$. However, this quantity is less technically convenient than $F(\mathbf{\Gamma})$ and we will not use it.

Instability parameters of the model (1.23).

Recall that f_q denotes the "free energy of the restricted ensemble" of (1.23) . Put

$$f = \min_{q \in Q} f_q \tag{2.8}$$

and define useful auxiliary quantities, for any $q \in Q$

$$u_q = f_q - f \quad ; \quad \min_{q \in Q} \; u_q = 0. \tag{2.9}$$

Note. A more traditional notation of these quantities ("Pirogov Sinai parameters") would be a_q but we already used, in accordance with [BZE] and the references quoted there, the letter a for a different purpose in (1.27). A crude approximation to u_q is the quantity $e_q - \min_{q' \in Q} e_{q'}$. Recall that we may assume for a simplicity that $\min_q e_q = 0$ in these considerations.

Small cubes and small contour systems.

We introduce now some other important notions, helping to handle also the *entropy* of extremal contour systems $\mathbf{\Gamma}_{\odot}$. Namely, these systems still can be rather "disconnected inside of $\mathbf{\Gamma}$" if they have a large number of mutually distant interior contours.

The forthcoming definitions deviate a little bit from that of [HZ] and [ZR]. While [HZ] works with a more delicate situation of "stratified" systems the paper [ZR] followed, somehow arbitrarily, the definition of a small cube from [HZ]. The estimates here are somewhat sharper and more systematic (concerning the bounds for cluster expansion terms) than the corresponding ones of [ZR].

First rewrite (1.27) as follows, introducing the new reference constants Ω and b

$$\tilde{L} \sum_{\underline{\mathbf{\Gamma}} \ni i} e^{-E(\mathbf{\Gamma})} \; e^{(a+\epsilon+\omega)|\underline{\mathbf{\Gamma}}|} \; \leq \; e^{-\Omega} \; b \tag{2.10}$$

where $\tilde{L} \doteq 1$ is another auxiliary constant (which will be fixed below, in (3.3), imagine that $\tilde{L} = 1$) and where we take Ω and b such that (2.10) would be just the relation (1.27) but not with $b\; e^{-\Omega} = \tilde{L}\; \omega$ but with a slightly bigger constant b such that

$$b\; e^{-\Omega} = \tilde{L}\omega(1+\tilde{\omega})$$

where the value of $\tilde{\omega} \ll 1$ will be introduced later, in (2.22) and the term $1+\tilde{\omega}$ will appear in (2.28). Recall also the bound (1.28), which will be written now analogously as (2.10)

$$\sum_{\underline{\mathbf{\Gamma}} \ni i : \operatorname{diam} \mathbf{\Gamma} = N} e^{-E(\mathbf{\Gamma})} \; e^{(a+\epsilon+\omega)|\underline{\mathbf{\Gamma}}|} \quad \leq e^{-\Omega} \; \frac{b_N}{\tilde{L}} \quad \leq \; K \; \kappa^N \; ; \; \sum_N b_N = b. \tag{2.11}$$

It will be useful to assume also the following trivial inequality, see Corollary 2 below

$$b < c \leq a. \tag{2.12}$$

See (1.25) for the value of c. If c was too large, such that $c > a$ lower it now to $c = a$.

The new constant Ω will play a substantial role[7] in our definition of a small contour system. Let $\Box$ be a cube of the type $\Box = I^{\nu}$ where I is some, closed from below and open from above, interval in $\mathbb{Z}$. Say that it is a q–small cube of the model (1.23) if the volume of the cube is

$$u_q |\Box| \leq \Omega \tag{2.13}$$

in other words if the length of a side of such a cube is

$$\operatorname{diam} \Box \leq \left(\frac{\Omega}{u_q} \right)^{\frac{1}{\nu}}. \tag{2.14}$$

We will speak simply about a small cube, if no ambiguity arises concerning the value q. Finally we will say that a contour resp. contour system $\mathbf{\Gamma}_{\odot} = \mathbf{\Gamma}^q_{\odot}$ is *small* if there exists a q– small "packing cube" $\Box$ of $\mathbf{\Gamma}_{\odot}$ such that $\underline{\mathbf{\Gamma}_{\odot}} \subset \Box$.

This notion suggests the way how we will organize our expansions. Namely, by choosing *small*, extremal contours resp. contour systems, as objects to be expanded. Recall that for extremal small $\mathbf{\Gamma}_{\odot}$, no small contours "smaller than $\mathbf{\Gamma}_{\odot}$" exist (they were expanded before).

[7] Any large enough value of a "testing constant" $\Omega \leq \tau |\underline{\mathbf{\Gamma}}|$ would do a similar job. The test could be applied directly to contours instead of cubes, with values $C|\operatorname{diam} \mathbf{\Gamma}|$ or $C|\underline{\mathbf{\Gamma}}|$ like in [ZR] or [Z].

The skeleton of a system $\mathbf{\Gamma}_\odot$. Peierls condition for a small, extremal $\mathbf{\Gamma}_\odot$.

The forthcoming lower bound for $F(\mathbf{\Gamma}_\odot)$ will be combined with some entropy estimates developed later. Recall that even a small, extremal contour $\mathbf{\Gamma}$ may contain a lot of interior, nonsmall contours forming, together with the external contour $\mathbf{\Gamma}$, an extremal system $\mathbf{\Gamma}_\odot$. The following construction will help to treat their entropy in a technically convenient way.

For any $\tilde{q} \in Q$, fix a decomposition $\mathcal{K}_{\tilde{q}}$ of the lattice Λ into mutually disjoint non $\tilde{q}$–small cubes of a *smallest possible* diameter. The union, for all possible colors $\tilde{q} \in Q$, of all cubes $\square$, from a corresponding $\mathcal{K}_{\tilde{q}}$, which are *subsets* of $\operatorname{int}_{\tilde{q}} \mathbf{\Gamma}_\odot$[8] will be called the *skeleton* of $\mathbf{\Gamma}_\odot$. The cardinality of this system of cubes (taken over all q) will be denoted by $\operatorname{conn}(\mathbf{\Gamma}_\odot)$.

Notes. 1) Here, the fact that we are on a finite torus and not on $\mathbb{Z}^\nu$ causes some minor technical inconveniences, which are however solvable in an obvious way. For example it is technically convenient to assume that the diameter of the torus Λ is divisible by the diameters of all the considered "smallest" cubes. We omit a discussion on how to polish the formulation of the forthcoming constructions in the situations when such an assumption is not valid.

2) Of course, complicated contour systems with *nonzero* $\operatorname{conn}(\mathbf{\Gamma}_\odot)$ usually will not figure at the crudest approximations of the model. Nevertheless they play a decisive role in the more delicate technical aspects of the theory, and they should be treated carefully! Here we present a treatment which somehow deviates from the treatment of [ZR]. It brings some simplifications, related to more systematic usage of the concept of a partially expanded model.

Theorem 1. *Let $\mathbf{\Gamma}_\odot$ be a small, extremal contour system. Then*

$$F(\mathbf{\Gamma}_\odot) \geq E(\mathbf{\Gamma}_\odot) - (\omega + \epsilon)|\underline{\mathbf{\Gamma}_\odot}| + \Omega \operatorname{conn}(\mathbf{\Gamma}_\odot) - \Omega. \tag{2.15}$$

Proof. We will show more generally that any extremal (not necessarily small) system $\mathbf{\Gamma}_\odot = \mathbf{\Gamma}_\odot^q$ with the external color q satisfies an inequality of the type

$$F(\mathbf{\Gamma}_\odot) \geq E(\mathbf{\Gamma}_\odot) - (\omega + \varepsilon)|\underline{\mathbf{\Gamma}_\odot}| + \Omega \operatorname{conn}(\mathbf{\Gamma}_\odot) - u_q |\operatorname{int} \mathbf{\Gamma}_\odot| \tag{2.16}$$

and this will yield (2.15) for *small* systems $\mathbf{\Gamma}_\odot$, because smallness of $\mathbf{\Gamma}_\odot$ implies

$$u_q |\operatorname{int} \mathbf{\Gamma}_\odot| \leq u_q |\square| \leq \Omega. \tag{2.17}$$

Recall the expression (2.5) of $F(\mathbf{\Gamma}_\odot)$ for a system $\mathbf{\Gamma}_\odot$ with external color q namely

$$F(\mathbf{\Gamma}_\odot) = E(\mathbf{\Gamma}_\odot) + \sum_{\tilde{q} \neq q} (u_{\tilde{q}} - u_q) |\operatorname{int}_{\tilde{q}} \mathbf{\Gamma}_\odot| + \Delta(\mathbf{\Gamma}_\odot) \tag{2.18}$$

(notice that $f_q - f_{\tilde{q}} = u_q - u_{\tilde{q}}$) where the "error term" $\Delta(\mathbf{\Gamma}_\odot)$ satisfies

$$|\Delta(\mathbf{\Gamma}_\odot)| \leq (\omega + \varepsilon)|\underline{\mathbf{\Gamma}_\odot}|).$$

[8] For a small system $\mathbf{\Gamma}_\odot^q$, the sets $\operatorname{int}_{\tilde{q}} \mathbf{\Gamma}_\odot^q$ may contain such squares only for $\tilde{q} \neq q$.

So it really suffices to estimate from below the middle term in (2.18). Ignore the term $u_q \sum_{\tilde q\in Q} |\operatorname{int}_{\tilde q} \mathbf{\Gamma}_\odot|$ which was just estimated in (2.17) and notice that the (colored, by various $\tilde q \in Q$) cubes of the skeleton make a disjoint decomposition of a suitable "coarse grained" *subset* of the corresponding $\operatorname{int}_{\tilde q} \mathbf{\Gamma}_\odot$. Imagine that the "volume energy excess $u_{\tilde q}|\Box|$ of any such cube is converted into the value Ω" . Then we have the obvious bound

$$\sum_{\tilde q\in Q} u_{\tilde q} |\operatorname{int}_{\tilde q} \mathbf{\Gamma}_\odot| > \Omega \operatorname{conn}(\mathbf{\Gamma}_\odot). \tag{2.19}$$

Note. The estimate above, namely the "conversion of the volume energy excess of the interior of $\mathbf{\Gamma}_\odot$ to a skeleton energy capable to beat its entropy" conveniently replaces and strengthens an older estimate,"Main Lemma" of [Z84]. It already appeared in [HZ] and [ZR].

Enriched contour system $\mathbf{\Gamma}_\odot^\Box$, interpreted as a tree.

Below, we will expand the small, extremal contour systems $\mathbf{\Gamma}_\odot$. Recall that these systems have always one external contour, denoted below always by symbol $\mathbf{\Gamma}$, but they may contain (though not typically) many interior *nonsmall* contours.

In principle, we could have two alternatives now, either to expand simultaneously all the extremal systems $\mathbf{\Gamma}_\odot$ with the same, "small" external contour $\mathbf{\Gamma}$, or to expand all the systems $\mathbf{\Gamma}_\odot$ together, with a fixed external contour $\mathbf{\Gamma}$. The latter approach would require a use of a factorized, "external" contour functional (2.7) $F_f(\mathbf{\Gamma})$. While the latter, more traditional approach is simpler conceptionally the simultaneous expansion of the systems $\mathbf{\Gamma}_\odot$ is actually more natural technically, and it gives sharper control of the expansions.

We will rely here on the notion of a small contour system, and on the expression (2.5) and the bound (2.15). They will be combined with the observation, elaborated now, namely that "adding of the cubes of the skeleton makes any extremal system $\mathbf{\Gamma}_\odot$ almost connected".

Denote by $\mathbf{\Gamma}_\odot^\Box$ the contour system $\mathbf{\Gamma}_\odot$ enriched by the collection of all the cubes (having appropriate colors) of its skeleton. It will be useful to construct a structure of a *tree* on $\mathbf{\Gamma}_\odot^\Box$, with a root $\mathbf{\Gamma}$, for any simple extremal contour system $\mathbf{\Gamma}_\odot$ having an external contour $\mathbf{\Gamma}$. Such a tree will be defined as a suitable extension of a forest $\mathcal{T}_{\mathbf{\Gamma}_\odot}$ constructed now on the vertex family of all contours $\mathbf{\Gamma}$ resp. $\mathbf{\Gamma}_i$ of $\mathbf{\Gamma}_\odot = (\mathbf{\Gamma}, \{\mathbf{\Gamma}_i\})$, and all cubes $\Box$ of its skeleton.

Consider first an auxiliary tree $\mathcal{T}_{aux}$ whose vertices are all contours of $\mathbf{\Gamma}_\odot$ and also all *components* C of $\operatorname{int} \mathbf{\Gamma}_\odot$, and whose (provisionally) ordered bonds are characterized just as "go inside". Then "forget" the vertices of the type "component" and from the ordered bonds $(\mathbf{\Gamma}', \mathbf{\Gamma}'')$ thus arising from $\mathcal{T}_{aux}$ keep only those bonds satisfying a relation of commensurability

$$\operatorname{dist}(\underline{\mathbf{\Gamma}}', \underline{\mathbf{\Gamma}}'') \le 2 \operatorname{diam} \underline{\mathbf{\Gamma}}''. \tag{2.20}$$

Thus we get some forest $\mathcal{T}_{\mathbf{\Gamma}_\odot}$ of trees on the collection of contours of $\mathbf{\Gamma}_\odot$, not necessarily a connected one if the interior contours of $\mathbf{\Gamma}_\odot$ are very "sparse".

To make a tree from such $\mathcal{T}_{\mathbf{\Gamma}_\odot}$, denoted below by $\mathcal{T}_{\mathbf{\Gamma}_\odot}^\Box$ and having as set of vertices the family $\mathbf{\Gamma}_\odot^\Box$, we have to add some other (yet unordered) bonds ("forget" now the ordering of previously constructed bonds, just keep in mind that $\mathbf{\Gamma}$ will be the root) going also *through the cubes* of the skeleton. These new bonds of the type contour | cube resp. cube | cube will be unordered and defined as follows. Among the objects, either contours $\mathbf{\Gamma}'$ or cubes $\square$ in the same component C of $\operatorname{int}\mathbf{\Gamma}_\odot$ we consider a relation of commensurability

$$AcB \iff \operatorname{dist}(A,B) \le 2\min(\operatorname{diam} A, \operatorname{diam} B). \tag{2.21}$$

Clearly, when adding all these bonds to the above forest $\mathcal{T}_{\mathbf{\Gamma}_\odot}$ we get a *connected graph.* Then we take a subtree of this graph, by possibly erasing some of the newly added bonds. The construction of such a tree is of course nonunique, its only aim is to formalize the property of "almost connectedness" of the enriched system $\mathbf{\Gamma}_\odot^\square$.

Bounds on the decay of $e^{-F(\mathbf{\Gamma}_\odot)}$ for extremal contour systems $\mathbf{\Gamma}_\odot$.

Now we will transcript (2.10) into an analogous inequality for the sum over *systems* $\mathbf{\Gamma}_\odot$

$$\sum_{\mathbf{\Gamma}_\odot\, \iota\, i} e^{-F(\mathbf{\Gamma}_\odot)+a|\underline{\mathbf{\Gamma}_\odot}|}$$

where the symbol $\mathbf{\Gamma}_\odot\, \iota\, i$ ("i is incompatible with $\mathbf{\Gamma}_\odot$") means that i overlaps the support $\underline{\mathbf{\Gamma}_\odot}$.

Actually, we will still have to lower the constant a (which tests the "correlation length" of the gas with fugacities $e^{-F(\mathbf{\Gamma}_\odot)}$) slightly, see below for details. Take

$$a' = a - \tilde\omega \qquad \text{where} \qquad \tilde\omega \approx \omega + \xi\ e^{-\Omega} \qquad \text{and} \qquad \xi = \max_q \left(\frac{u_q}{\Omega}\right)^{\frac{1}{d}}; \tag{2.22}$$

the constant ξ being proportional to the maximal inverse value of a diameter of a cube in a skeleton, see more precisely below.

Theorem 2. *For any $i \in \Lambda$ we have the bound (compare (2.10) and (2.11))*

$$\sum_{\mathbf{\Gamma}_\odot\, \iota\, i : \operatorname{diam}\mathbf{\Gamma}_\odot = N} e^{-F(\mathbf{\Gamma}_\odot)}\ e^{a'|\underline{\mathbf{\Gamma}_\odot}|} \ \le\ \frac{b_N}{\tilde L} \tag{2.23}$$

where b_N decay geometrically like $b_N \le\ K\ \kappa^N$ and

$$\sum_N b_N = b \qquad \textit{with} \qquad b < c < a'.$$

Proof. We will treat this sum as a sum over $\mathbf{\Gamma}_\odot^\square$, and identify the latter objects with trees described above, denoted by $\mathcal{T}_\square^{\mathbf{\Gamma}_\odot}$. Write the quantity on the r. h. s. of (2.15) as

$$F(\mathbf{\Gamma}_\odot^\square) \equiv E(\mathbf{\Gamma}_\odot) - (\omega+\epsilon)|\underline{\mathbf{\Gamma}_\odot}| + \Omega(\operatorname{conn}(\mathbf{\Gamma}_\odot) - 1). \tag{2.24}$$

Then we can write the following estimate for the sum over all $\mathbf{\Gamma}_\odot$ having external contour $\mathbf{\Gamma}$:

$$\sum_{\mathbf{\Gamma}_\odot \ni \mathbf{\Gamma}} e^{-F(\mathbf{\Gamma}_\odot)+a'|\underline{\mathbf{\Gamma}_\odot}|} \le \sum_{\mathbf{\Gamma}_\odot \ni \mathbf{\Gamma}} e^{-F(\mathbf{\Gamma}_\odot^\square)+a'|\underline{\mathbf{\Gamma}_\odot}|)} \le \sum_{\mathbf{\Gamma}_\odot \ni \mathbf{\Gamma}} e^{-E(\mathbf{\Gamma}_\odot)}\, e^{(a'+\omega+\epsilon)|\underline{\mathbf{\Gamma}_\odot}|}\, e^{\Omega(1-|\operatorname{conn}\mathbf{\Gamma}_\odot|)}$$

$$\le e^{\Omega} \sum_{\mathbf{\Gamma}_\odot \ni \mathbf{\Gamma}} \prod_{\mathbf{\Gamma}' \in \mathbf{\Gamma}_\odot} e^{-E(\mathbf{\Gamma}')}\, e^{(a+\omega+\epsilon)|\underline{\mathbf{\Gamma}'}|}\, e^{-\tilde\omega|\underline{\mathbf{\Gamma}'}|} \prod_{\square \in \mathbf{\Gamma}_\odot^\square} e^{-\Omega+\tilde\omega}\, e^{-\tilde\omega} \tag{2.25}$$

where last inequality is written such that we could use (2.10) for all contours $\mathbf{\Gamma}'$ of $\mathbf{\Gamma}_\odot$.

So, we are in a slightly generalized situation of the type (1.7)–(1.8). The non-root vertices of the tree $\mathcal{T}_\square^{\mathbf{\Gamma}_\odot}$ considered here are either 1) *nonsmall* internal contours of $\mathbf{\Gamma}_\odot$ or 2) *nonsmall* skeleton cubes in Λ (colored by values $q \in Q$). The bonds between these objects are as described above, and $\tilde\omega$ will estimate the sum of weights of these bonds. Let us write $\mathbf{\Gamma}' \to \mathbf{\Gamma}''$, $\mathbf{\Gamma}' \to \square$, $\square \to \square'$ etc. if there could be a bond between these two objects (contours or cubes of the skeleton) in some tree $\mathcal{T}_\square^{\mathbf{\Gamma}_\odot}$, in the direction "from the root".

The sums over the trees $\mathcal{T}_\square^{\mathbf{\Gamma}_\odot}$ will be now estimated by using a general tree estimate (1.8). While the root activity $w_{\mathbf{\Gamma}}$ will be kept temporarily aside, the activities of the remaining vertices of $\mathcal{T}_\square^{\mathbf{\Gamma}_\odot}$ will be considered as *bond* activities, and (1.8) will be used to estimate the sum over all trees with a given root $\mathbf{\Gamma}$. Notice that we have an additional "bond"activity $e^{-\Omega+\tilde\omega}$ for any cube of the skeleton while the terms $e^{-\tilde\omega|\mathbf{\Gamma}|}$ resp. $e^{-\tilde\omega}$ are assigned to the vertices (contours resp. cubes) of the tree $\mathcal{T}_\square^{\mathbf{\Gamma}_\odot}$ so that a bound of the type (1.7) should be achieved. Denote by

$$M = \min\left(\frac{\Omega}{u_q}\right)^{1/d}$$

the minimal size of a small contour resp. cube from $\mathbf{\Gamma}_\odot^\square$. Using (1.28) we have the following bounds for the sums of the above activities of bonds containing a given contour $\mathbf{\Gamma}'$ resp. a given cube $\square$. For any contour $\mathbf{\Gamma}'$ of the system $\mathbf{\Gamma}_\odot$ consider a sum (like that used in (1.7); the arrow means that there could be a bond of some tree $\mathcal{T}_\square^{\mathbf{\Gamma}_\odot}$ between these two objects)

$$\zeta_{\mathbf{\Gamma}'} = \sum_{\mathbf{\Gamma}' \to \mathbf{\Gamma}''} e^{-E(\mathbf{\Gamma}'')} e^{a+\omega+\epsilon)|\underline{\mathbf{\Gamma}''}|)} + \sum_{\mathbf{\Gamma}' \to \square} e^{-\Omega+\tilde\omega} \le |\underline{\mathbf{\Gamma}'}|\,(KM^d\kappa^M + K'\frac{e^{-\Omega+\tilde\omega}}{M}) \le \tilde\omega\,|\underline{\mathbf{\Gamma}}| \tag{2.26}$$

for a suitably small $\tilde\omega$. Here the first part of the estimate (the value $|\underline{\mathbf{\Gamma}'}|M^d$ bounds the number of possibilities how $\mathbf{\Gamma}''$ can be "anchored" in a vicinity of a $\mathbf{\Gamma}'$) follows immediately from (1.28) and the second part uses elementary upper bound $K'|\underline{\mathbf{\Gamma}'}|/M$ for the number of cubes of the skeleton sticked to $\mathbf{\Gamma}'$.

On the other hand, for any cube of the skeleton $\square$ we have the estimate

$$\zeta_\square = \sum_{\square \to \mathbf{\Gamma}''} e^{-E(\mathbf{\Gamma}^q)} e^{(a+\omega+\epsilon)|\underline{\mathbf{\Gamma}''}|)} + \sum_{\square \to \square'} e^{-\Omega+\tilde\omega} \le K''|\square|\,\kappa^{\operatorname{diam}\square} + 2de^{-\Omega+\tilde\omega} + \le \tilde\omega \tag{2.27}$$

where again the first part of the estimate bounds the weights of contours $\mathbf{\Gamma}''$ sticked to the cube $\square$ and it uses the important fact that the contours are nonsmall i.e. have a diameter at least diam $\square$. Then we have to choose suitable $\tilde{\omega}$ such that the final inequalities in these last two bounds are really valid. Let us emphasize once again that in both inequalities it was rather important that the size of possible $\mathbf{\Gamma}''$ is *big*, at least M resp. $1/2\,\mathrm{diam}\,\mathbf{\Gamma}'$.

Now we combine these estimates and our general tree estimate (1.8). In fact, we need a generalization of (1.8) (already mentioned there) for *site dependent* quantities ζ_i, here denoted as $\zeta_{\mathbf{\Gamma}'}$ and $\zeta_{\square}$. The final estimate produced from such a generalized estimate (1.8) is the following one, with the sum being over all extremal systems $\mathbf{\Gamma}_\odot$ having $\mathbf{\Gamma}$ as their external contour (notice that $2-e^{-\tilde{\omega}} \approx 1$, we include here also the case of an empty tree)

$$\sum_{\mathbf{\Gamma}_\odot\, \iota\, i : \mathbf{\Gamma}_\odot \ni \mathbf{\Gamma}} \prod_{\mathbf{\Gamma}' \in \mathbf{\Gamma}_\odot} e^{-E(\mathbf{\Gamma}')}\; e^{(a+\omega+\epsilon)|\underline{\mathbf{\Gamma}'}|}\;\; e^{-\tilde{\omega}|\underline{\mathbf{\Gamma}'}|} \prod_{\square \in \mathbf{\Gamma}_\odot^{\square}} e^{-\Omega} \;\le (2-e^{-\tilde{\omega}})\; e^{-E(\mathbf{\Gamma})} e^{(a+\omega+\epsilon)|\underline{\mathbf{\Gamma}}|}$$

and from this we finally get the relation, obviously proving (2.23) by (2.10)

$$\sum_{\mathbf{\Gamma}_\odot\, \iota\, i : \mathrm{diam}\, \mathbf{\Gamma}_\odot = N} e^{-F(\mathbf{\Gamma}_\odot)}\; e^{a'|\underline{\mathbf{\Gamma}_\odot}|} \;\le (1+\tilde{\omega}) e^{\Omega} \sum_{\mathbf{\Gamma}\, \iota\, i : \mathrm{diam}\, \mathbf{\Gamma} = N} e^{-E(\mathbf{\Gamma}) + (a+\omega+\epsilon)|\underline{\mathbf{\Gamma}}|}. \tag{2.28}$$

3. Expansion of small extremal subsystems $\mathbf{\Gamma}_\odot$ in an abstract P.S. model

This is the main step of Pirogov Sinai theory, if we understand the latter just as a method of (low temperature) cluster expansions. Before going to our actual situation, namely to a model with “matching compatible” contour systems and with external cluster field we explain briefly the “monochromatic” version of the problem, arising for the case $|Q| = 1$. This is of course uninteresting from the point of view of phase transitions, but it is still important as a source of some standard cluster expansion technology:

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We will now rephrase the second section of [BZE]. The notations will be taken in accordance with that paper (for an easy comparison) e.g. clusters (here also called chains) denoted by us later as T resp. $\mathbb{T}$ are now provisionally denoted by G resp. I and contours (here called just polymers) $\mathbf{\Gamma}$ by P. Later we will return to our usual notations. We present here a slightly less abstract (in the requirements on the constants a and b resp. c ; these are *not* considered to be general functions of polymers resp. clusters here) but also a slightly more general (in the requirements on cluster weights for clusters touching the polymers) variant of Theorem 2 of that paper. The proof is completely analogous to that given there, so we omit it.

Consider a “monochromatic” polymer model with external cluster field i.e. the model of the type (1.11) however with $|Q| = 1$ and $e_q \equiv 0$. Given *complex* fugacities $\{w_P\}$ and $\{w_G\}$ define the partition function in a volume $M \subset \Lambda$

$$Z(M) = \sum_{\{P_1,\dots,P_n\}_c : P_i \in \mathcal{P}} \prod_{i=1}^{n} w_{P_i}\; \exp\left(\sum_{G \sqsubset (\cup_i \underline{P}_i)^c} w_G \right) \tag{3.1}$$

where the first sum is over all families $\{P_1, \dots, P_n\}_c$ of pairwise compatible polymers in $\mathcal{P}$ and the second sum $\sum_G w_G$ is over all chains G touching the complement of the union of supports of all polymer P_i, the latter property being denoted as $G \sqsubset (\cup_i \underline{P}_i)^c$. In [BZE] and other references, only a sum over those G which are compatible with all P_i is usually considered. The quantities w_G are often assumed to be translation invariant. It is reasonable to assume the latter property only for those G in (3.1) which are compatible with all P_i.

Investigate the Taylor series of the logarithm of partition function (3.1)

$$\log Z(M) = \sum_{I \in \mathcal{I}(\mathcal{P} \cup \mathcal{G})} w_I \qquad \text{with} \qquad w_I = n_I \prod_{P \in \underline{I}} w_P^{I_P} \prod_{G \in \underline{I}} w_G^{I_G} \tag{3.2}$$

where the sum is over the collection $\mathcal{I} = \mathcal{I}(\mathcal{P} \cup \mathcal{G})$ of all multi-indices I (integer valued functions on $\mathcal{P} \cup \mathcal{G}$) and n_I are corresponding partial derivatives of $\log Z$ at $\{w_P = 0, w_G = 0\}$.

The *support* of a multi-index I is defined as $\underline{I} = \{P : I_P > 0\} \cup \{G : I_G > 0\}$. A collection $\mathcal{P} \cup \mathcal{G}$ is called a *cluster* if the graph with bonds $P \, \iota \, P'$ and $P \, \iota \, G$ is *connected.*

It can be easily shown that nonzero n_I appear only for clusters i.e. for connected, "indecomposable" multiindices I. The *decomposability* of a multiindex with a support $\mathcal{P} \cup \mathcal{G}$ $= (\mathcal{P}_1 \cup \mathcal{G}_1) \; \cup \; (\mathcal{P}_2 \cup \mathcal{G}_2)$ means that any polymer resp. chain from the first system is compatible with any one from the second one. Since all the chains G are mutually compatible, incompatibility can only occur between two polymers or between a polymer and a chain, and it is usually defined by the requirement that the supports of these objects "overlap".

Now define auxiliary constants $L \doteq 1$ and $\tilde{L} \doteq 1$ as follows, with a (small) δ given below

$$L = \frac{-\log(1-\delta)}{\delta}, \quad E = \frac{e^\delta - 1}{\delta}, \qquad \tilde{L} = L^2 E. \tag{3.3}$$

In the proof of the statements (3.9) and (3.10) below it will be important that

$$\tilde{L} > L \qquad \text{and} \qquad 1 + LE(e^{\frac{Lx}{\tilde{L}}} - 1) \le e^x \;, \; x \ge 0. \tag{3.4}$$

Theorem 3. On decay of the terms in (3.2). *Let δ be chosen such that the bounds*

$$|w_P| e^{a|\underline{P}|} \le \delta, \;\; |w_G| \le \delta \qquad \textit{and} \qquad (e^{|w_G|} - 1)\; (e^{b|\underline{G}|} - 1) \le \delta \tag{3.5}$$

hold for any $P \in \mathcal{P}$ and $G \in \mathcal{G}$. Assume moreover that there are some $a > b > 0$ such that

1) Polymer fugacities satisfy the condition: for any $i \in M$ we have[9]

$$\sum_{P \in \mathcal{P} : \underline{P} \ni i} |w_P| e^{a|\underline{P}|} \le \frac{b}{\tilde{L}}. \tag{3.6}$$

[9] One would like to have $\tilde{L} = L = 1$ here and everywhere. However, we do not know whether our result can be really sharpened in such a way, like (see [KP]) in the case of ordinary polymer models with $w_G \equiv 0$.

2) Chain fugacities w_G satisfy the condition: for any $i \in M$

$$\sum_{G \in \mathcal{G}: \underline{G} \ni i} |w_G| e^{b|\underline{G}|} \leq a - b. \tag{3.7}$$

Denote by $\mathcal{C}(\mathcal{P} \cup \mathcal{G})$ the collection of all clusters on $\mathcal{P} \cup \mathcal{G}$. Then the following estimates are valid for the expansion terms w_I (which are equal to w_P resp. w_G in the special cases when T is identified with a single polymer resp. chain) in the equation (3.2):

a) For the sum of all clusters $I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G})$ containing a polymer $P \in \mathcal{P}$

$$\sum_{I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G}):\, I \ni P} |w_I| \leq \; L \; |w_P| \; e^{a|\underline{P}|}. \tag{3.8}$$

b) For the sum of all clusters $I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G})$ containing a chain $G \in \mathcal{G}$

$$\sum_{I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G}):\, I \ni G} |w_I| \quad \leq \; |w_G| \; e^{b|\underline{G}|}. \tag{3.9}$$

c) For the sum of all clusters $I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G})$ containing a polymer P containing $i \in M$

$$\sum_{I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G}):\;\; I \ni P \ni i} |w_I| \; \leq \; b. \tag{3.10}$$

d) For the sum of all ("old" and "new"[10]) clusters $I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G})$ containing $i \in M$

$$\sum_{I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G}):\;\; I \ni i} |w_I| \; \leq \; a. \tag{3.11}$$

Proof. See [BZE]. Compare in particular (3.7) with (3.11) for the "old" resp. "new" clusters G resp. T. While we have to sacrifice the constant b in the exponent we may include all the new, nontrivial clusters $I \neq G$ in the final estimate (3.11). This will be our main philosophy also in all the forthcoming estimates, for sums of more complicated clusters.

It will be useful to formulate also the following obvious consequence of (3.11). Consider new auxiliary weights $\tilde{w}_P = w_P e^{\alpha|\underline{P}|}$ and $\tilde{w}_G = w_G e^{\alpha|\underline{G}|}$ and rewrite (3.11) using the obvious inequality $|\underline{I}| \leq |I|$ where $|I| = \sum_{P \in \underline{I}} I_P |\underline{P}| + \sum_{G \in \underline{I}} I_G |\underline{G}|$ and $|\underline{I}| = \sum_{P \in \underline{I}} |\underline{P}| + \sum_{G \in \underline{I}} |\underline{G}|$:

[10] "Old" clusters are just chains G containing i, "new" clusters contain a polymer containing i

Corollary 1. *Let the weights* w_P *and* w_G *satisfy the assumptions*

$$\sum_{P\in\mathcal{P}:\underline{P}\ni i}|w_P|e^{(a+\alpha)|\underline{P}|}\le\frac{b}{\tilde{L}}\qquad and\qquad\sum_{G\in\mathcal{G}:\underline{G}\ni i}|w_G|e^{(b+\alpha)|\underline{G}|}\le a-b\tag{3.12}$$

where $\alpha\ge 0$*. Then we have the estimate*

$$\sum_{I\in\mathcal{C}(\mathcal{P}\cup\mathcal{G}):\ I\ni i}|w_I|\ e^{\alpha|\underline{I}|}\quad\le\sum_{I\in\mathcal{C}(\mathcal{P}\cup\mathcal{G}):\ I\ni i}|w_I|\ e^{\alpha|I|}\quad\le\ a.\tag{3.13}$$

This Corollary will be now modified to be useful also in the cases, studied below, when a *sequential* expansion of polymers from $\mathcal{P}$ is performed. Though the idea of an expansion of $\mathcal{P}$ in more than one step looks unnecessary and arbitrary here (in the case of polymer models) the same idea will be indispensable later, in the more complex situation of P.S. theory.

Sequential version of the above expansion.

Let us decompose the family $\mathcal{P}$ into subclasses, $\mathcal{P}=\cup_{n=1}^{\infty}\mathcal{P}_n$, where typically $\mathcal{P}_n$ is a collection of all polymers from $\mathcal{P}$ which have a (suitably defined) "size" n. Then we may apply the above result inductively; the first step being the cluster expansion, in the style of above theorem, of the "mixed" model employing all the clusters G of the original model but *only those* polymers P which are from the subfamily $\mathcal{P}_1$. The second step will then be to allow, and fix temporarily, a compatible collection of polymers from $\cup_{k\ge 3}\mathcal{P}_k$ and to perform the cluster expansion of the "rest" of the configurations (already partially expanded by the first step, containing still nonexpanded polymers from $\mathcal{P}_2$ and also "first level" clusters G). Then another "mixed model" of the type above will be obtained, with polymers from $\cup_{k\ge 3}\mathcal{P}_k$ and one can expand it again (fixing now a compatible family of polymers from $\cup_{k\ge 4}\mathcal{P}_k$) etc.

It is important to require that the control over these expansions should not be lost during these expansion processes (requiring an infinite number of steps in the thermodynamic limit $\Lambda\to\mathbb{Z}^\nu$) more specifically that the bounds (3.13) survive (with an appropriate choice of constants) such a sequential process: Denote by $\mathcal{G}_{<n}$ the collection of clusters constructed in the $n-$ th step of such a construction. We will assume for simplicity of notations that all the considered classes of contours and clusters are translation invariant; then the quantities a, b, c etc. are just constants and not functions of points of Λ. Looking at (3.12) it is clear how to decompose b into a suitable sum $b=\sum_n b_n$ so that the assumptions of the following Corollary 2 are also satisfied: We can take b_n as follows

$$b_n\ =\ \tilde{L}\sum_{P\in\mathcal{P}_n:\underline{P}\ni i}|w_P|e^{(a+\alpha)|\underline{P}|}\ ;\ b=\sum_{m=1}^{\infty}b_n\ ;\quad d_n=\sum_{m=n}^{\infty}b_n=b_n+d_{n+1}.\tag{3.14}$$

The new auxiliary parameter $\alpha\ge 0$ will be used to measure the correlation lengths of the expanded model, and we may imagine that $\alpha=0$ for a brevity.

Corollary 2. *Consider the situation as above. Put* $a_n = a - d_{n+1}$. *Assume that the estimates (for polymers from* $\mathcal{P}_n$ *and clusters from* $\mathcal{G}_{<n}$ *formed in the previous steps)*

$$\sum_{P \in \mathcal{P}_n : \underline{P} \ni i} |w_P| e^{(\alpha + a_n)|\underline{P}|} \leq \frac{b_n}{\tilde{L}} \quad \textit{and} \quad \sum_{G \in \mathcal{G}_{<n} : \underline{G} \ni i} |w_G| e^{(\alpha + d_n)|\underline{G}|} \leq a_n - b_n \tag{3.15}$$

are valid. Then, denoting by $\mathcal{G}_{\leq n} \equiv \mathcal{G}_{<n+1} = \mathcal{C}(\mathcal{P}_n \cup \mathcal{G}_{<n})$ *we have also the bound (the sum is also over the new clusters formed from clusters from* $\mathcal{G}_{<n}$ *and polymers from* $\mathcal{P}_n$*)*

$$\sum_{I \in \mathcal{G}_{\leq n} \ I \ni i} |w_I| \ e^{(\alpha + d_{n+1})|\underline{I}|} \ \leq \ a_n \ = a_{n+1} - b_{n+1}. \tag{3.16}$$

Proof. Immediate from Corollary 1. It is instructive to notice also the fact that because we have $|w_P| e^{(\alpha+a)|\underline{P}|} > |w_P| e^{(\alpha+a_n)|\underline{P}|}$, the assumptions of the first part of (3.15) are actually weaker (if summed together for all n) than that of (3.6). So we come to a conclusion that a sequential construction of the cluster expansion gives even a slightly more detailed control of its convergence than the usual expansion "at once".

Towards a cluster expansion of "polychromatic", abstract P.S. models.

We return back now to our previous situation of an abstract P.S. model, and to previous notations. Take again a model of the type (1.23). Assume that for each q some collection $\mathcal{H}^q \subset \mathcal{A}^q$ of *small contours* $\mathbf{\Gamma} = \mathbf{\Gamma}^q$ is selected, with the properties described above i.e. such that any extremal contour system $\mathbf{\Gamma}_\odot$ with the external contour $\mathbf{\Gamma}$ contains no contour from $\cup_{q' \in Q} \mathcal{H}^{q'}$ other than $\mathbf{\Gamma}$. Assume that the choice of $\{\mathcal{H}^q\}$ is nontrivial i.e. $\cup_{q \in Q} \mathcal{H}^q \neq \emptyset$; then we will say that our model is not yet fully expanded. Typically we take for $\mathcal{H}^q$ the class of all smallest existing small contours $\mathbf{\Gamma}^q$ of the model.

If $\cup_{q \in Q} \mathcal{H}^q = \emptyset$ holds (for a given finite torus Λ) and moreover if $\mathcal{R}^q$ is already *empty* for all $q \in Q$ minimizing the value f_q we call our model *fully expanded in* Λ. This is of course our goal, and below we explain how to reach it. First we formulate and analyze our basic expansion step – Theorem 4. A more detailed discussion (and modification) of the scheme (1.22), and how it changes during the application of expansion steps, will be then given in the forthcoming chapter.

The following result will be a crucial one. We already formulated above its preparatory variant (Corollary 2) for the "monochromatic" case of polymer models. The idea of a partially expanded *polymer* model was already used there and now we have to adapt it to the specific situation of P.S. theory. However, the expansion of contour systems having the activity $e^{-F(\mathbf{\Gamma}_\odot)}$ will require considerably more care than in the monochromatic case – when the relevant quantity to expand was just $e^{-E(\mathbf{\Gamma})}$. Also, the expansion of a typical (non-symmetric) P.S. model can *not* be done in a single step. An infinite number of steps will be needed.[11]

[11] Actually, if there is a unique Gibbs state then the number of necessary expansion steps can be made finite (even in $\mathbb{Z}^\nu$) – but it increases to infinity when we approach the coexistence regime.

It is very important to realize the idea that before investigating the expandability of some $\mathbf{\Gamma}_\odot = (\mathbf{\Gamma}, \{\mathbf{\Gamma}_i\}$ we need to expand carefully the model "with contours smaller than $\mathbf{\Gamma}$". Actually, one cannot even compute exactly $F(\mathbf{\Gamma}_\odot)$ before doing it! Another important observation is that nonexpandable contours $\mathbf{\Gamma}^q \in \mathcal{R}^q$ may survive in the model forever (for unstable q) but *only* if they are not "packed" by a bigger contour $\mathbf{\Gamma}^{q'}$ having a "more stable" external color q' than q. Namely, such a "conglomerate" could be possibly expanded in some later step. It is important to realize then (Theorem 2) that the entropy of the large contours $\mathbf{\Gamma}^q$ inside such a $\mathbf{\Gamma}^{q'}$ cannot beat the energy density excess of the unstable color q inside.

Notations. In the formulation of the forthcoming result we use the concept of a $(\mathcal{H}^q_\odot, \mathcal{T}^q)$ cluster introduced above. The "nuclei" of a cluster are extremal contour systems $\mathbf{\Gamma}_\odot = \mathbf{\Gamma}^q_\odot$ such that the external contour $\mathbf{\Gamma}^q$ of the system is from $\mathcal{H}^q$. Recall that $\mathbf{\Gamma}_\odot$ are formed such that the external contour $\mathbf{\Gamma}^q$ is possibly "filled" by a system of contours from $\cup_{\tilde{q}\in Q}\mathcal{R}^{\tilde{q}}$.

When summing also over all such new clusters with nuclei we will convert the estimates (1.25), (2.10) we already have for the "old" cluster weights w_T into analogous estimates for both the old and new cluster weights w_T and $w_{\mathbb{T}}$. We reserve the new symbol $\mathbb{T}$ only for the new clusters and keep the notation T for the "old" ones, from (1.23), (1.24).

Basic expansion step for a partially expanded, abstract P.S. model (1.23).

Theorem 4. (Main Lemma of Pirogov Sinai theory) A) Algebraic part. *Take a model (1.23) satisfying (1.13) and (1.14). Choose, for each $q \in Q$, a subsystem (not empty for at least some q) $\mathcal{H}^q \subset \mathcal{A}^q$ of small contours $\mathbf{\Gamma}^q$, such that the restriction of the relation $\prec$ to $\mathcal{H} = \cup_{q\in Q}\mathcal{H}^q$ is empty. Denote by $\mathcal{R}^q = \mathcal{A}^q \setminus \mathcal{H}^q$ the collection of all contours $\mathbf{\Gamma}$ from $\mathcal{A}^q \setminus \mathcal{H}^q$ ("the rest of $\mathcal{A}^q$"). Then the partition function $Z(\Lambda)$ in (1.23) can be expressed as follows (the sets Λ_q depend on $\{\mathbf{\Gamma}_i\}$; notation in (3.26) and (3.27) will be more explicit)*

$$Z(\Lambda) = \sum_{\{\mathbf{\Gamma}_1,\dots\mathbf{\Gamma}_m\}:\mathbf{\Gamma}_i\in\mathcal{A}} \prod_{i=1}^m w_{\mathbf{\Gamma}_i} \prod_q \exp(-e_q|\Lambda_q|)\ \exp\Big(\sum_{T_q:\underline{T}_q\sqsubset\Lambda_q} w_{T_q}\Big) = \tag{3.17}$$

$$= \sum_{\{\mathbf{\Gamma}_1,\dots\mathbf{\Gamma}_m\}:\mathbf{\Gamma}_i\in\mathcal{R}} \prod_{i=1}^m w_{\mathbf{\Gamma}_i} \prod_q \exp(-e_q|\Lambda_q|)\ \exp\Big(\sum_{T_q:\underline{T}_q\sqsubset\Lambda_q} w_{T_q} + \sum_{\mathbb{T}_q:\underline{\mathbb{T}}_q\sqsubset\Lambda_q} w_{\mathbb{T}_q}\Big). \tag{3.18}$$

The weights of the newly formed clusters $\mathbb{T}$ are given, in the simplest case $\mathbb{T} = \mathbf{\Gamma}_\odot$, by

$$w_{\mathbb{T}} = e^{-F(\mathbf{\Gamma}_\odot)}. \tag{3.19}$$

Analogous formulas are obtained, by applying (3.2) to (3.28), also for more complicated $\mathbb{T}$.

Defining the "free energies" f_q from (2.3), or alternatively the "entropies" s_q by

$$\begin{aligned} f_q &= e_q - \sum_{T^q\,\iota\,i} \frac{w_{T^q}}{|\underline{T}_q|} = e_q - s_q \qquad \textit{and} \\ f^\circ_q &= e_q - \sum_{T^q\,\iota\,i} \frac{w_{T^q}}{|\underline{T}_q|} - \sum_{\mathbb{T}_q:\underline{\mathbb{T}}_q\,\iota\,i} \frac{w_{\mathbb{T}_q}}{|\underline{\mathbb{T}}_q|} = e_q - s^\circ_q \end{aligned} \tag{3.20}$$

(quantities f_q°, s_q° are new counterparts to old f_q, s_q) we can write the relations (3.17), (3.18) also as follows, with suitably defined new correcting terms $\tilde{w}_{T_q}$ resp. $\tilde{w}_{\mathbb{T}_q}$ which have the same decay as w_{T_q} and are now "alive" only if the corresponding cluster touches $\{\mathbf{\Gamma}_i\}$

$$Z(\Lambda) \;=\; \sum_{\{\mathbf{\Gamma}_1,\dots\mathbf{\Gamma}_m\}:\mathbf{\Gamma}_i\in\mathcal{A}} \;\prod_{i=1}^{m} w_{\mathbf{\Gamma}_i}\;\prod_q \exp(-f_q|\Lambda_q|)\;\exp\big(\sum_{T_q:\underline{T}_q\,\iota\,\{\mathbf{\Gamma}_i\}} \tilde{w}_{T_q}\big) \;=$$

$$=\sum_{\{\mathbf{\Gamma}_1,\dots\mathbf{\Gamma}_m\}:\mathbf{\Gamma}_i\in\mathcal{R}} \;\prod_{i=1}^{m} w_{\mathbf{\Gamma}_i}\prod_q \exp(-f_q^\circ|\Lambda_q|)\;\exp\Big(\sum_{T_q:\underline{T}_q\,\iota\,\{\mathbf{\Gamma}_i\}} \tilde{w}_{T_q}) + \sum_{\mathbb{T}_q:\underline{\mathbb{T}}_q\,\iota\,\{\mathbf{\Gamma}_i\}} \tilde{w}_{\mathbb{T}_q}\Big). \tag{3.21}$$

Analogous expressions are valid also for more general, conditioned partition functions $Z_{\mathcal{C}}(\Lambda)$ of the type (1.24) where $\mathcal{C}$ is a fixed limitation, expressed in terms of contours of some subsystem $\mathcal{S}$ of $\mathcal{R}=\cup_q\mathcal{R}^q$. The new quantities $w_{\mathbb{T}}$ are then also affected by such a limitation, but only in the case when $\mathbb{T}$ is incompatible with some contour of $\mathcal{S}$.

The new clusters $\mathbb{T}_{\shortparallel}$ are defined as $(\mathcal{H}^q_\odot,\mathcal{T}^q)$ clusters where $\mathcal{T}=\cup_q\mathcal{T}_q$ denotes the family of all clusters used in the definition of the original model (3.17), with the system $\mathcal{A}=\cup_q\mathcal{A}^q$ of allowed contours) and $\mathcal{H}_\odot=\cup_q\mathcal{H}^q_\odot$ denotes the family of all simple extremal contour systems $\mathbf{\Gamma}_\odot=(\mathbf{\Gamma},\{\mathbf{\Gamma}_l\})$ whose "external frame" $\mathbf{\Gamma}$ is from $\mathcal{H}=\cup_q\mathcal{H}^q$.

B) Bounds for the newly defined weights $w_{\mathbb{T}}$. *The smallness of extremal contour systems $\mathbf{\Gamma}_\odot$ assures a quick convergence of the new cluster weights $w_{\mathbb{T}}$, expressed by the following formulas: Assume that all $\mathbf{\Gamma}\in\mathcal{H}^q$ have the same diameter* $\operatorname{diam}\mathbf{\Gamma}=N$, *for simplicity of notations. Let the estimate (2.10) and therefore also (2.23) be valid, with (2.12). Write it as follows: Choose $\alpha\ge 0$ and another new constant $a>0$ related to a' from (2.23) such that*

$$a'-b\equiv\; a+\alpha \qquad \textit{where} \quad b=\sum_{n=1}^{\infty} b_n. \tag{3.22}$$

Put $a_N=a+\sum_{n=1}^N b_n=a+b-d_N$ where $d_N\equiv\sum_{n=N}^{\infty}b_n$. Asssume that

$$\sum_{\mathbf{\Gamma}_\odot\in\mathcal{H}^q_\odot:\mathbf{\Gamma}_\odot\,\iota\, i} e^{-F(\mathbf{\Gamma}_\odot)}\;e^{(\alpha+a_N)|\underline{\mathbf{\Gamma}_\odot}|}\;\le\;\frac{b_N}{\tilde{L}} \tag{3.23}$$

which is a weaker form of (2.23). Assume that the estimate (1.25) was already transformed, using Corollary 2 and previous expansion steps, into an estimate (compare (3.15), (3.16))

$$\sum_{T\ni i}|w_T|\;e^{\alpha+d_N|\underline{T}|}\;\le\; a_N-b_N\;. \tag{3.24}$$

Then the new and old cluster weights $w_{\mathbb{T}}$ satisfy a new estimate of the same type as (3.24)

$$\sum_{T\ni i}|w_T|e^{\alpha+d_{N+1}|\underline{T}|}+\sum_{\mathbb{T}\ni i}|w_{\mathbb{T}}|\;e^{\alpha+d_{N+1}|\underline{\mathbb{T}}|}\;\le\; a_N\;=\;a_{N+1}-b_{N+1}. \tag{3.25}$$

Proof of Theorem 4. It is clear that the bounds (3.23)–(3.25) are like in Corollary 2 above. So it suffices to clarify, and this will be an important step, that by employing the contour functionals $F(\mathbf{\Gamma}^{\odot})$ one really gets a situation analogous to that of Corollary 2.

Rewrite (3.17) more precisely as

$$
\begin{aligned}
Z(\Lambda) &= \sum_{\{\mathbf{\Gamma}_i\}:\mathbf{\Gamma}_i\in\mathcal{R}} \prod_i w_{\mathbf{\Gamma}_i} \sum_{\{\mathbf{\Gamma}^j_{\odot}\}} \exp(-E(\mathbf{\Gamma}^j_{\odot}) \prod_q \exp(-e_q|\Lambda^{\odot}_q|) \ \exp(\sum_{T_q:\underline{T}_q\sqsubset\Lambda^{\odot}_q} w_{T_q}) \\
&= \sum_{\{\mathbf{\Gamma}_i\}:\mathbf{\Gamma}_i\in\mathcal{R}} \prod_i w_{\mathbf{\Gamma}_i} \ Z_{\{\mathbf{\Gamma}_i\}}(\Lambda) \qquad \text{where} \qquad w_{\mathbf{\Gamma}} = e^{-E(\mathbf{\Gamma})} \qquad \text{and} \qquad (3.26)
\end{aligned}
$$

$$
Z_{\{\mathbf{\Gamma}_i\}}(\Lambda) = \prod_q \exp(-e_q|\Lambda^{\odot}_q|) \sum_{\{\mathbf{\Gamma}^j_{\odot}\}} \prod_j \exp(-E(\mathbf{\Gamma}^j_{\odot})) \ \exp(\sum_{T_q:\underline{T}_q\sqsubset\Lambda^{\odot}_q} w_{T_q}). \qquad (3.27)
$$

The sum in (3.27) is over all possible small, "inner" systems $\{\mathbf{\Gamma}^j_{\odot}\}$ compatible with $\{\mathbf{\Gamma}_i\}$. In the above relation, we use a more precise notation $\Lambda^{\odot}_q = \Lambda_q(\{\mathbf{\Gamma}_i\}\cup\{\mathbf{\Gamma}^j_{\odot}\})$ for the collection of all correct points of the full contour configuration $\{\mathbf{\Gamma}_1,\ldots,\mathbf{\Gamma}_m\}$ written now as $\{\mathbf{\Gamma}_i\}\cup\{\mathbf{\Gamma}^j_{\odot}\}$.

So, to emphasize the possible presence of a subcollection $\{\mathbf{\Gamma}^j_{\odot}\}$ of small systems we denote, in this proof, the corresponding set Λ_q more precisely as $\Lambda^{\odot}_q = \Lambda^{\odot}_q(\{\mathbf{\Gamma}_i\}\cup\{\mathbf{\Gamma}^j_{\odot}\})$ while using the notation Λ_q for the sets $\Lambda_q(\{\mathbf{\Gamma}_i\})$ with the collection $\{\mathbf{\Gamma}^j_{\odot}\}$ being erased. Now, $\cup_q\Lambda^{\odot}_q$ is a subset[12] of $\cup_q\Lambda_q$ and the above partition function (3.27), over the inner systems $\{\mathbf{\Gamma}^j_{\odot}\}$ compatible with a fixed collection $\{\mathbf{\Gamma}_i\}$, can be written as follows.

This is a crucial step of our construction: Rewrite (3.27) using the definition (2.1) of $F(\mathbf{\Gamma}^{\odot}_j)$. Notice that in the forthcoming equation we replace $\Lambda^{\odot}_q$ by Λ_q everywhere i.e. we really "recolor" the interiors of all $\mathbf{\Gamma}^j_{\odot}$ by the *external colors* of the contour systems $\mathbf{\Gamma}^{\odot}_j$

$$
Z_{\{\mathbf{\Gamma}_i\}}(\Lambda) = \prod_q \exp(-e_q|\Lambda_q|) \sum_{\{\mathbf{\Gamma}^j_{\odot}\}} \prod_j \exp(-F(\mathbf{\Gamma}^j_{\odot})) \ \exp(\sum_{T_q:\underline{T}_q\sqsubset\Lambda_q} w_{T_q}) \qquad (3.28)
$$

The partition function (3.28) can be now treated as a *polymer* partition function, more precisely as a *product* (over $q\in Q$) of corresponding "q chromatic" polymer partition functions, having also extra exponential terms $\exp(-e_q|\Lambda^q|)$. Really, we have an expression

$$
\begin{aligned}
Z_{\{\mathbf{\Gamma}_i\}}(\Lambda) &= \prod_q \exp(-e_q|\Lambda_q|) \ \ Z^q_{\{\mathbf{\Gamma}_i\}}(\Lambda) \qquad \text{where} \\
Z^q_{\{\mathbf{\Gamma}_i\}}(\Lambda) &= \sum_{\{\mathbf{\Gamma}_{\odot}{}^q_k\}} \exp(-F(\mathbf{\Gamma}_{\odot}{}^q_k)) \ \exp(\sum_{T_q:\underline{T_q}\sqsubset\Lambda_q} w_{T_q}) \qquad (3.29)
\end{aligned}
$$

and where we denote by symbols $\mathbf{\Gamma}_{\odot}{}^q_k$ those extremal simple systems $\mathbf{\Gamma}^j_{\odot}$ which have the external color q. The weights of the contour systems $\mathbf{\Gamma}_{\odot}$ are given by $w_{\mathbf{\Gamma}_{\odot}} = \exp(-F(\mathbf{\Gamma}_{\odot}))$.

The partition functions (3.29) are, however, "monochromatic" polymer models (the "mixed" ones, with polymers $\mathbf{\Gamma}_{\odot}$ and with clusters T) of the type considered in Corollary 2 above. Now is suffices to apply to them the classical cluster expansion explained above. Inspection of the bounds stated in Corollary 2 then finishes the proof of Theorem 4.

[12] It may happen that inside of Λ_q, components of some other $\Lambda^{\odot}_{q'}$ appear.

In the next chapter, we will investigate what happens when our expansion result – Theorem 4 – is used iteratively (infinitely) many times simultaneously for all tori Λ (having in mind especially the case $\Lambda \to \mathbb{Z}^\nu$). However, before doing it we have to solve one technical problem namely that by expanding only the small systems $\mathbf{\Gamma}_\odot$ we introduced some *discontinuity* into our constructions (of f_q^o, for example). Such discontinuities are, of course, natural for quantities defined in terms of "yes/no" constructions – but they are uncomfortable for a study of a phase diagram. We will remedy the situation in this chapter. We will replace the previous constructions by their, suitably defined, "smoothened" variants:

Smoothened version of Theorem 4.

Now we will present a simple auxiliary construction whose aim is to modify the result of Theorem 4 (and, in particular, the construction of f_q^o) such that all the resulting quantities become continuous, even *Lipschitz* functions of the Hamiltonian (1.23).

The inherent discontinuity in the decision small/nonsmall (concerning any system $\mathbf{\Gamma}_\odot$) makes the expansion technology explained above directly useful *only* for the analysis of the structure of Gibbs states for a *fixed* Hamiltonian (1.23). The aim is now to "smoothen the jumps in the expand/nonexpand dichotomy" i.e. to replace the Boolean function (with values Yes,No meaning small,nonsmall) of $\mathbf{\Gamma}_\odot$ by a continuous function keeping the value Yes whenever it appeared before – but modifying continuously the value No.

Consider the situation of Theorem 4. Recall the relation (2.18)

$$F(\mathbf{\Gamma}_\odot) \geq E(\mathbf{\Gamma}_\odot) + \sum_{\tilde{q}\neq q} u_{\tilde{q}} |\operatorname{int}_{\tilde{q}} \mathbf{\Gamma}_\odot| - (\omega+\varepsilon)|\underline{\mathbf{\Gamma}_\odot}| - \Omega \tag{4.1}$$

which we proved to be valid for any extremal small system $\mathbf{\Gamma}_\odot$. For other extremal systems we cannot, of course, expect such a relation to be valid in general. Define, then, for any such $\mathbf{\Gamma}_\odot$ a provisional continuous quantity $F_{cut}(\mathbf{\Gamma}_\odot)$, a "cut" of $F(\mathbf{\Gamma}_\odot)$

$$F_{cut}(\mathbf{\Gamma}_\odot) \ = \max(F(\mathbf{\Gamma}_\odot)\ ,\ E(\mathbf{\Gamma}_\odot) - (\omega+\epsilon)|\underline{\mathbf{\Gamma}}| + u_{\tilde{q}} |\operatorname{int}_{\tilde{q}} \mathbf{\Gamma}_\odot| - \Omega). \tag{4.2}$$

This is a continuous function, yielding continuous functions f_q^o if also nonsmall systems $\mathbf{\Gamma}_\odot$ are expanded with the "artificial weight" $e^{-F_{cut}(\mathbf{\Gamma}_\odot)}$ (which is of course equal to $e^{-F(\mathbf{\Gamma}_\odot)}$ for a small $\mathbf{\Gamma}_\odot$). However, we will need a more refined variant of such a cut satisfying suitable Lipschitz condition. Namely, our phase diagram will be computed from quantities of the type f_q^o using the implicit function theorem, and their usage requires stronger properties of f_q than mere continuity. In fact, in the same way one could construct even infinitely smooth variants of f_q^o but an attempt to construct locally analytic variant of such quantities requires another approach [ZA]. We are preparing a new article dealing with local analyticity of the phase diagram, from the point of view of an abstract P.S. setting (1.23).

Smoothened contour functional $\bar{F}(\mathbf{\Gamma}_\odot)$.

Define an operation of a "smooth maximum" $\operatorname{maxs}(x,1)$ as a 1– Lipschitz function which is equal to $\max(x,1)$ outside of some neighborhood of $x=1$ and moreover taken (infinitely) smooth. Define the function $\operatorname{maxs}(x,c) = c\operatorname{maxs}(x/c,1)$ and put

$$\bar{F}(\mathbf{\Gamma}_\odot) \ = \operatorname{maxs}(F(\mathbf{\Gamma}_\odot)\ ,\ E(\mathbf{\Gamma}_\odot) - (\omega+\epsilon)|\underline{\mathbf{\Gamma}}| + u_{\tilde{q}}|\operatorname{int}_{\tilde{q}}\mathbf{\Gamma}_\odot| - \Omega). \tag{4.3}$$

It is clear that if $F(\mathbf{\Gamma})$ (and other terms in (4.3)) depend on some parameter λ then we have

$$|\frac{\partial \bar{F}(\mathbf{\Gamma}_\odot)}{\partial\lambda}| \ \le \ |\frac{\partial F(\mathbf{\Gamma}_\odot)}{\partial\lambda}|. \tag{4.4}$$

Partial expansion of a nonsmall extremal system $\mathbf{\Gamma}_\odot$.

We have to explain what will be meant by a *partial expansion* of a *nonsmall* system $\mathbf{\Gamma}_\odot$: Just expand $\mathbf{\Gamma}_\odot$ as having the value $\bar{F}(\mathbf{\Gamma}_\odot)$ instead of its true value $F(\mathbf{\Gamma}_\odot)$ –which would possibly violate (2.23). Of course, this introduces some arbitrariness into our constructions, and the expansion we obtain in such a way will in principle affect, in a complicated way, all future expansions of objects "containing" such a partially expanded $\mathbf{\Gamma}_\odot$.

We introduce here the following modification of Theorem 4. It keeps the notion of smallness with the roughly the same meaning (with slightly different quantities f^q and u_q than before) but it "smoothens" in a convenient way the expansion algorithm of Theorem 4, at the same time redefining the concept of the "remainder" $\mathcal{R}$, of contours which cannot be expanded.

The systems $\mathbf{\Gamma}_\odot$ which cannot be expanded are not treated, as before, as conglomerates of contours from $\mathcal{R}$ but as *new elementary objects*. We denote by $\mathcal{P}$ the collection of all such "partially expanded" $\mathbf{\Gamma}_\odot$. To any such $\mathbf{\Gamma}_\odot$ we assign a new[13] energy $E_{rest}(\mathbf{\Gamma}_\odot) > E(\mathbf{\Gamma}_\odot)$ which is computed, as a new $E(\mathbf{\Gamma})$, from the r.h.s. of (2.1) if we replace $F(\mathbf{\Gamma}_\odot)$ on its l.h.s. by the new quantity $F_{rest}(\mathbf{\Gamma}_\odot)$ whose exponential is given by the relation

$$e^{-F(\mathbf{\Gamma}_\odot)} = e^{-\bar{F}(\mathbf{\Gamma}_\odot)} + e^{-F_{rest}(\mathbf{\Gamma}_\odot)}. \tag{4.5}$$

Then we have the following result, which will conveniently replace Theorem 4 in everything what follows and concerns the *phase diagram* of the model. (Only for a detailed analysis of Gibbs states of a fixed Hamiltonian, it is still recommended to stay in the setting of Theorem 4.) Extremality of $\mathbf{\Gamma}_\odot$ will be meant, from now on, in the sense that no contour system $\mathbf{\Gamma}'_\odot$ with a *finite* energy $E_{rest}(\mathbf{\Gamma}'_\odot)$ can be compatibly placed inside $\mathbf{\Gamma}_\odot$.

Theorem 5 (Smoothening of Theorem 4). *Assume a situation like in Theorem 4 but relaxing the condition that contours from $\mathcal{H} \subset \mathcal{A}$ are small. (Just suppose that $\mathcal{H}$ is a subset of $\mathcal{A}$.) Rephrase the constructions of Theorem 4 with $F(\mathbf{\Gamma}_\odot)$ being replaced by the quantity $\bar{F}(\mathbf{\Gamma}_\odot)$ from (4.3), for any extremal $\mathbf{\Gamma}_\odot$ whose external contour is from $\mathcal{H}$. Denote by $\mathcal{P}$ the class of all such $\mathbf{\Gamma}_\odot$. Put $\mathcal{R} = \mathcal{A}\setminus\mathcal{H}$. Take the new model (3.18), enrich it again by all configurations $x_\Lambda \cup \{{\mathbf{\Gamma}_\odot}^i\}$ where x_Λ is a configuration from (3.18) such that $\{{\mathbf{\Gamma}_\odot}^i\}$ can be*

[13] As the system $\mathbf{\Gamma}_\odot$ will be "partially expanded" now, the energy of the "remainder" will increase. All this will be done in such a way that partition functions of interest remain the same.

compatibly placed into it. Define the values $w_{\mathbb{T}}$ of new cluster weights analogously as before, with $F(\mathbf{\Gamma}_\odot)$ replaced by $\bar{F}(\mathbf{\Gamma}_\odot)$ everywhere. Then we get a model $\mathcal{R}$–equivalent to the original one. In particular the relation (we do not introduce any new notations for quantities like f_q^o and $\tilde{w}_T$, and symbol T_q here denotes both the "old" and "new" clusters T_q and $\mathbb{T}_q$)

$$Z(\Lambda) = \sum_{\{\mathbf{\Gamma}_i\}\cup\{\mathbf{\Gamma}_\odot^j\}:\mathbf{\Gamma}_i\in\mathcal{R},\mathbf{\Gamma}_\odot^j\in\mathcal{P}} \prod_i w_{\mathbf{\Gamma}_i} \prod_j w_{\mathbf{\Gamma}_\odot^j} \prod_q \exp(-f_q^\circ|\Lambda_q|)\ \exp(\sum_{T_q\wr\{\mathbf{\Gamma}_i\}\cup\{\mathbf{\Gamma}_\odot^j\}} \tilde{w}_{T_q}) \quad (4.6)$$

is valid, with $w_{\mathbf{\Gamma}} = e^{-E(\mathbf{\Gamma})}$ for any $\mathbf{\Gamma}\in\mathcal{R}$ and with $w_{\mathbf{\Gamma}_\odot} = e^{-E_{rest}(\mathbf{\Gamma}_\odot)}$ for any $\mathbf{\Gamma}_\odot\in\mathcal{P}$.

Notes. 1) Of course in practise we will be interested in very special choices of $\mathcal{H}$, see below. However, it is impossible to describe $\mathcal{H}$ precisely as family of small contours if we want to apply Theorem 5 simultaneously also to perturbations of a given model (of a type (1.23)). Namely, small contours of the original model may become nonsmall in the perturbed model. 2) Notice the difference w.r. to the situation in (3.17) – where energies $E(\mathbf{\Gamma}))$ were used for all nonexpanded contours $\mathbf{\Gamma}\in\mathcal{R}$. Now, the systems $\mathbf{\Gamma}_\odot$ with extremal contours from $\mathcal{H}$ (any $\mathbf{\Gamma}_\odot$ will be treated, from now on, always as a single unit) survive in the model but they increase their energy $E(\mathbf{\Gamma}_\odot)$ to $E_{rest}(\mathbf{\Gamma}_\odot)$. For small $\mathbf{\Gamma}_\odot$ we then have $E_{rest}(\mathbf{\Gamma}_\odot)=\infty$.

Proof. This is rather straightforward. Recall the relation (3.29) and modify it as follows (we do not write the index q at $\mathbf{\Gamma}_\odot$ and we denote by I the index set for various systems ${\mathbf{\Gamma}_\odot}^i$)

$$Z_{\{\mathbf{\Gamma}_i\}}(\Lambda) = \prod_q \exp(-e_q|\Lambda_q|) \sum_{\{\mathbf{\Gamma}_\odot^i\}} \prod_{i\in I}(\exp(-\bar{F}(\mathbf{\Gamma}_\odot^i)) + \exp(-F_{rest}({\mathbf{\Gamma}_\odot}^i))\ \exp(\sum_{T_q:\underline{T}_q\sqsubset\Lambda_q} w_{T_q})$$

$$= \prod_q \sum_{\{\mathbf{\Gamma}_\odot^j\}} \sum_{J\subset I} \exp(-e_q|\Lambda_q|) \prod_{j\in J}\exp(-\bar{F}(\mathbf{\Gamma}_\odot^j) \prod_{j\in I\setminus J} \exp(-F_{rest}(\mathbf{\Gamma}_\odot^j)\exp(\sum_{T_q:\underline{T}_q\sqsubset\Lambda_q} w_{T_q}) =$$

$$= \prod_q \sum_{\{\mathbf{\Gamma}_\odot^j\}} \sum_{J\subset I} \exp(-e_q|\Lambda_q^J|) \prod_{j\in J}\exp(-\bar{F}(\mathbf{\Gamma}_\odot^j) \prod_{j\in I\setminus J} \exp(-E_{rest}(\mathbf{\Gamma}_\odot^j))\exp(\sum_{T_q\sqsubset\Lambda_q^J} w_{T_q}) \quad (4.7)$$

where $\Lambda_q^J \neq \Lambda^q$ is the set of the q–correct points of $\{\mathbf{\Gamma}_i\}\cup\{\mathbf{\Gamma}_\odot^j, j\in J\}$. Notice that we do not expand the "$e^{-F_{rest}(\mathbf{\Gamma}_\odot)}$ part" of the weight $e^{-F(\mathbf{\Gamma}_\odot)}$! On the contrary, for $j\in I\setminus J$ we *cancel* the idea of going to $F(\mathbf{\Gamma}_\odot)$ and we go back to the original polychromatic expression (1.23) inside those $\mathbf{\Gamma}_\odot^j$, by introducing the new artificial energy $E_{rest}(\mathbf{\Gamma}_\odot) > E(\mathbf{\Gamma}_\odot)$.

The following result will be crucial for the investigation of a phase diagram:

Theorem 6. *The smoothened quantities f_q^o are Lipschitz functions of the parameters in (1.23). More precisely, if the terms of (1.23) depend on some parameter λ in such a way that $|\partial/\partial\lambda\ e_q|\le C$ and also the derivatives $\partial/\partial\lambda\ w_{\mathbf{\Gamma}}$ and $\partial/\partial\lambda\ w_T$ on the l.h.s of (1.13) and (1.14) obey similar bounds as the original quantities $w_{\mathbf{\Gamma}}$ and w_T (with enough large g, c and enough small ϵ,ω) then we have an estimate, with analogous notations as in (3.25)*

$$\sum_{T\ni i} |\frac{\partial}{\partial\lambda} w_T|\ e^{\alpha+d_{N+1}} + \sum_{\mathbb{T}\ni i} |\frac{\partial}{\partial\lambda} w_{\mathbb{T}}|\ e^{\alpha+d_{N+1}|\mathbb{T}|}\ \le\ a_N \quad (4.8)$$

where the constants a_N have similar meaning as in Theorem 4 resp. 5. In particular we have

$$|\frac{\partial}{\partial\lambda}(f_q - e_q)| \ \leq \ \varepsilon \qquad with \quad \varepsilon \approx e^{-C'a} + \omega + \epsilon. \tag{4.9}$$

Proof. This is not difficult; here we only outline how this could be done. The main point is the observation that when taking the derivatives w.r. to λ of quantities like $w_{\mathbb{T}}$, only the term $w_{\mathbf{\Gamma}_\odot} = e^{-F(\mathbf{\Gamma})}$ has nontrivial behaviour. In particular, volume terms $|\operatorname{int}\mathbf{\Gamma}_\odot|$ will appear in front of $w_{\mathbf{\Gamma}_\odot}$ in the corresponding estimates. Really, if we take the derivative in (2.1)

$$\frac{\partial}{\partial\lambda}F(\mathbf{\Gamma}_\odot) = \frac{\partial}{\partial\lambda}E(\mathbf{\Gamma}_\odot) + \sum_{\tilde{q}\neq q}\frac{\partial}{\partial\lambda}\Big(e_{\tilde{q}} - e_q)|\operatorname{int}_{\tilde{q}}\mathbf{\Gamma}_\odot| + \sum_{T_{\tilde{q}}\subset \operatorname{int}_{\tilde{q}}\mathbf{\Gamma}_\odot} w_{T_{\tilde{q}}} - \sum_{T_q\subset \operatorname{int}_{\tilde{q}}\mathbf{\Gamma}_\odot} w_{T_q}\Big) \tag{4.10}$$

for a system $\mathbf{\Gamma}_\odot = \mathbf{\Gamma}_\odot^q$ then we get something like

$$|\frac{\partial}{\partial\lambda}F(\mathbf{\Gamma}_\odot)| \leq |\frac{\partial}{\partial\lambda}E(\mathbf{\Gamma}_\odot)| + |\operatorname{int}\mathbf{\Gamma}_\odot|(\sum_q|\frac{\partial}{\partial\lambda}e_q)| + \sum_{T\ni 0}|\frac{\partial}{\partial\lambda}w_T|).$$

However, the terms $|\operatorname{int}\mathbf{\Gamma}_\odot|\ e^{-C|\underline{\mathbf{\Gamma}_\odot}|}$ are smaller than 1 for large enough C and decay very quickly. So it is no problem, by possibly strengthening our assumptions on c, g, to rewrite all the bounds considered before so that they would remain valid also for the derivatives of the weights $w_{\mathbf{\Gamma}_\odot}$ and w_T. So, inserting inequalities like

$$|\operatorname{int}\mathbf{\Gamma}_\odot| w_{\mathbf{\Gamma}_\odot} e^{(a-C)|\underline{\mathbf{\Gamma}_\odot}|} \leq w_{\mathbf{\Gamma}_\odot} e^{a|\underline{\mathbf{\Gamma}_\odot}|} \tag{4.11}$$

into the series of derivatives of $w_{\mathbf{\Gamma}_\odot}$ in (2.23), the estimates of these tails will be very much like before, of course with suitably changed ω_N and strengthened assumptions on a and c.

Notice that the first two terms on the r.h.s. of the relation (4.10) are responsible for most of the change of $F(\mathbf{\Gamma}_\odot)$ resp. $\tilde{F}(\mathbf{\Gamma}_\odot)$, and a control of the terms $\frac{\partial}{\partial\lambda}E(\mathbf{\Gamma}_\odot)e^{-F(\mathbf{\Gamma}_\odot)}$ follows from our assumptions on sufficient smallness of $\frac{\partial}{\partial\lambda}e^{-E(\mathbf{\Gamma}_\odot)}$.

On the choice of $\mathcal{H}$, $\mathcal{R}$ and $\mathcal{P}$ in the expansion procedures of Theorem 5.

The partially expanded objects $\mathbf{\Gamma}_\odot$ do not disappear from our model now. They survive in it but are treated, from now on, as conglomerates $\mathbf{\Gamma}_\odot$ never separated into single contours again. Of course, if $E_{rest}(\mathbf{\Gamma}_\odot) = \infty$ then such a contour system $\mathbf{\Gamma}_\odot$ still "effectively disappears" from the expanded model.[14]

We make an agreement that the class $\mathcal{P}$ of all these systems $\mathbf{\Gamma}_\odot$ will be added to the family $\mathcal{R}$ (of contours not used as external contours of some $\mathbf{\Gamma}_\odot$) in future expansion steps. This is

[14]Nonexpandability of $\mathbf{\Gamma}_\odot$ i.e. the event $E_{rest}(\mathbf{\Gamma}_\odot) < \infty$ is rather a delicate thing. In spite of the fact that it is defined in somehow artificial way (arbitrariness in the choice of (4.3) and in the choice of Ω, for example) it gives some information on the tendency to phase transitions in volumes bigger than $\operatorname{int}\mathbf{\Gamma}_\odot$.

a little bit more abstract approach than in the iterations of Theorem 4, where future families $\mathcal{R}$ were just subsets $\mathcal{R} \setminus \mathcal{H}$ of the previous families $\mathcal{R}$.

All the elements $\mathbf{\Gamma}_\odot$ of such a newly formed family $\mathcal{P}$ can be then included, as interior subsystems, into more complex systems $\widetilde{\mathbf{\Gamma}_\odot}$ created in the next expansion steps; in such a way that we "wrap" the collections $\mathbf{\Gamma}_\odot$ by external contours taken from some newly chosen family $\mathcal{H}$ of "small" contours. Then, in a new application of Theorem 5, these bigger systems $\widetilde{\mathbf{\Gamma}_\odot}$ will be (partially) expanded, too. And so on.

In such a setting, which is different from those used in Theorem 4, it is not necessary to care whether the elements of $\mathcal{H}$ are small in the sense of (2.13). However, the latter notion is still crucial when we want to interpret the result of these expansions. See the next chapter.

Additional notes. 1) For very big nonsmall $\mathbf{\Gamma}_\odot$ we have $E_{rest}(\mathbf{\Gamma}_\odot) \approx E(\mathbf{\Gamma}_\odot)$. (Imagine a contour appearing around a boundary of a very large finite volume M, if we impose an unstable boundary color q around it.) Then the expanded part $e^{-\bar{F}(\mathbf{\Gamma}_\odot)}$ is just a small correction to the "basic" situation with $E_{rest}(\mathbf{\Gamma}_\odot)$. The latter situation cannot be expanded. A detailed analysis of what exactly happens under unstable boundary conditions in finite volumes cannot be done with the help of expansions only. In fact we do not address such a question here, investigation of what happens precisely under unstable boundary condition in finite volumes is not a goal of this article. In thermodynamical limit the situation is more trivial even for unstable boundary conditions. See [Z] where it is proven (for more restricted models) that no other translation invariant Gibbs states, than those corresponding to stable boundary conditions, exist in the thermodynamic limit.

2) In general, there is always a lot of freedom in the choice of $\{\mathcal{H}^q\}$, and there could be also other variants of the notion of "smallness" of a contour system. See [ZR], where the final product (the fully expanded model) leaves less contours left unexpanded than in our approach here (assuming, however, sharper Peierls bounds in principle). Anyway, the freedom in the choice of the families $\mathcal{H}^q$ does not affect the physics of the problem i.e. the determination of final, *stable* values of colors $q \in Q$. For the stable values q, all the contours $\mathbf{\Gamma}^q$ are finally expanded, in any of the above mentioned approaches.

5. Iterated expansions. Construction of the phase diagram

First we have to modify and make more precise the expansion plan (1.22), originally formulated in setting of Theorem 4. We will now choose, in the n–th step of expansion by Theorem 5, the family $\mathcal{H}_n$ as the collection of all contours from the original family $\mathcal{A}$ (of all contours of the starting model (1.23)) having a diameter equal to n. Thus we get a decomposition

$$\mathcal{A} = \cup_n \mathcal{H}_n. \tag{5.1}$$

We put $\mathcal{R}_n \equiv \cup_n^\infty \mathcal{H}_m$ and define recursively the new families of contour systems

$$\mathcal{P}_n \equiv \{\; \mathbf{\Gamma}_\odot = (\mathbf{\Gamma}, \{\mathbf{\Gamma}_\odot^i\}) \;:\; \mathbf{\Gamma} \in \mathcal{H}_n,\; \mathbf{\Gamma}_\odot^i \in \cup_1^{n-1} \mathcal{P}_k \}. \tag{5.2}$$

Thus, in the n–th step of expansion we consider a model (1.23) with a a family $\mathcal{H}_n^{\odot}$ of systems $\mathbf{\Gamma}_{\odot}$ formed from an external contour from $\mathcal{H}_n$, and from some other interior objects $\mathbf{\Gamma}'_{\odot} \in \mathcal{P}_{n-1}^{\odot}$ which were already *partially* expanded before. (Hence the notation "P" of the family $\mathcal{P}_{n-1}$, which was empty in Theorem 4 and which we now add to $\mathcal{R}_n$.) Some of these systems $\mathbf{\Gamma}'_{\odot}$ were actually expanded fully i.e. left with the energy $E_{rest}(\mathbf{\Gamma}'_{\odot}) = \infty$.

Notes. 1) A somewhat arbitrary, but convenient fact is that with our choice of mutually disjoint families $\mathcal{H}_n$ (of external contours of systems $\mathbf{\Gamma}_{\odot}$ which have to be expanded in the n–th step) an attempt to expand some $\mathbf{\Gamma}_{\odot}$ will be never be repeated again, in future expansion steps. Always new portions $\mathcal{H}_n = \cup_q \mathcal{H}_n^q$ of $\mathcal{A} = \cup_n \mathcal{H}_n$ will be used as external contours of objects $\mathbf{\Gamma}_{\odot}$, partially expanded in the n- th use of Theorem 5.

2) The systems $\mathcal{R}$ resp. $\mathcal{A}$ are more complicated now, than in Theorem 4. The family $\mathcal{A}_n$ of allowed contours in a model expanded in the n–th expansion step (by Theorem 5) looks like follows: $\mathcal{A}_n = \mathcal{A} \setminus (\cup_{m=1}^n \mathcal{H}_m) \cup \mathcal{P}_n$ where $\mathcal{A}$ is the starting family of contours and $\mathcal{P}_n$ is the collection of systems $\mathbf{\Gamma}_{\odot}$ created and partially expanded in previous steps. The "contours" $\mathbf{\Gamma}_{\odot}$ from the family $\mathcal{P}_n$ carry the energy $E_{rest}(\mathbf{\Gamma}_{\odot})$ now, not $E(\mathbf{\Gamma}_{\odot})$.

3) One has also to stress that the construction of a skeleton of any $\mathbf{\Gamma}_{\odot} = (\mathbf{\Gamma}, \{\mathbf{\Gamma}^i_{\odot}\})$ where $\mathbf{\Gamma} \in \mathcal{H}_n$ and $\mathbf{\Gamma}^i_{\odot} \in \mathcal{P}_n$ goes, of course, also inside of the interior systems $\mathbf{\Gamma}^i_{\odot}$. The argument remains the same as before, the bound (2.23) being rephrased without any change (actually, with $E_{rest}(\mathbf{\Gamma}_{\odot}) > E(\mathbf{\Gamma}_{\odot})$) also here, when we employ systems $\mathbf{\Gamma}^i_{\odot}$ partially expanded before and then left unexpanded with the remaining weight $e^{-E_{rest}(\mathbf{\Gamma}'_{\odot})}$.

4) When we discussed the energy/entropy contest inside $\mathbf{\Gamma}_{\odot}$ in (2.23), we had to avoid that small contours would appear as interior contours of $\mathbf{\Gamma}_{\odot}$. The same care is, of course, needed here for the modified expansion procedure. Small contours do not disappear now but trivialize (add an infinite energy to) any conglomerate $\mathbf{\Gamma}_{\odot}$ containing them.

5) The family $\mathcal{P}_n$ grows with n. It is our aim to show that for some "stable" q, any element of this (complicated!) family has the energy $E_{rest}(\mathbf{\Gamma}_{\odot}) = \infty$, even in the limit $n \to \infty$. Then only the configuration "constant color q" will survive. (Below we develop corollaries of such an important fact.) Recall also that up to now, all expansions were done simultaneously on all tori Λ. So it is necessary to make a comparison; for example we would like to show that there is at least one $q \in Q$ for which we finally get stability on any *large enough* torus Λ:

The fully expanded model in a fixed torus Λ.

Denote by f_q^n the quantity f_q^o from the n–th application of Theorem 5, as described above. Introduce also the following notations (for any Λ, we do not mark the dependence on it)

$$f^n = \min_{q \in Q} f_q^n, \quad f_q^{\infty} = \lim_n f_q^n, \quad f^{\infty} = \min_{q \in Q}\ f_q^{\infty} \ = \lim_n f^n. \tag{5.3}$$

We will see below that the quantity f^{∞} equals to the free energy of (1.23) for $\Lambda \to \mathbb{Z}^{\nu}$.

For a comparison of the results of various expansion steps it is crucial to realize that once a contour $\mathbf{\Gamma}^q$ with a diameter $\operatorname{diam} \mathbf{\Gamma}^q = n$ becomes nonsmall in the n–th expansion step i.e.

once we have the relation, for the smallest "packing cube" $\square \supset \underline{\mathbf{\Gamma}}^q$ of $\mathbf{\Gamma}^q$

$$u_q^n |\square| > \Omega \tag{5.4}$$

then such a colour q *never attains a minimum* of the values $\{f_q^m, q \in Q\}$ later, for any $m > n$. Really, all the future quantities f_q^m differ from f_q^n very little:

Proposition. *Let f_q^n be the quantity f_q^o computed from Theorem 5 at a moment when contour functionals $F(\mathbf{\Gamma}_\odot)$ of extremal systems $\mathbf{\Gamma}_\odot$, of a size* $\operatorname{diam} \mathbf{\Gamma}_\odot = n$ *were just (partially) expanded. Let f_q^m be an analogous quantity constructed in a later, m–th step; $m > n$. Then*

$$|f_q^n - f_q^m| \le \gamma_n \qquad \textit{where} \quad \gamma_n \approx e^{-Can}. \tag{5.5}$$

Proof. Analyze the terms in (2.23) which make principal contributions to the terms $w_{\mathbb{T}}$ formed by subsequent applications of Theorem 5. Notice that the cores of the new clusters have diameters of the order $\ge \operatorname{diam} \mathbf{\Gamma}$; the smaller systems being already partially expanded before. It is ony here where one needs the bound (1.28).

Then we easily get the following useful consequence, for any expansion process of an abstract P.S. model (1.23) which was organized according to scheme desribed above.

Corollary. *A color q for which some contour $\mathbf{\Gamma}^q$ was classified nonsmall at least once during the expansion process will never become stable later.*

Proof. Denote by $\square$ the smallest packing cube of $\mathbf{\Gamma}^q$. Clearly, a difference of two quantities where the first one (the n–th value u_q) is $\ge \Omega/|\square|$ while the second one (future correction to such a u_q) is like $e^{-Ca \operatorname{diam} \square}$ only cannot be equal to 0 (for a reasonable choice of $\Omega \approx \tau$).

Therefore, a nontrivial result of the decomposition (4.5) always signalizes that the external color of $\mathbf{\Gamma}_\odot$ will become finally *unstable* in the thermodynamical limit. Even if such a system $\mathbf{\Gamma}_\odot$ becomes finally small then it will not be necessary to expand it (fully), except that it will appear as a subsystem of some bigger system $\widetilde{\mathbf{\Gamma}_\odot}$ having a stable external color.

So we see that the colors q for which some nonsmall $\mathbf{\Gamma}^q$ appear during the expansion procedures are the sure "candidates for unstability" in the limit. But how is it with possible "candidates for stability" in Q ? It is natural to look for them at the collection

$$\widetilde{Q_n} = \{q \in Q : f_q^n \le f^n + \gamma_n\} \quad \text{where} \qquad \gamma_n \approx e^{-Can}. \tag{5.6}$$

One can actually manage that $\widetilde{Q_n}$ decays monotonically with $n \to \infty$, for a suitable choice of γ_n. Moreover, if $f_q^n = f^n$ is a temporary minimum then f_q^∞ satisfies a relation, by (5.5)

$$f_q^\infty \le f^\infty + \tilde{\gamma}_n \tag{5.7}$$

with another $\tilde{\gamma}_n$. So, we conclude that the limit of (5.6) is really the collection of all values $q \in Q$, for which we have the limit relation $f_q^\infty = f^\infty$.

Comparison of fully expanded models in different tori, and in $\mathbb{Z}^\nu$.

The limit of the expansion procedures of Theorem 5, discussed above will be called the *fully expanded model*. Colors realizing the minimum of $\{f_q^\infty\}$ will be called the *stable* ones.

Notice that all these constructions were still done not on $\mathbb{Z}^\nu$ but *simultaneously on all tori* Λ. Of course, one would like to compare now all these quantities, for different tori Λ of arbitrarily large size. An easy inspection of the expansions then shows that the quantities f_q^Λ resp. $f_q^{\Lambda'}$ constructed in Λ resp. in Λ' differ as (a very little quantity indeed!)

$$|f_q^\Lambda - f_q^{\Lambda'}| \ \le \ e^{-C\tau \min\{\operatorname{diam}\Lambda, \operatorname{diam}\Lambda'\}}. \tag{5.8}$$

Then we can look for a possible coexistence of *several* stable q in a thermodynamical limit. In finite volumes, a "coexistence of phases" does not have a sharply defined meaning, but still the notion of "almost stability" (5.6) has some practical sense, see below.

Of course, the coexistence of phases is in principle a *very delicate* phenomenon, something which can never be decided with a full accuracy (for nonsymmetric models) on finite volumes of any size. We may be able to expand all the contour systems $\mathbf{\Gamma}_\odot^q$ in a given torus Λ but still a slight instability of such an "almost stable" q can be found in bigger "universes" Λ'.

We will see that if q is stable in the thermodynamic limit, then a Gibbs state "of the type q" always exists for the model (1.23), on tori of *any* size. By the latter we mean the finite volume Gibbs state conditioned at "infinity" (see the remark made in the next section) by the value q. This gives the usual limit q–th Gibbs state (if interpreted in such a way, on the corresponding spin configuration space) in the thermodynamic limit $\Lambda \to \mathbb{Z}^\nu$.

A conclusion is that there is always a nonempty collection of *stable* colors q (namely those minimizing the value f_q^∞) for which "everything is expanded" and on any torus Λ, *only one configuration of a type q survives all the expansions* (we do not look here at systems $\mathbf{\Gamma}_\odot$ with $E_{rest}(\mathbf{\Gamma}_\odot) = \infty$); namely the constant configuration "q everywhere".

Let us conclude this discussion and state some important consequences :

Extremal Gibbs states for stable colors q. Take the Hamiltonian (1.23), select a given stable $q \in Q$ and define a new auxiliary value "infinitesimally bigger than" e_q

$$e_q^* = e_q \ + \ \frac{C_q(\Lambda)}{|\Lambda|} \qquad \text{where} \quad \lim_{\Lambda\to\mathbb{Z}^\nu} \frac{C_q(\Lambda)}{|\Lambda|} = 0 \quad \text{and} \quad \lim_{\Lambda\to\mathbb{Z}^\nu} C_q(\Lambda) = \infty. \tag{5.9}$$

A probability measure $P_*^{C_q(\Lambda)}$, defined by such a changed Hamiltonian (using the value e_q^* instead of e_q for this particular color q, everything else in (1.23) left unchanged) on the configuration space of (1.23) will be called an *extremal q–th Gibbs state* of the model (1.23), on a torus Λ. Of course, there is some ambiguity (the choice of e_q^*) in the definition of this notion but this asmbiguity disappears with $\Lambda \to \mathbb{Z}^\nu$:

Theorem 7. *The limit*[15], *for* $\Lambda \to \mathbb{Z}^\nu$, *of* $P_*^{C_q(\Lambda)}$ *does not depend, under (5.9), on the choice of* $C_q(\Lambda)$. *Translated to a spin language, it is equal to the extremal q–th Gibbs state on* $\mathbb{Z}^\nu$.

Residual contours.

For a model (1.23) we introduce another notion, characteristic for this approach to Pirogov–Sinai theory and having no analogy in the theory of polymer models.

This will be relevant for a detailed study of structure of Gibbs states on any torus Λ, under any constraint $\mathcal{C}$. We are concerned here with a fixed Hamiltonian and so it is much more convenient to *return to the setting of Theorem 4* and to forget any partial expansions made later. An analogy of such a construction in the context of Theorem 5 can be also made, but it is a little bit artificial, less useful then the more direct approach of Theorem 4.

So, we take here again the scheme (1.22) and we use repeatedly (infinitely many times) Theorem 4. The limit will be again called a *fully expanded model*. It is important to notice that the quantities f_q^∞ are the same as before for all stable q, and stability of q means the same *in both approaches*. This follows from Theorems 5 and 4; namely both results produce a model $\mathcal{R}$–equivalent to the original one.

For nonstable q, the quantities f_q^∞ are not the same as before. However, the quantities $u_q^\infty = f_q^\infty - f^\infty$ are of the same order in both approaches. (We omit a detailed discussion.) For these values of q, there are nontrivial families $\mathcal{R}_q^\infty \subset \mathcal{A}^q$ of contours $\mathbf{\Gamma} = \mathbf{\Gamma}^q$ "surviving all the expansions". Let us call the elements of $\mathcal{R}^\infty = \cup_q \mathcal{R}_q^\infty$ (the union is over $q : u_q^\infty > 0$) the *residual contours* of the model (1.23). Of course there is again some arbitrariness in the definition of all these notions; they depend on the concrete choice of Ω and other things.

Residual contours are important when investigating unstable boundary conditions of a type "unstable color q outside a given volume $M \subset \Lambda$". More general boundary conditions in spin models cannot be directly addressed in full generality in the language of (1.23). The setting (1.23) allows only a formulation of somehow more restricted questions like above; most important case being that of an ensemble where a given "distant" region Λ_∞ of Λ (interpreted as the "infinity") is kept outside of all contours, at a fixed constant color q. For stable q, our expansions then give a very precise control of such a situation, and of resulting extremal q–th Gibbs states arising if we take the thermodynamic limit $\Lambda \setminus \Lambda_\infty \to \mathbb{Z}^\nu$.

For unstable q (kept fixed in "infinity") the problem is more complicated. If Λ_∞ is relatively small set then "astronomical" contours can also play an important role but we do not want to look at such situations. If $\Lambda \setminus \Lambda_\infty$ is relatively small subvolume of Λ then astronomical contours are excluded by definition, and problem has a direct description in the language of (1.23). This is a most natural situation, like if $\Lambda \setminus \Lambda_\infty$ is a finite subvolume of $\Lambda \equiv \mathbb{Z}^\nu$.

If we fully expand the latter situation then residual contours separating the region "of type q" around Λ_∞ from the big "bubbles" (of "phases" having another, more stable color)

[15] We omit a more detailed discussion of these notions here. However, an appropriately general and exhaustive reference on these general questions is still missing, in the context of abstract P.S. models.

do appear for large enough $\Lambda \setminus \Lambda_\infty$, and their statistics has to be analyzed. This problem was addressed in [Z] but its fuller treatment is also missing. However, this is a problem essentially complementary to what we call the P.S. technology. Detailed statistics of residual contours (having expanded the rest of the model) is generally a (spin) model dependent task, and so is the question on the "completeness of the P.S. phase picture" i.e. question whether there are some "exotic" translation invariant phases, other than those corresponding to stable colors q. The answer is *no* in usual (spin) models, but we do not address in detail such a question here. Nor we try to expain here why the uniqueness of a stable q implies that there are *no* other (even translation noninvariant) Gibbs states of such a Hamiltonian on $\mathbb{Z}^\nu$. Another paper dealing with these questions is in preparation.

6. Cluster expansion of conditioned models. Structure of Gibbs states

Here again, it is more convenient to stay in the setting of Theorem 4. We start with an important remark that an analogous construction (to that of the full expansion of $Z(\Lambda)$) can be done *also in the conditioned ensemble (1.24)*, with a fixed limitation $\mathcal{C}$ i.e. at a presence of some fixed conditioning in the ensemble (1.23).

To understand this, analyze in what sense we actually needed the translation invariance of the model (1.23) resp. more generally how much "local" was the expansion of Theorem 4. The answer is: cluster weights $w_{\mathbb{T}}$ depend only on objects (contours and clusters) incompatible with $\mathbb{T}$. This implies that translation invariancy of the constructed expansion is not violated "outside of $\mathcal{C}$". However, translation invariance is essential only to get invariant quantities f_q, which otherwise would be defined as site dependent densities. (This would generally cause a disaster in formulas (2.5), and the estimates of small contours following it.)

So, the constraint $\mathcal{C}$ does not effect construction of clusters outside it. More precisely, if such a limitation is expressed in terms of contours from some subsystem $\mathcal{S} \subset \mathcal{A}$ then the values of $w_{\mathbb{T}}$ for clusters from objects *compatible* with any contour from $\mathcal{S}$ are independent of $\mathcal{S}$. So, while the estimates of the convergence of clusters remain essentially unchanged even at the presence of $\mathcal{C}$, the translation invariancy of the newly formed w_T will be lost only for clusters which touch the system $\mathcal{S}$. In practise, $\mathcal{C}$ will be often formulated as a condition on "q–valued spins at infinity". Then the iterative application of Theorem 4 gives a cluster expansion, for q stable, of an extremal Gibbs state "of a type q" in the limit $\Lambda \to \mathbb{Z}^\nu$.

So it has a prefect sense to apply, iteratively, the expansions of Theorem 4 at a presence of some fixed limitation $\mathcal{C}$, expressed in terms of contours from some $\mathcal{S}$. We may expand iteratively all the small extremal systems $\mathbf{\Gamma}_\odot$ (arising during the iterative process) compatible with $\mathcal{C}$, but *only those systems*. Of course, also systems $\mathbf{\Gamma}_\odot$ bigger than contours from $\mathcal{S}$.

This clearly requires some adaptation of the scheme (1.22), by reducing it only to the objects *untouched* by $\mathcal{C}$. Contour systems incompatible with some contour of a family $\mathcal{S}$ (in which the limitation $\mathcal{C}$ is formulated) are left unexpanded, of course. However, for stable colors q such a restriction creates only very minor, easily controllable changes (when compared to the full expansion in the unconditioned ensemble (1.23)).

It is important to realize that such expansions actually will give a very compact *formula* *for Gibbs probability* of an event "an island $\mathcal{D}$ with external color q appears in the $q-$ sea", both in the full ensemble (1.23) resp. in the conditioned ensemble (1.24). Let us denote by $P(\mathcal{D})$ the probability of such an event, namely that there is no contour encircling $\mathcal{D}$. The knowledge of this quantity is especially useful in the following two cases:

i) if the external color q of the contour system $\mathcal{D}$ is *stable* and all the interior contours of $\mathcal{D}$ are residual in the model (1.23)

ii) if $\mathcal{D} = \mathbf{\Gamma}$ is a single residual contour –or a collection of residual contours– with an unstable external color q, in a volume Λ which is not "too big" (resp. under additional constraints outside $\mathcal{D}$ which make the probability of such event an interesting quantity, even if the instability of the external color q would be otherwise strongly felt in the exterior of $\mathcal{D}$).

We prefer to write $\mathcal{D}$ instead of $\mathbf{\Gamma}_\odot$ because we do not assume its "extremality" here.

Take a limitation $\mathcal{C}$ "having the value q at infinity" in the ensemble (1.24). Introduce, in addition to the probability (1.17), here denoted as $P^q(\mathcal{D})$ also the quantity $P^q_{ref}(\mathcal{D})$ measuring the probability, in the ensemble (1.24), of the event that "there is a free place around $\underline{\mathcal{D}}$ and moreover $\underline{\mathcal{D}}$ has color q". More precisely, in the latter case we investigate the event that a configuration ($\equiv$ compatible contour system) $\{\mathbf{\Gamma}_i\}$ with a prescribed external color q is such that $\mathrm{dist}(\mathbf{\Gamma}_i, \underline{\mathcal{D}}) \geq 2$ for each $\mathbf{\Gamma}_i$. We will see that the quantity $F(\mathcal{D})$ is really "tailored" to answering the above probabilistic questions. Namely, we will have the relation

$$P^q\mathcal{D}) \doteq P^q_{ref}(\mathcal{D}) \ \exp(-F(\mathcal{D}))$$

more precisely in place of $F(\mathcal{D})$ we have to take quantity $F_\infty(\mathcal{D})$ described below. The term $P^\circ_\Lambda(\mathcal{D})$ will be approximately equal to 1 in many cases of interest. Then we really get a useful formula – assuming that we establish the Peierls condition for $F(\mathcal{D})$:

Contour functional $F^\infty(\mathcal{D})$ and the probability $P^q(\mathcal{D})$. The Peierls argument.

It remains to give a precise definition of the "limit" contour functional $F_\infty(\mathcal{D})$, as compared to "temporary" (inductively defined) contour functionals $F(\mathbf{\Gamma}_\odot)$ used in Theorem 4. While the latter quantities were constructed at a stage when "everything smaller than $\mathbf{\Gamma}_\odot$ and expandable (i.e. extremal small) was expanded" the quantity $F_\infty(\mathcal{D})$ is a limit one, defined only at the very end of the expansion procedure. Even after expanding "everything expandable and smaller than $\mathcal{D}$" one continues the process of expansion of the conditioned, by the required presence of $\mathcal{D}$, model. At the end of the expansion we get cluster weights which we denote by w^∞_T (as compared to temporary values w_T and $w_{\mathbb{T}}$ used in (2.1). These quantities are translation invariant except of the situation when $T \ \iota \ \mathcal{D}$.

The limit contour functional $F^\infty(\mathcal{D})$ of a contour system $\mathcal{D}$.

Let $\mathcal{D}$ be a contour system with an external color q. Put

$$F^\infty(\mathcal{D}) = E(\mathcal{D}) + \sum_{\tilde{q} \neq q} \Big((e_{\tilde{q}} - e_q) |\operatorname{int}_{\tilde{q}} \mathcal{D}| - \sum_{T_{\tilde{q}} \subset \operatorname{int}_{\tilde{q}} \mathcal{D}} w^\infty_{T_{\tilde{q}}} + \sum_{T_q \subset \operatorname{int}_{\tilde{q}} \mathcal{D}} w^\infty_{T_q} \Big). \tag{6.1}$$

Clearly, $F^\infty(\mathcal{D})$ is almost equal to the "temporary" $F(\mathcal{D})$ used in Theorem 4. The difference, caused by "future" corrections to $F(\mathcal{D})$, is very small, of the order (compare (2.6))

$$|F^\infty(\mathcal{D}) - F(\mathcal{D})| \le Ce^{-C'\tau \operatorname{diam} \mathcal{D}}. \tag{6.2}$$

Theorem 8. *Let $\mathcal{E}(\mathcal{D})$ be an event, in an ensemble given by the partition function (1.24) (conditioned by "color q at infinity") that a given contour system $\mathcal{D}$ appears as a subsystem of the contour system x_Λ. Then the probability of this event is equal to*

$$P^q_\Lambda(\mathcal{E}(\mathcal{D})) = \exp(-F^\infty(\mathcal{D})) \exp(\sum_{T:T\iota\mathcal{D}} \bar{w}^\infty_T) \tag{6.3}$$

where $\bar{w}^\infty_T$ decay similarly as in Theorem 4.

Proof. By analogous arguments as those used in our crucial relation (3.28). Here, one has to compare cluster terms w^∞_T resp. $\tilde{w}^\infty_T$ arising in the fully expanded model defined at the presence of the constraint "$\mathcal{D}$ is a subsystem of x_Λ" resp. at the presence of the constraint "no contour of x_Λ touches $\underline{\mathcal{D}}$, and the color remains that of $\operatorname{ext}(\mathcal{D})$ there". Similar formulas hold for more complicated constraints $\mathcal{C}$ (influencing then the values of w_T for $T\iota\mathcal{C}$).

Conclusion.

Let us shortly summarize the additional information we derived in these last sections. Denote by $Z^r(\Lambda)$ the partition function over all configurations which have at least one "astronomically large" contour $\mathbf{\Gamma}$. (Obviously, these configurations are negligible in very large volumes Λ.) Consider finally the *full ensemble* with the partition function $Z_{full}(\Lambda) = Z(\Lambda) + Z^r(\Lambda)$. We conclude the paper by stating its main result.

Main Theorem. *For any model (1.23) satisfying the assumptions (1.13) and (1.14) with $p \approx \tau$, $c \approx \tau$ and with τ large enough*[16] *we construct the limit cluster weights w^∞_T. The construction of these quantities is recursive, starting with (1.23) and then using Theorem 4, thus incorporating also all the small contour systems $\mathbf{\Gamma}_\odot$ into the clusters T constructed in later expansion steps. Denote by f^∞_q the "metastable free energy" (2.3)*

$$f^\infty_q \;=\; e_q + \sum_{T_q\subset\mathbb{Z}^\nu:\ T_q\ \iota\ i} \frac{w^\infty_{T_q}}{|\underline{T}_q|}\ . \tag{6.4}$$

Then the free energy of the model is equal to $f^\infty = \min f^\infty_q$. Moreover the following is true:

1) The quantities w^∞_T are exponentially decaying, see also (3.24), with $\alpha \approx \tau$

$$\sum_{T:\underline{T}\ni i} w^\infty_T\ e^{\alpha|\underline{T}|} \ll 1. \tag{6.5}$$

[16] In particular, the result is valid for any model of a type (1.1) satisfying (1.5). It is not the aim of this article to investigate in detail the relation of these abstract P.S. models to concrete spin models.

2) For "stable" values of q, i.e. such that $f_q^\infty = f^\infty$, a translation invariant, extremal q–th Gibbs state on $\mathbb{Z}^\nu$ exists. Almost any its configuration belongs to the class of "q– like configurations" $X_p^q(\Lambda) = \{x_\Lambda : |\Lambda_q| > p|\Lambda|\}$ where p is rather close to 1, $p \to 1$ as $\tau \to \infty$.

3) On any finite torus Λ, the support of the Gibbs state is contained, with an accuracy $\approx e^{-C\tau \operatorname{diam}\Lambda} + e^{-a|\Lambda|}$, in the union of those $X_p^q(\Lambda)$ for which q is "almost" stable, such that $f_q^\infty \le f^\infty + a$. In any subvolume of a size $\le \frac{C\tau}{a}$ (where C is a universal constant) and under such a boundary color q, the conditioned Gibbs probability has analogous properties as in 2).

4) The probability, in the ensemble (1.23), of the event "system $\mathcal{D}$ appears" is given by

$$P_\Lambda^q(\mathcal{D}) = \exp(-E(\mathcal{D}))\ \exp(\ \xi_\Lambda(\mathcal{D}) + \zeta_\Lambda(\mathcal{D})\ ) \qquad \textit{see more precisely (6.3)} \tag{6.6}$$

$$\textit{where} \qquad |\xi_\Lambda(\mathcal{D})| \le \epsilon\ |\underline{\mathcal{D}}| \qquad \textit{and} \qquad \zeta_\Lambda(\mathcal{D})\ \le\ (f_q^\infty - f^\infty)\ |\operatorname{int}\mathcal{D}| +\ e^{-C\tau \operatorname{diam}\Lambda} \tag{6.7}$$

where q denotes the external color of $\mathcal{D}$. While the term $\xi_\Lambda(\mathcal{D})$ can be interpreted as some small "surface tension" of $\mathcal{D}$ the term ζ_Λ in (6.7) accounts also the missing "astronomical" contours in (6.3) and, much more importantly, the possible instability of q: Then $f_q^\infty - f^\infty > 0$, and for very large volumes Λ the term $\zeta_\Lambda(\mathcal{D})$ increases linearly with $|\operatorname{int}\mathcal{D}|$ and formula (6.6) loses any useful information. Big "bubbles" of stable phases may then appear in volumes $M \subset \Lambda$, under a fixed external color $q_{\Lambda\setminus M}$. We do not study their statistics here.

5) The events considered in (6.6) are rare for stable q, and also for instable q in tori such that $(f_q^\infty - f^\infty)|\Lambda| \ll \tau \operatorname{diam}\Lambda$. Then they have a probability decaying like $e^{-C\tau|\underline{\mathcal{D}}|}$. Typical configuration in the Gibbs state corresponding to (1.23) on Λ then looks, for some of these q, like a "sea of color q with rare and small islands $\mathcal{D}$" (bordered by external contours of $\mathcal{D}$).

6) The properties of the **phase diagram** *of the model*

$$\{\ H \to \{q \in Q : f_q^\infty = f^\infty\}\} \tag{6.8}$$

can be derived from the properties of the mapping, constructed with the help of Theorem 5

$$\{\ H \quad \to \quad \{f_q^\infty, q \in Q\}\}. \tag{6.9}$$

To get a useful consequence from it, we strengthen the assumptions (1.13), (1.14) as follows. Let all the weights of (1.23) depend on another (vector) parameter λ such that the estimates

$$\sum_{T \ni x_i} |\frac{\partial}{\partial\lambda} w_T|\ e^{c|\underline{T}|} \quad \le\ \epsilon \ll 1 \ll\ c \tag{6.10}$$

and (recall that $w_{\mathbf{\Gamma}} = e^{-E(\mathbf{\Gamma})}$)

$$\sum_{\mathbf{\Gamma}:\underline{\mathbf{\Gamma}}\ni i} |\frac{\partial}{\partial\lambda} w_{\mathbf{\Gamma}}|\ |\operatorname{int}\mathbf{\Gamma}|\ e^{g|\underline{\mathbf{\Gamma}}|}\ \le\ \omega\ \ll 1 \ll g \tag{6.11}$$

are valid, with appropriately stronger requirements on ε, ω than before. Then

$$|\frac{\partial}{\partial\lambda}\ (f_q^\infty - e_q)|\ \le\ \varepsilon$$

and this condition implies, by implicit function theorem, the existence and differentiability of the manifolds of coexistence $M_{\tilde{Q}} = \{q \in \tilde{Q} \iff f_q^\infty = f^\infty\}$, *if a suitable "degeneracy removing condition" on* $e_q(\lambda)$ *is imposed. See [PS], [Z]. In case of (almost) linear dependence of the quantities* e_q *and* $E(\mathbf{\Gamma})$ *on* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ *where* $|Q| = n+1$ *this condition can be formulated in terms of the invertibility of the matrix of partial derivatives* $\{\partial f_q^\infty / \partial \lambda_i\}$, *complemented by a column of values* 1. *If the matrix norm of such an inverse matrix is smaller enough than* τ, *then we can conclude that the phase diagram (6.8) of the model (1.23) is a small perturbation of the "zero temperature" diagram*

$$\{\ H \to \{q \in Q : e_q = e\}\} \qquad \textit{where} \qquad e = \min_{\tilde{q} \in Q}\ e_{\tilde{q}}. \tag{6.12}$$

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