ON THE PRESERVATION OF ABSOLUTELY CONTINUOUS SPECTRUM FOR SCHRÖDINGER OPERATORS

SERGEY A. DENISOV

Abstract. We present general principles for the preservation of a.c. spectrum under weak perturbations. The Schrödinger operators on the strip and on the Caley tree (Bethe lattice) are considered.

In this paper, we consider the problem of preservation of a.c. spectrum for Schrödinger operators when the perturbation (potential) is decaying slowly at infinity. In a recent years, lots of results in this direction (including analysis of Jacobi matrices and Dirac operators) were obtained [10, 2, 9, 13, 17, 21, 3]. The proofs were of analytical flavor and had roots in the approximation theory, in particular, the theory of orthogonal polynomials of one variable [22, 12]. In the one-dimensional case, most of the questions were answered and we have a rather complete understanding of the picture. In the higher dimension, we know much less. In a series of papers [14, 15, 4, 5], authors were partially motivated by the following conjecture

Conjecture. *(B. Simon,* [19]*)*

Consider

$$H = -\Delta + V(x),\ x \in \mathbb{R}^d$$

with

$$\int\limits_{\mathbb{R}^d} \frac{V^2(x)}{|x|^{d-1}+1} dx < +\infty$$

Show that $\sigma_{ac}(H) = \mathbb{R}^+$.

In spite of some progress in the field, this conjecture is still an open problem.

In the current paper, we consider analogous problems, but on different domains. In the first section, we prove one general result on preservation of absolutely continuous spectrum for block matrices. We also apply this result to Schrödinger operator on the strip. The second section is devoted to the discrete Schrödinger operator on Caley tree (Bethe lattice) and the corresponding L^2 conjecture. We

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denote by $\mathcal{J}^p$ the standard Schatten-von Neumann class of operators. Symbols $\mathbb{T}$ and $\mathbb{D}$ stand for the unit circle (disk) on the complex plane.

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1. One corollary of Kato-Rosenblum theorem.

The famous Kato-Rosenblum theorem [7] says that the wave operators $W_\pm(H_0, H_0+V)$ exist for any self-adjoint bounded H_0 as long as $V \in \mathcal{J}^1, V^* = V$. As a corollary, $\sigma_{ac}(H_0) = \sigma_{ac}(H_0+V)$. We prove the following theorem

Theorem 1.1. *Let H_1, H_2 be two self-adjoint operators in the Hilbert space $\mathcal{H}$, both bounded from below. Assume that $\sigma_{ess}(H_2) \subseteq [b, +\infty]$ and $[a,b] \subseteq \sigma_{ac}(H_1), (a<b)$. Then, for any $V \in \mathcal{J}^2, V^*=V$, we have that $[a,b] \subseteq \sigma_{ac}(H)$, where*

$$H = \begin{bmatrix} H_1 & V \\ V & H_2 \end{bmatrix}$$

Remark 1.2. *In the case $\sigma_{ess}(H_1) \subseteq (-\infty, c), c<b$, the statement follows from the arguments in* [1]*. The case when the spectra of H_1 and H_2 overlap is more involved but the effect is more or less the same: the influence of H_2 on $[a,b]$ is small and the Hilbert-Schmidt off-diagonal perturbation does not change a.c. spectrum of H on $[a,b]$.*

Proof. Take any $\epsilon > 0$. Let us prove that $[a, b-2\epsilon] \subseteq \sigma_{ac}(H)$. Since finite-rank perturbations do not change a.c. spectrum and $\sigma_{ess}(H_2) \subseteq [b,+\infty)$, we can always assume that

$$\sigma(H_2) \subseteq [b-\epsilon/2, +\infty)$$

We can also arrange for $\|V\|_2$ to be as small as we wish. The actual estimate on $\|V\|_2$ will follow from the arguments below.

Denote $R_z^{(1)} = (H_1-z)^{-1}, R_z^{(2)} = (H_2-z)^{-1}$. Consider $R_z = (H-z)^{-1}$

$$\begin{cases} (H_1-z)x + Vy = f \\ Vx + (H_2-z)y = g \end{cases} \quad \begin{cases} x = R_z^{(1)}(f - VR_z^{(2)}g + VR_z^{(2)}Vx) \\ y = R_z^{(2)}g - R_z^{(2)}Vx \end{cases} \tag{1.1}$$

Without loss of generality, we can assume $b-\epsilon$ is not an eigenvalue of H. Then,

$$P_{(-\infty, b-\epsilon)} \begin{pmatrix} f \\ g \end{pmatrix} = \frac{1}{2\pi i} \int\limits_\gamma R_z \begin{pmatrix} f \\ g \end{pmatrix} dz$$

where γ is a contour, say, rectangle, that crosses the real axes at the point $b-\epsilon$ with its right side and "encircles" the parts of $\sigma(H_1)$ and $\sigma(H)$ below the point $b-\epsilon$. P_Δ denotes the orthoprojector of H onto the set Δ. The integral converges in principal value near $b-\epsilon$. Denote projections to the first/second component in $\mathcal{H}\times\mathcal{H}$ by $\Pi_{1(2)}$. Let $x(z)=\Pi_1 R_z(f,g)^t$, $y(z)=\Pi_2 R_z(f,g)^t$. From (1.1),

$$P_{(-\infty,b-\epsilon)}\begin{pmatrix} f\\ g\end{pmatrix}=\frac{1}{2\pi i}\begin{bmatrix} \int\limits_\gamma x(z)dz\\ -\int\limits_\gamma R_z^{(2)}Vx(z)dz\end{bmatrix}$$

Using the first resolvent identity, we have

$$\int\limits_\gamma R_z^{(2)}Vx(z)dz=R_{b-\epsilon}^{(2)}V\int\limits_\gamma x(z)dz+\int\limits_\gamma [z-(b-\epsilon)]R_z^{(2)}R_{b-\epsilon}^{(2)}Vx(z)dz$$

Due to the regularization at $z=b-\epsilon$, the second integral converges in the norm and we can write projector as

$$P_{(-\infty,b-\epsilon)}\begin{pmatrix} f\\ g\end{pmatrix}=\begin{pmatrix} A_1f+B_1g\\ K_1(A_1f+B_1g)+D_1f+D_2g\end{pmatrix}$$

where $K_1,D_1,D_2\in\mathcal{J}^2$. Notice that if $(f,g)^t\in\Omega\equiv RanP_{(-\infty,b-\epsilon)}$, then

$$\begin{cases} f=A_1f+B_1g\\ g=K_1(A_1f+B_1g)+D_1f+D_2g\end{cases}$$

So, $g=K_1f+D_1f+D_2g$. Taking $\|V\|$ small enough, we can always assume that $\|D_2\|<1$ and

$$g=(1-D_2)^{-1}(K_1+D_1)f=Kf \text{ with } K\in\mathcal{J}^2$$

That means any $(f,g)^t\in\Omega$ can be written as $(f,Kf)^t$. Let Ω_1 denote $\Pi_1\Omega$. Ω_1 is a subspace in $\mathcal{H}$. Ω reduces H. Thus,

$$\begin{pmatrix} H_1 & V\\ V & H_2\end{pmatrix}\begin{pmatrix} f\\ Kf\end{pmatrix}=\begin{pmatrix} Tf\\ KTf\end{pmatrix},\text{ where }\quad Tf=(H_1+VK)f$$

and T is reduced by Ω_1. Being equipped with new the inner product

$$<f_1,f_2>=((1+K^*K)f_1,f_2)$$

subspace Ω_1 is isometrically isomorphic to Ω and $H_{|\Omega}$ (i.e. H restricted on Ω) is unitarily equivalent to T, which acts in $(\Omega_1,<.>)$ as a self-adjoin operator.

Now let us show that $P^{(1)}_{(-\infty,b-2\epsilon)}$, the spectral projection for H_1, has the following property: $Ran\ P^{(1)}_{(-\infty,b-2\epsilon)} \subseteq \Omega_1$. Indeed, assume that $h \in\ Ran\ P^{(1)}_{(-\infty,b-2\epsilon)}$. Then,

$$P^{(1)}_{(-\infty,b-2\epsilon)}\Pi_1 P_{(-\infty,b-\epsilon)}\begin{pmatrix} h \\ 0 \end{pmatrix} = h + \frac{1}{2\pi i}P^{(1)}_{(-\infty,b-2\epsilon)}\int\limits_{\gamma} R^{(1)}_z V R^{(2)}_z V x_h(z)dz = h + Eh$$

where $\|E\| < 1$ if $\|V\|$ is small enough. So, $Ran\ (1+E) = Ran\ P^{(1)}_{(-\infty,b-2\epsilon)}$. For any $f \in RanP^{(1)}_{(-\infty,b-2\epsilon)}$, there is an $h \in RanP^{(1)}_{(-\infty,b-2\epsilon)}$ such that for

$$\Pi_1 P_{(-\infty,b-\epsilon)}\begin{pmatrix} h \\ 0 \end{pmatrix} \in \Omega_1$$

we have

$$P^{(1)}_{(-\infty,b-2\epsilon)}\Pi_1 P_{(-\infty,b-\epsilon)}\begin{pmatrix} h \\ 0 \end{pmatrix} = f$$

That implies $Ran\ P^{(1)}_{(-\infty,b-2\epsilon)} \subseteq \Omega_1$. Since $K \in \mathcal{J}^2$ and $VK \in \mathcal{J}^1$, we have by Kato-Rosenblum theorem that $(a, b-2\epsilon) \subseteq \sigma_{ac}(T)$ and thus $(a, b-2\epsilon) \subseteq \sigma_{ac}(H)$. $\square$

It is easy to generalize this result to the case when $H_{1(2)}$ act in different Hilbert spaces and V is a relative Hilbert-Schmidt perturbation. That allows to apply theorem 1.1 to continuous Schrödinger operator on the strip $\Pi = \{0 < x < \infty, 0 < y < \pi\}$. Let

$$H = -\Delta + V(x,y)$$

and impose Dirichlet conditions on the boundary of Π. Then

Corollary. *If* $\underset{0\le y\le\pi}{sup}|v(x,y)| \in L^2(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$, *then* $[1,4] \subseteq \sigma_{ac}(H)$.

This result follows from the matrix representation of H. For $f(x,y) \in L^2(\Pi)$,

$$f(x,y) = \sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\sin(ny)f_n(x),\ f_n(x) = \sqrt{\frac{2}{\pi}}\int\limits_0^\pi f(x,y)\sin(ny)dy$$

$$H = \begin{bmatrix} -\dfrac{d^2}{dx^2} + V_{11}(x) + 1 & V_{12}(x) & \dots \\ V_{21}(x) & -\dfrac{d^2}{dx^2} + V_{22}(x) + 4 & \dots \\ \dots & \dots & \dots \end{bmatrix} \tag{1.2}$$

$$V_{lj}(x) = \frac{2}{\pi}\int\limits_0^\pi V(x,y)\sin(ly)\sin(jy)dy$$

Since $V_{11} \in L^2(\mathbb{R}^+)$ we can use [2] to conclude $[1,4] \subseteq \sigma_{ac}(H)$. In [18], the case of matrix-valued potential was considered. Using this technique, it is conceivable (and authors of [18] intend to do that) that L^2 conjecture can be proved for Schrödinger operators with matrix-valued potential of the form $V = D+v$, where $D-$ diagonal and $\|v\| \in L^2(\mathbb{R}^+)$. That would allow to conclude that $\sigma_{ac}(H) = [1,\infty)$.

2. L^2 spectral conjecture for Schrödinger operator on the Caley tree (Bethe lattice).

In this section, we consider the Caley tree (Bethe lattice) $\mathbb{B}$ and the discrete Laplacian on it

$$(H_0 u)_n = \sum_{|i-n|=1} u_i$$

We remind that the Caley tree is an infinite graph with no closed loops and a fixed degree (number of nearest neighbors) at each vertex. We will denote by $|x-y|$ the lenght of the shortest path connecting x and y. Assume, for simplicity, that the degree at each point is equal to 3. All arguments can be easily adjusted to the general case and even to some modifications of the Laplacian. It is both known and easy to see that $\sigma(H_0) = [-2\sqrt{2}, 2\sqrt{2}]$ and the spectrum is purely absolutely continuous. Let $H = H_0 + V$, where V is a potential. The natural question to ask is under which conditions on V the a.c. spectrum of H is preserved. The Bethe lattice has a great advantage compared to $\mathbb{Z}^d$: there is only one path connecting any two points. That usually makes it possible to set up a recursion relation to control different quantities (see, e.g., the proof of delocalization for the Anderson model [11]). The result below and its proof has their roots in the theory of orthogonal polynomials on the unit circle [8] and on the real line. One could call it the multidimensional step-by-step sum rule (using the term recently introduced into the subject [9, 20]). At the same time, the idea has a clear physical meaning, as will be discussed below. We hope that it might give some insight to the solution (positive or negative) of L^2 conjecture on $\mathbb{Z}^d$. Scattering on the special graphs and networks was studied before (see, e.g., [16, 6]).

Consider any vertex O. It is connected to its neigbors by three edges. Delete one edge together with the corresponding part of the tree stemming from it. What is left will be called B_O. The degree of O within $\mathbb{B}_0$ is equal to 2. The main result of this section is the following

Theorem 2.1. *If*

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{x \in \mathbb{B}_0, |x-O|=n} V^2(x) < +\infty \tag{2.1}$$

then

$$[-2\sqrt{2}, 2\sqrt{2}] = \sigma_{ac}(H_{|\mathbb{B}_0}) \subseteq \sigma_{ac}(H)$$

Proof. Since $\mathbb{B}_0$ is connected to the other part of $\mathbb{B}$ by just one edge, the rank-two perturbation argument allows us to focus on $H_{|\mathbb{B}_0}$ only, i.e. we only need to show that $\sigma_{ac}(H_{|\mathbb{B}_0}) = [-2\sqrt{2}, 2\sqrt{2}]$. Assume that $V_{|\mathbb{B}_0}$ has a finite support. We introduce

$$m(\lambda) = ((H_{|\mathbb{B}_0} - \lambda)^{-1}\delta_O, \delta_O)$$
$$m_1(\lambda) = ((H_{|\mathbb{B}_1} - \lambda)^{-1}\delta_{O_1}, \delta_{O_1})$$
$$m_2(\lambda) = ((H_{|\mathbb{B}_2} - \lambda)^{-1}\delta_{O_2}, \delta_{O_2})$$

where $O_{1(2)}$ - neighbors of O in $\mathbb{B}_0$, $H_{|\mathbb{B}_j}$ - restrictions of $H_{|\mathbb{B}_0}$ to the branches $\mathbb{B}_j$ having "roots" at $O_j, j = 1, 2$, and δ_X is a discrete delta-function with support at the vertex X. The following is straightforward

$$-m_1 m - m_2 m = -V(O)m + \lambda m + 1$$

and

$$m(\lambda) = -\frac{1}{m_1(\lambda) + m_2(\lambda) - V(O) + \lambda} \tag{2.2}$$

Recurrence relation (2.2) has the fundamental importance. If $V \equiv 0$, we have

$$m_0(\lambda) = -\frac{1}{2m_0(\lambda) + \lambda}, \quad m_0(\lambda) = \frac{-\lambda + \sqrt{\lambda^2 - 8}}{4}$$

(2.2) can be also used to obtain an asymptotical expansion of $m(\lambda)$ as $\lambda \to \infty$. We will need only a couple of coefficients in this expansion. Clearly, $m_j(\lambda) = -\lambda^{-1} + O(\lambda^{-2}), j = 1, 2$ and

$$m(\lambda) = -\frac{1}{-V(O) + \lambda - 2\lambda^{-1} + O(\lambda^{-2})} =$$

$$= -\frac{1}{\lambda[1 - (V(O)\lambda^{-1} + 2\lambda^{-2} + O(\lambda^{-3})]} =$$

$$= -\lambda^{-1}[1 + V(O)\lambda^{-1} + 2\lambda^{-2} + V^2(O)\lambda^{-2} + O(\lambda^{-3})] =$$

$$= -[\lambda^{-1} + V(O)\lambda^{-2} + (2 + V^2(O))\lambda^{-3} + O(\lambda^{-4})]$$

One can iterate (2.2) to get an interesting analog of the continued fraction. Knowing $m_0(\lambda)$ and recursion relation, we conclude that $m(\lambda)$ is meromorphic in $\mathbb{C}\backslash\{[-2\sqrt{2}, 2\sqrt{2}]\}$. Introduce $M(\lambda) = m_1 + m_2 - V(O) + \lambda$. Then

$$m(\lambda) = -\frac{1}{M(\lambda)} \tag{2.3}$$

In the next paragraph, we repeat the calculation done by B. Simon [20]. Let us map $\mathbb{C}$ to the unt disk $\mathbb{D}$ by considering $f(z) = -m(\sqrt{2}(z+z^{-1}))$ and $F(z) = -M(\sqrt{2}(z+z^{-1}))$. Notice that in the free case $f_0(z) = z/\sqrt{2}$. Let z_k^+ and p_k^+ be positive zeroes/poles of $f(z)$, respectively. Denote the negative zeroes/poles by z_k^- and p_k^-. We order them as follows

$$0 < z_1^+ < z_2^+ < \ldots < 1, 0 < p_1^+ < p_2^+ < \ldots < 1,$$
$$0 > z_1^- > z_2^- > \ldots > -1, 0 > p_1^- > p_2^- > \ldots > -1$$

We have

$$f(z) = -\frac{1}{F(z)}, \qquad \frac{\Im f(z)}{\Im F(z)} = |f(z)|^2 \tag{2.4}$$

The function $f(z)$ has positive/negative imaginary parts in $\mathbb{C}^\pm \cap \mathbb{D}$. Then, by Simon's theorem [20], we have a multiplicative representation for $f(z)$

$$f(z) = zB(z) \exp\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{t+z}{t-z} \ln|f(t)|d\theta\Big), t = e^{i\theta} \tag{2.5}$$

$$B(z) = B_1(z)B_2^{-1}(z)$$

$$B_1(z) = \prod_k \frac{|z_k|}{z_k}\cdot\frac{z_k - z}{1-z_k z},\ B_2(z) = \prod_k \frac{|p_k|}{p_k}\cdot\frac{p_k - z}{1-p_k z}$$

Let us divide both sides of (2.5) by z, take ln, and expand into the Taylor series around 0. We have

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{2}}\frac{z}{z^2+1} + \frac{V(O)}{2}\cdot\frac{z^2}{(z^2+1)^2} + \frac{2+V^2(O)}{2\sqrt{2}}\cdot\frac{z^3}{(z^2+1)^3} + O(z^4) \\ &= \frac{1}{\sqrt{2}}z + \frac{V(O)}{2}z^2 + \frac{V^2(O)}{2\sqrt{2}}z^3 + O(z^4); \end{aligned}$$

$$\ln\Big(\frac{f(z)}{z}\Big) = -\frac{1}{2}\ln 2 + \frac{V(O)}{\sqrt{2}}z + \frac{V^2(O)}{4}z^2 + \ldots;$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{t+z}{t-z}\ln|f(t)|d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\ln|f(t)|(1+2zt^{-1}+2z^2t^{-2}+\ldots)d\theta;$$

$$\ln\left(\frac{z_k^+ - z}{1 - z_k^+ z}\right) = \ln[(z_k^+ - z)(1 + z_k^+ z + {z_k^+}^2 z^2 + \ldots)] =$$

$$= \ln z_k^+ + z(z_k^+ - {z_k^+}^{-1}) + z^2({z_k^+}^2 - {z_k^+}^{-2})/2 + O(z^3)$$

Comparing the first and the third coefficients, we obtain

$$-\frac{1}{2}\ln 2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|f(t)|d\theta + \sum_k \ln|z_k| - \sum_k \ln|p_k| \tag{2.6}$$

and

$$\frac{V^2(O)}{4} = \frac{1}{\pi}\int_{-\pi}^{\pi} \ln|f(t)|t^{-2}d\theta + \sum_k \frac{|z_k|^2 - |z_k|^{-2}}{2} - \sum_k \frac{|p_k|^2 - |p_k|^{-2}}{2} \tag{2.7}$$

Take (2.6)-1/2 $\cdot$ (2.7). We then have

$$-\frac{1}{2}\ln 2 - \frac{V^2(O)}{8} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|f(t)|(1 - t^{-2})d\theta - \Big\{\sum_k Y(\widetilde{E}_k) - \sum_k Y(E_k)\Big\}$$

where

$$Y(-E) = Y(E) = \frac{|z|^2 - |z|^{-2}}{4} - \ln|z|, E = \sqrt{2}(|z| + |z|^{-1}), |z| < 1, E > 2\sqrt{2}$$

and

$$E_k^\pm = \pm\sqrt{2}(|p_k^\pm| + |p_k^\pm|^{-1}), \widetilde{E}_k^\pm = \pm\sqrt{2}(|z_k^\pm| + |z_k^\pm|^{-1})$$

Recall (2.4)

$$\frac{1}{2\pi}\int_0^{\pi} \frac{\ln\,\Im f}{\ln\Im F}(1 - \cos 2\theta)d\theta =$$

$$= -\frac{1}{2}\ln 2 - \frac{V^2(O)}{8} + \sum_k Y(\widetilde{E}_k^+) - \sum_k Y(E_k^+) + \sum_k Y(\widetilde{E}_k^-) - \sum_k Y(E_k^-)$$

For the spectral measures,

$$\Im m(\lambda) = \pi\sigma_O'(\lambda), \quad \lambda \in [-2\sqrt{2}, 2\sqrt{2}]$$

$$\Im M(\lambda) = \Im m_1(\lambda) + \Im m_2(\lambda) = \pi(\sigma_{O_1}'(\lambda) + \sigma_{O_2}'(\lambda))$$

Then, the multidimensional step-by-step sum rule is

$$\frac{1}{8\pi}\int\limits_{-2\sqrt{2}}^{2\sqrt{2}}\sqrt{8-\lambda^2}\ln[\pi\sigma'(\lambda)]d\lambda = \frac{1}{8\pi}\int\limits_{-2\sqrt{2}}^{2\sqrt{2}}\sqrt{8-\lambda^2}\ln\left(\pi\frac{\sigma_1'+\sigma_2'}{2}\right)d\lambda$$

$$-\frac{V^2(O)}{8}+\sum_k Y(\widetilde{E}_k)-\sum_k Y(E_k)$$

Let us show that

$$\sum_k Y(\widetilde{E}_k) \ge \sum_k Y(E_k) \tag{2.8}$$

Indeed, $Y(2\sqrt{2})=0$, $Y(E)$ is even, negative, and decreasing in $E>2\sqrt{2}$. Notice that E_k are poles of $m(\lambda)$, $\widetilde{E}_k$ are zeroes of $m((\lambda)$ (and poles of $M(\lambda)$). From (2.3) and integral representation of $m(\lambda)$ we have the interlacing property

$$E_1^- < \widetilde{E}_1^- < E_2^- < \ldots < -2\sqrt{2}$$

$$2\sqrt{2} < \ldots < E_2^+ < \widetilde{E}_1^+ < E_1^+$$

So, (2.8) holds and

(2.9)

$$\frac{1}{\pi}\int\limits_{-2\sqrt{2}}^{2\sqrt{2}}\sqrt{8-\lambda^2}\ln[\pi\sigma_O'(\lambda)]d\lambda \ge \frac{1}{\pi}\int\limits_{-2\sqrt{2}}^{2\sqrt{2}}\sqrt{8-\lambda^2}\ln\left(\pi\frac{\sigma_{O_1}'(\lambda)+\sigma_{O_2}'(\lambda)}{2}\right)d\lambda - V^2(O)$$

We have

$$\frac{1}{\pi}\int\limits_{-2\sqrt{2}}^{2\sqrt{2}}\sqrt{8-\lambda^2}\ln[\pi\sigma_O'(\lambda)]d\lambda \ge$$

$$\ge \frac{1}{2}\Big[\frac{1}{\pi}\int\limits_{-2\sqrt{2}}^{2\sqrt{2}}\sqrt{8-\lambda^2}\ln[\pi\sigma_{O_1}'(\lambda)]d\lambda + \frac{1}{\pi}\int\limits_{-2\sqrt{2}}^{2\sqrt{2}}\sqrt{8-\lambda^2}\ln[\pi\sigma_{O_2}'(\lambda)]d\lambda\Big] - V^2(O)$$

Iterate this estimate till we leave the support of V. Then, using the precise expression for $m_0(\lambda)$ and $\sigma_0(\lambda)=\pi^{-1}\Im m_0(\lambda)$, one has

$$\frac{1}{\pi}\int\limits_{-2\sqrt{2}}^{2\sqrt{2}}\sqrt{8-\lambda^2}\ln\left(\frac{4\sigma'(\lambda)}{\sqrt{8-\lambda^2}}\right)d\lambda \ge -\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{|j-O|=n}V^2(j)$$

It is instructive to introduce the notion of relative entropy [9, 20]

$$S(\mu|\nu) = -\int \ln\left(\frac{d\mu}{d\nu}\right) d\mu$$

and write

$$S(\sigma_0|\sigma) \geq -\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{|j-O|=n} V^2(j) \tag{2.10}$$

The bound (2.10) can be extended to all potentials satisfying (2.1). This is a standard argument that uses semicontinuity of the entropy and the fact that $d\sigma_O^{(k)}$ converges weakly to $d\sigma_O$ as $k \to \infty$. Here $d\sigma_O^{(k)}$ corresponds to the potential truncated to the ball in $\mathbb{B}_0$ of radius k centered at O. The estimate (2.10) implies that $\sigma_{ac}(H_{|\mathbb{B}_0}) = [-2\sqrt{2}, 2\sqrt{2}]$.

$\square$

In (2.9), we could have used

$$\frac{1}{\pi}\int\limits_{-2\sqrt{2}}^{2\sqrt{2}} \sqrt{8-\lambda^2}\ln[\pi\sigma'_O(\lambda)]d\lambda \geq \frac{1}{\pi}\int\limits_{-2\sqrt{2}}^{2\sqrt{2}} \sqrt{8-\lambda^2}\ln\left(\pi\frac{\sigma'_{O_1}(\lambda)}{2}\right)d\lambda - V^2(O)$$

This inequality has a perfect physical meaning. If the branches $\mathbb{B}_1$ and $\mathbb{B}_2$ are not connected and $H_{|\mathbb{B}_1}$ is such that a particle can scatter from O_1, then it can scatter from O also, regardless of what happens at the other branch $\mathbb{B}_2$. This is a general fact, which can be adjusted to any tree and to different modifications of the corresponding Laplacian. In particular, it is true that under the conditions of theorem 2.1, the a.c. component of the spectral measure corresponding to any vertex in $\mathbb{B}$ is supported by $[-2\sqrt{2}, 2\sqrt{2}]$, i.e. the scattering is possible from any vertex of the lattice.

For Caley trees, the right quantity to study is an entropy $S(\sigma_0|\sigma)$ and an accurate bound is (2.10). The method seems to be adjustable to the decorated trees, for which there are many paths connecting two points. Using the rank-one perturbation technique, one can calculate how the "entropy at the site" changes upon making the potential 0 at that site. This simple calculation reveals some analytic structure. It certainly says nothing about the global scattering picture, for this argument does not feel the lattice itself. We do not discuss these issues here. The proof of theorem 2.1 is rather easy, but it is very different from what was used before to deal with multidimensional scattering problems. We believe it might be of some interest.

The theorem 2.1 is optimal in a sense. One can easily find [10] the spherically symmetric potential $V(x) = V(|x|) \in \ell^p(\mathbb{Z}^+), p > 2$, for which the spectral measure corresponding to the origin O has no a.c. component. That can be done using the reduction to the one-dimensional discrete Schrödinger operator.

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E-mail address: `denissov@math.ias.edu`

School of Mathematics, Institute for Advanced Study, 1 Einstein Drive, Princeton, NJ, 08540, USA.