Generating spectral gaps by geometry

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Abstract. Motivated by the analysis of Schrödinger operators with periodic potentials we consider the following abstract situation: Let Δ_X be the Laplacian on a non-compact Riemannian covering manifold X with a discrete isometric group Γ acting on it such that the quotient X/Γ is a compact manifold. We prove the existence of a finite number of spectral gaps for the operator Δ_X associated with a suitable class of manifolds X with *non-abelian* covering transformation groups Γ. This result is based on the non-abelian Floquet theory as well as the Min-Max-principle. Groups of type I specify a class of examples satisfying the assumptions of the main theorem. (15th July 2005)

1. Introduction

It is a well known fact that a Schrödinger operator $-\Delta + V$ on $\mathbb{R}^d$, $d \geq 2$, with a suitable periodic potential V has gaps in its spectrum. This is a quite natural situation in solid state physics, where — for example in insulators — the particles described by the Schrödinger operator have some unreachable energy regions (gaps). This behaviour is ensured by the following two crucial properties: first, the fact that V is periodic. This means that there is a basis $\{\varepsilon_i\}_{i=1}^d$ of $\mathbb{R}^d$ such that the potential satisfies

$$V(x + \varepsilon_i) = V(x)\,, \quad i = 1, \dots, d\,.$$

In other words, the periodicity of V introduces an action of the discrete abelian group $\mathbb{Z}^d$ on $\mathbb{R}^d$ and the potential is completely specified on a fundamental domain $D \subset X$. A typical example for a fundamental domain is the parallelepiped $D = (0,1)\varepsilon_1 + \cdots + (0,1)\varepsilon_d$. Second, the potential V has a high barrier near the boundary of D. In this way, the potential V essentially decouples the fundamental domain D from the neighbouring domains $\varepsilon_i + D$, $i = 1, \dots, d$ (see [**HeP03**] for an overview).

A natural question in this context is whether one can replace the effect of the periodic potentials on the spectrum of the Laplace operator by using geometry. Specifically, can we replace $\mathbb{R}^d$ with some Riemannian manifold X with a suitable discrete group action on it, such that the corresponding Laplace operator Δ_X also has gaps in its spectrum (which is purely essential spectrum)? In other words, has $\operatorname{spec} \Delta_X$ more than one component as a subset of $[0, \infty)$? A positive answer to this question was given in the context of abelian groups in [**P03**] (see also the references cited therein). The intuitive idea is that the junctions of the fundamental domains that build up X are small enough (see Figure 1 below). This has a similar effect on the energy of the particles as the high barriers of the potential in the case of the Schrödinger operator on $\mathbb{R}^d$. Note that the case $d = 1$ is uninteresting in our context since every Laplacian on a one-dimensional (non-compact) Riemannian manifold is unitary equivalent to the standard one, which has no gaps (cf. [**DaH87**]). Moreover, for one-dimensional Schrödinger operators any non-constant potential produces gaps [**RS78**, Theorem XIII.91].

We will show in this paper that this simple idea of scaling down the junctions also works for many non-abelian discrete groups Γ. The analysis in the non-abelian case is more involved because the structure of the dual object $\widehat{\Gamma}$ (i.e., the set of equivalence classes of unitary irreducible representations of Γ) is less transparent from an algebraic and measure theoretic point of view. The purpose of the present paper is to stress the fundamental ideas that allow to extend the previous result to non-abelian group actions. Technical details and further developments will be published in [**LP05**]. We will not study the nature of the spectrum outside the gaps. Some papers related to the problem of the band-gap structure of elliptic operators on covering manifolds are e.g. [**BSu92**] or [**Gr01**]. In contrast with the Schrödinger operator case, the present geometric setting allows the freedom to choose the dimension d of the manifold and the number of "period directions" r (i.e., number of generators of Γ) *independently* from each other. This observation can probably be useful in further investigations on spectral properties common for periodic Schrödinger operators and Laplacians on manifolds.

The paper is organised as follows: In the following section we set up the problem, present the geometrical context and state some results that will be needed later. In Section 3 we introduce in detail the non-abelian Floquet theory, which is at the basis of our analysis. We will also illustrate the general formulas in the special case $\Gamma = \mathbb{Z}^d$ and $X = \mathbb{R}^d$. In the next section we prove our main result: the existence of spectral gaps in the spectrum of Δ_X for suitable manifolds X. In Section 5 we specify a family of discrete groups, so-called groups of type I, that satisfy the assumptions of our theorem. For convenience of the reader we have included in an appendix a short review of the main results concerning direct integral decompositions of unitary group representations. This technique is crucial for the Floquet theory.

2. Notation and background

2.1. Periodic manifolds and Laplacians. We begin fixing our notation and recalling some results that will be useful later on. We denote by X a *non-compact* Riemannian manifold of dimension $d \geq 2$. We also assume the action on X of a finitely generated, discrete group Γ of isometries of X such that the quotient $M := X/\Gamma$ is a *compact* Riemannian manifold which also has dimension d. In other words, X is a *periodic manifold* or *Riemannian covering space* of M with covering transformation group Γ. Moreover, we fix a *fundamental domain*

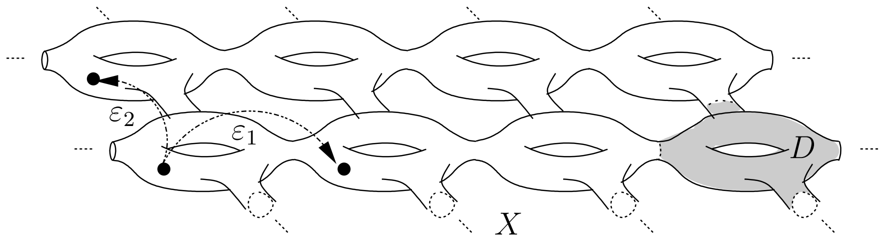

FIGURE 1. A periodic (or covering) manifold X with group Γ generated by two elements $\varepsilon_1, \varepsilon_2$ and fundamental domain D (in grey). Here, the group Γ is abelian.

D, i.e., an open set $D \subset X$ such that γD and $\gamma' D$ are disjoint for all $\gamma \neq \gamma'$ and $\bigcup_{\gamma\in\Gamma} \gamma \overline{D} = X$ (cf. Figure 1).

As a prototype for an elliptic operator we consider the Laplacian Δ_X on X acting on a dense subspace of the Hilbert space $L_2(X)$ with norm $\|\cdot\|_X$. The positive self-adjoint operator Δ_X can be defined in terms of a suitable quadratic form q_X (see e.g. [**K95**, Chapter VI], [**RS80**] or [**Da96**]). Concretely we have

$$q_X(u) := \|du\|_X^2 = \int_X |du|^2 dX\ , \quad u \in C_c^\infty(X)\ . \tag{2.1}$$

In coordinates we write the pointwise norm of the 1-form du as

$$|du|^2 = \sum_{i,j} g^{ij} \partial_i u\, \partial_j \overline{u}\,,$$

where (g^{ij}) is the inverse of the metric tensor (g_{ij}) in a chart. Taking the closure of the quadratic form we can extend q_X onto the Sobolev space

$$\mathcal{H}^1(X) = \{\, u \in L_2(X) \,|\, q_X(u) < \infty \,\}\,.$$

As usual the operator Δ_X is related with the quadratic form by the formula $\langle \Delta_X u, u\rangle = q_X(u)$, $u \in C_c^\infty(X)$. Since the metric on X is Γ-invariant, the Laplacian Δ_X (i.e., its resolvent) commutes with the translations on X given by

$$(T_\gamma u)(x) := u(\gamma^{-1}x), \quad u \in L_2(X), \gamma \in \Gamma. \tag{2.2}$$

Operators with this property are called *periodic.*

For an open, relatively compact subset $D \subset X$ with sufficiently smooth boundary ∂D (e.g. Lipschitz) we define the Dirichlet (respectively, Neumann) Laplacian Δ_D^+ (resp., Δ_D^-) via its quadratic form q_D^+ (resp., q_D^-) associated to the closure of q_D on $C_c^\infty(D)$, the space of smooth functions with compact support, (resp. $C^\infty(\overline{D})$, the space of smooth functions with continuous derivatives up to the boundary). We also use the notation $\mathring{\mathcal{H}}^1(D) = \operatorname{dom} q_D^+$ (resp., $\mathcal{H}^1(D) = \operatorname{dom} q_D^-$). Note that the usual boundary condition of the Neumann Laplacian occurs only in the *operator* domain via the Gauß-Green formula. Since $\overline{D}$ is compact, Δ_D^+ has purely discrete spectrum λ_k^+, $k \in \mathbb{N}$. It is written in ascending order and repeated according to multiplicity. The same is true for the Neumann Laplacian and we denote the corresponding purely discrete spectrum by λ_k^-, $k \in \mathbb{N}$.

One of the advantages of the quadratic form approach is that one can easily read off from the inclusion of domains an order relation for the eigenvalues. In fact, by the the *min-max principle* we have

$$\lambda_k^\pm = \inf_{L_k} \sup_{u \in L_k \setminus \{0\}} \frac{q_D^\pm(u)}{\|u\|^2}\,, \tag{2.3}$$

where the infimum is taken over all k-dimensional subspaces L_k of the corresponding *quadratic* form domain $\operatorname{dom} q_D^\pm$. Then the inclusion

$$\operatorname{dom} q_D^+ = \mathring{\mathcal{H}}^1(D) \subset \mathcal{H}^1(D) = \operatorname{dom} q_D^- \tag{2.4}$$

implies $\lambda_k^+ \geq \lambda_k^-$, i.e., the Dirichlet eigenvalue is in general larger than the Neumann eigenvalue and this justifies the choice of the labels $+$, respectively, $-$.

2.2. Spectral gaps in the abelian case. Due to the previous inequality relating the Dirichlet and Neumann eigenvalues for any $k \in \mathbb{N}$, we may introduce the following intervals

$$I_k := [\lambda_k^-, \lambda_k^+]\,, \qquad k \in \mathbb{N}\,. \tag{2.5}$$

Here, $\lambda_k^\pm$ denotes the k-th Dirichlet/Neumann eigenvalue on a fundamental domain D.

In this context the existence of spectral gaps of Δ_X can be reduced to the question whether $I_k \cap I_{k+1} = \emptyset$ for some k. A class of manifolds with abelian group actions satisfying the previous intersection condition is specified in [**P03**]:

THEOREM 2.1. *Given a finitely generated abelian group Γ we can always find a covering space $X \to X/\Gamma =: M$ and a fundamental domain D with the following property: For every $K \in \mathbb{N}$ there exists a metric $g = g_K$ on M such that*

$$I_k \cap I_{k+1} = \emptyset \tag{2.6}$$

for at least K indices $k \in \mathbb{N}$, where I_k is defined as in (2.5) *for the manifold D with metric $g = g_K$. In particular, the Laplacian Δ_X corresponding to the lifted metric on X has at least K gaps.*

It is important to note that the construction of the covering space X and the metric g only depends on the quotient M, not on the cover X. It is therefore independent whether Γ is abelian or not (see the sketch of the proof of Theorem 4.2). Roughly speaking, we have replaced the *high* potential barrier in the case of Schrödinger operators on $\mathbb{R}^d$ by *small* junctions between the fundamental domains. We can say that now geometry is partly decoupling one fundamental domain from its neighbours.

3. Non-abelian Floquet theory

In this section we will introduce in several steps the Floquet theory for non-abelian groups. The main idea is to use the group action on X and a partial Fourier transformation to decompose the Hilbert space $L_2(X)$ and the periodic operators on it into a direct integral of simpler components that can be analysed more easily. For convenience of the reader we have summarized in the appendix the main results on the direct integral decompositions of unitary group representations.

3.1. Non-abelian Fourier transformation. Consider first the right, respectively, left regular representation R, resp., L on the Hilbert space $\ell_2(\Gamma)$:

$$(R_\gamma a)_{\widetilde{\gamma}} = a_{\widetilde{\gamma}\gamma}, \qquad (L_\gamma a)_{\widetilde{\gamma}} = a_{\gamma^{-1}\widetilde{\gamma}}, \qquad a = (a_\gamma)_\gamma \in \ell_2(\Gamma), \quad \gamma, \widetilde{\gamma} \in \Gamma. \tag{3.1}$$

Let $\mathcal{R}$ be the von Neumann algebra generated by all unitaries R_γ, $\gamma \in \Gamma$, i.e.,

$$\mathcal{R} = \{\, R_\gamma \mid \gamma \in \Gamma \,\}'',$$

and denote by $\mathcal{R}'$ the commutant of $\mathcal{R}$ in $\mathcal{B}(\ell_2(\Gamma))$; similarly, we define $\mathcal{L} = \{\, L_\gamma \mid \gamma \in \Gamma \,\}''$.

In this context, we may generalise the Fourier transformation to the unitary map

$$F \colon \ell_2(\Gamma) \longrightarrow \int_Z^\oplus H(z)\, dz \tag{3.2}$$

that transforms the right regular representation R into the following direct integral representation

$$\widehat{R}_\gamma = F R_\gamma F^{-1} = \int_Z^\oplus R_\gamma(z)\, dz, \qquad \gamma \in \Gamma. \tag{3.3}$$

By a suitable choice of the measure space (Z, dz) corresponding to an maximal abelian algebra $\mathcal{A}$ in $\mathcal{R}'$ (see the appendix) we can assume that the unitary representations $R_\gamma(z)$ are irreducible on the Hilbert space $H(z)$ a.e. In addition, operators commuting with all L_γ ($\gamma \in \Gamma$), i.e., operators in $\mathcal{L}'$, are decomposable, since one can show that $\mathcal{L}' = \mathcal{R}$ and therefore $\mathcal{L}' \subset \mathcal{A}'$ (cf. the appendix).

3.2. Equivariant Laplacians. We will introduce next a new operator that lies "between" the Dirichlet and Neumann Laplacians and that will play an important role in the following section. Consider on almost each fibre smooth $R(z)$-*equivariant functions*, i.e., smooth functions $h \colon X \longrightarrow H(z)$ satisfying

$$h(\gamma x) = R_\gamma(z) h(x), \qquad \gamma \in \Gamma, x \in X. \tag{3.4}$$

We denote the corresponding space of smooth $R(z)$-equivariant functions *restricted to a fundamental domain* D by $C^\infty_{\mathrm{eq}}(D, H(z))$. Note that we need *vector-valued* functions $h \colon X \longrightarrow H(z)$ since the representation $R(z)$ acts on the Hilbert space $H(z)$. If Γ is non-abelian, then some of its unitary irreducible representations must be of dimension greater than one.

We introduce next the so-called *equivariant Laplacian* (w.r.t. the representation $R(z)$) on $L_2(D, H(z)) \cong L_2(D) \otimes H(z)$. Consider the quadratic form given by

$$\|dh\|_D^2 := \int_D \|dh(x)\|_{H(z)}^2 dX \tag{3.5}$$

for $h \in C_{\mathrm{eq}}^\infty(D, H(z))$, where the integrand is locally specified by

$$\|dh(x)\|_{H(z)}^2 = \sum_{i,j} g^{ij}(x)\, \langle \partial_i h(x), \partial_j h(x)\rangle_{H(z)}, \qquad x \in D.$$

This generalises Eq. (2.1) to the case of vector-valued functions. We denote the closure by q_D^{eq} and its domain by $\mathcal{H}^1_{\mathrm{eq}}(D, H(z))$. The corresponding non-negative operator on $L_2(D, H(z))$, the so-called $R(z)$-*equivariant Laplacian*, will be denoted by $\Delta_D(z)$.

3.3. Non-abelian Floquet transformation. Next, we analyse the Floquet transformation

$$U \colon L_2(X) \longrightarrow \int_Z^\oplus L_2(D, H(z))\, dz,$$

which is the composition of the following three unitary transformations (denoted with horizontal arrows)

$$\begin{array}{ccccccc}
\overset{T_\gamma}{\curvearrowright} & & \overset{L_\gamma \otimes \mathbb{1}}{\curvearrowright} & & \overset{\widehat{L}_\gamma \otimes \mathbb{1}}{\curvearrowright} & & \\
L_2(X) & \to & \ell_2(\Gamma) \otimes L_2(D) & \overset{F\otimes \mathrm{id}}{\to} & \int_Z^\oplus H(z)\, dz \otimes L_2(D) & \to & \int_Z^\oplus L_2(D, H(z))\, dz \\
u & \mapsto & \sum_\gamma \delta_\gamma \otimes (T_{\gamma^{-1}} u \restriction_D) & & b \otimes f & \mapsto & (b(\rho) f)_\rho \,,
\end{array}$$

where $(\delta_\gamma)_\gamma$ is the canonical orthogonal basis of $\ell_2(\Gamma)$ and T_γ is the translation by γ of functions on X given by (2.2). Each of these transformations intertwines with the unitary representation of Γ which are denoted with curved arrows in the previous diagram. The first horizontal unitary transformation just splits a function on $L_2(X)$ into a sequence of γ-translates over the fundamental domain D. The second horizontal unitary is essentially the Fourier transformation on the group part and the last horizontal unitary is clear, since $L_2(D)$ is independent of $z \in Z$.

Note that periodic operators on $L_2(X)$, i.e, operators commuting with T_γ, are those commuting with $L_\gamma \otimes \mathbb{1}$ on $\ell_2(\Gamma) \otimes L_2(D)$. Therefore, periodic operators are also decomposable (recall that $\mathcal{L}' = \mathcal{R} \subset \mathcal{A}'$). Note in addition that if Γ is not abelian then $\widehat{L}_\gamma = F L_\gamma F^{-1}$ does not decompose with respect to the direct integral specified in Section 3.1.

The Floquet transformation U is given explicitely in the following theorem (see also [**Su88**] and [**RS78**, Section XIII.16]):

THEOREM 3.1. *The map*

$$(Uu)(z)(x) = \sum_{\gamma \in \Gamma} u(\gamma x)\, R_{\gamma^{-1}}(z) v(z), \quad \text{where } v := F\delta_e, \tag{3.6}$$

is a unitary transformation that intertwines the representations T *and* $\int_Z^{\oplus} R(z)\, dz$. *In addition,* U *maps* $C_c^\infty(X)$ *into* $\int_Z^{\oplus} C_{eq}^\infty(D, H(z))\, dz$ *and operators on* $L_2(X)$ *commuting with all* T_γ*'s are decomposable when transformed onto the direct integral. In particular, the Laplacian* Δ_X *is unitary equivalent to* $\int_Z^{\oplus} \Delta_D(z)\, dz$ *and*

$$\operatorname{spec} \Delta_X \subseteq \overline{\bigcup_{z \in Z} \operatorname{spec} \Delta_D(z)}. \tag{3.7}$$

3.4. The special case $\Gamma = \mathbb{Z}^d$ and $X = \mathbb{R}^d$. In this case the dual is simply given by the d-dimensional torus, i.e., $\widehat{\Gamma} = \mathbb{T}^d$. Therefore, we can choose $Z = \mathbb{T}^d$ with Lebesgue measure $d\theta$. The Fourier transformation (3.2) reduces to the standard formula

$$F \colon \ell_2(\mathbb{Z}^d) \longrightarrow L_2(\mathbb{T}^d) = \int_{\mathbb{T}^d}^{\oplus} H(\theta)\, d\theta\,, \quad \text{with } (Fa)(\theta) = \sum_\gamma \mathrm{e}^{-\mathrm{i}\theta\cdot\gamma} a_\gamma$$

where $H(\theta) = \mathbb{C}$ for a.e. $\theta \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $a = (a_\gamma) \in \ell_2(\mathbb{Z}^d)$. Here, we only need scalar functions since every irreducible representation of an *abelian* group is one-dimensional. In this case, we can decompose *both* the left *and* the right regular representation simultaneously; each fibre of R, resp., L is the multiplication with a phase factor

$$\widehat{R}_\gamma(\theta) = \mathrm{e}^{\mathrm{i}\theta\cdot\gamma}, \quad \text{resp.,} \quad \widehat{L}_\gamma(\theta) = \mathrm{e}^{-\mathrm{i}\theta\cdot\gamma}$$

The equivariant condition becomes

$$h(x+\gamma) = \mathrm{e}^{-\mathrm{i}\theta\cdot\gamma} h(x) \tag{3.8}$$

for all $x \in \mathbb{R}^d$ and $\gamma \in \mathbb{Z}^d$. A fundamental domain is the cube $D := (0,1)^d$ and the Floquet transformation is given by

$$(Uu)(\theta)(x) = \sum_{\gamma \in \mathbb{Z}^d} u(x+\gamma)\mathrm{e}^{\mathrm{i}\theta\cdot\gamma}, \qquad x \in D, \theta \in \mathbb{T}^d.$$

4. Existence of spectral gaps for non-abelian group actions

We will present in this section a method to show the existence of finitely many spectral gaps for the Laplace operator Δ_X in the case of *non-abelian* group actions. Our main assumption on the group is the fact that the irreducible representation appearing in the decomposition (3.2) are finite-dimensional a.e., in other words

$$\dim H(z) < \infty \qquad \text{for a.e. } z \in Z. \tag{4.1}$$

The operators $\Delta_D^+(z)$, $\Delta_D(z)$, resp., $\Delta_D^-(z)$ corresponding to the quadratic form (3.5) on $\mathring{\mathcal{H}}^1(D, H(z))$, $\mathcal{H}^1_{\mathrm{eq}}(D, H(z))$, resp., $\mathcal{H}^1(D, H(z))$ have purely discrete spectrum which we denote by $\lambda_m^+(z)$, $\lambda_m(z)$, resp., $\lambda_m^-(z)$, $m \in \mathbb{N}$. Recall that the space $\mathring{\mathcal{H}}^1(D, H(z))$ is the $\mathcal{H}^1$-closure of the space of smooth functions $h \colon D \longrightarrow H(z)$ with compact support and $\mathcal{H}^1(D, H(z))$ is the closure of the space of smooth functions with derivatives continuous up to the boundary (cf. Section 2).

As in (2.4) we obtain from the inclusion of the three domains

$$\mathcal{H}^1(D, H(z)) \supset \mathcal{H}^1_{\mathrm{eq}}(D, H(z)) \supset \mathring{\mathcal{H}}^1(D, H(z))$$

that the corresponding eigenvalues satisfy the following reverse inequalities

$$\lambda_m^-(z) \le \lambda_m(z) \le \lambda_m^+(z)$$

for all $m \in \mathbb{N}$ and a.e. $z \in Z$.

From the definition of the quadratic form in the Dirichlet, resp., Neumann case we have that the corresponding vector-valued Laplacians are a direct sum of the scalar operators since there is no coupling between the components on the boundary. In particular, if $n = \dim H(z)$, then $\Delta_D^\pm(z)$ is a n-fold direct sum of the scalar operators $\Delta_D^\pm$ on $L_2(D)$. Therefore the eigenvalues of the corresponding vector-valued Laplace operators consist of n-times repeated eigenvalues of the scalar Laplacians. We can therefore arrange the former in the following way:

$$\lambda_m^\pm(z) = \lambda_k^\pm, \qquad m = (k-1)n + 1, \dots, kn\,,$$

where $\lambda_k^\pm$ denotes the (scalar) k-th Dirichlet/Neumann eigenvalue on D.

Recall the definition of the intervals $I_k := [\lambda_k^-, \lambda_k^+]$ in Eq. (2.5). We may now collect the n eigenvalues of $\Delta_D(z)$ which lie in I_k:

$$B_k(z) := \{\, \lambda_m(z) \,|\, m = (k-1)n + 1, \dots, kn \,\} \subset I_k, \qquad n := \dim H(z)\,. \tag{4.2}$$

Moreover we put together all eigenvalues corresponding to operators over the base point $z \in Z$ that act on Hilbert spaces with the same dimension:

$$B_k(n) := \bigcup_{z \in Z,\, \dim H(z) = n} B_k(z) \subset I_k. \tag{4.3}$$

THEOREM 4.1. *Let Γ be a finitely generated (in general non-abelian) group satisfying* $\dim H(z) < \infty$ *for a.e. $z \in Z$ in the decomposition* (3.2). *Then*

$$\operatorname{spec} \Delta_X \subseteq \bigcup_{k \in \mathbb{N}} I_k.$$

In particular, the spectrum of Δ_X has a gap between I_k and I_{k+1} provided

$$I_k \cap I_{k+1} = \emptyset\,.$$

PROOF. The proof is a consequence of the following chain of inclusions

$$\operatorname{spec} \Delta_X \subseteq \overline{\bigcup_{z \in Z} \operatorname{spec} \Delta_D(z)} = \overline{\bigcup_{z \in Z} \bigcup_{k \in \mathbb{N}} B_k(z)} = \overline{\bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} B_k(n)} \subseteq \bigcup_{k \in \mathbb{N}} I_k\,. \tag{4.4}$$

For the first inclusion, we have applied (3.7). For the second equality we have used $\dim H(z) < \infty$. Note finally that $\bigcup_k I_k$ is closed since $\lambda_k^\pm \to \infty$ for $k \to \infty$ implies that $(I_k)_k$ is a locally finite family of closed sets. $\square$

We formulate our main result, i.e., an analogue of Theorem 2.1 in the case of non-abelian groups Γ satisfying the conditions stated in Section 2:

Theorem 4.2. *Let Γ be a finitely generated (in general non-abelian) group satisfying $\dim H(z) < \infty$ for a.e. $z \in Z$ in the decomposition (3.2). Then we can always find a covering space $X \to X/\Gamma$ and a fundamental domain D with the following property: For each $K \in \mathbb{N}$ there exist a Riemannian covering space $X \to X/\Gamma =: M$ with metric $g = g_K$ such that the Laplacian on X has at least K gaps, i.e., $\operatorname{spec} \Delta_X \subset [0, \infty)$ has at least K components.*

Sketch of the proof: Suppose that Γ has r generators ε_i. To construct X we begin with a d-dimensional compact manifold M with at least r handles which we may take diffeomorphic to $(0, 1) \times \mathbb{S}^{d-1}$. Then $D \subset M$ is the open subset obtained from M by removing a section $\{1/2\} \times \mathbb{S}^{d-1}$ from each of the r handles. The set D has therefore $2r$ cylindrical ends, for each generator a "left" and a "right" one. Then there exists a covering $X \to X/\Gamma \cong M$ with fundamental domain D: Intuitively, one can build up X by glueing Γ copies $(\gamma D)_\gamma$ of D, where one has to identify properly the points on the boundary (cf. Fig. 1 in the case $r = 2$). Concretely, we identify the "left" boundary part of the i-th cylindrical end of $\gamma_1 D$ with the "right" one of $\gamma_2 D$ iff $\gamma_2 = \varepsilon_i \gamma_1$. Finally, we change the metric on the handles in order to scale down the junctions between neighbouring copies of the fundamental domain (cf. [**P03**]). Note that the metric depends on the minimal number K of gaps. This implies that Eq. (2.6) is satisfied for at least K indices $k \in \mathbb{N}$ and the proof is concluded by Theorem 4.1.

Note that we have spectrum in each interval I_k, i.e., $\operatorname{spec} \Delta_X \cap I_k \neq \emptyset$ for all k, since a group satisfying (4.1) is amenable (cf. the next section); therefore, $\operatorname{spec} \Delta_M \subset \operatorname{spec} \Delta_X$ (see e.g. [**Su88**, Prop. 7–8]). Finally, $\operatorname{spec} \Delta_M \cap I_k \neq \emptyset$. $\square$

Note that the previous statement does not give information on the *maximal* number of spectral gaps. It still remains an open question if there are (connected) covering spaces X with an *infinite* number of spectral gaps. This problem is related to the so-called *Bethe-Sommerfeld conjecture* (cf. [**Sk87**]).

5. Examples

We begin defining a class of a discrete groups that have particularly simple properties (cf. [**Th64**]). A discrete group Γ is of type I iff there is an exact sequence

$$0 \longrightarrow A \longrightarrow \Gamma \longrightarrow \Gamma_0 \longrightarrow 0,$$

where A is a finitely generated *abelian* normal subgroup of Γ and $\Gamma_0 = \Gamma/A$ is a *finite* group (cf. [**Th64**]). Simple examples of groups of type I are abelian groups (in this case Γ_0 is trivial) or direct, resp., semi-direct products of an abelian group with a finite (in general non-abelian) group.

For these type of groups we have that all irreducible representations are finite dimensional. In addition, such groups are also amenable (as extensions of amenable groups), cf. [**Br81**]. Note that the converse is also true: A discrete group such that all irreducible representations are finite dimensional is of type I (cf. [**Mo72**]).

Recall that this was an important assumption in Section 4. Moreover in the direct integral decomposition of Subsection 3.3 we may take as measure space Z the dual $\widehat{\Gamma}$ of Γ. For these reasons we have

Proposition 5.1. *Suppose Γ is a finitely generated group of type I. Then for each $K \in \mathbb{N}$ there exist a Riemannian covering space $X \to X/\Gamma$ with metric $g = g_K$ such that the Laplacian on X has at least K gaps, i.e.,* $\operatorname{spec} \Delta_X \subset [0, \infty)$ *has at least K components.*

Of course, not all groups are of type I. For example, free groups with more than one generator are not of type I. In [**LP05**] we provide different methods and further classes of groups including the free groups for which the conclusions of the above proposition remain true.

Appendix: Direct integral decomposition of unitary group representations

In the present appendix we will describe in more detail the direct integral decomposition of the right regular representation given in Eqs. (3.1) and (3.2) of Section 3. It is an application of the direct integral decomposition of von Neumann algebras. General references are e.g. [**W92**, Chapter 14] or [**M76**, Chapter 2].

In this appendix we will consider a more general frame that includes the particular situation considered in Eqs. (3.2) and (3.3), where Γ is a discrete group satisfying the conditions of Section 2 and R is the right regular representation on $\ell_2(\Gamma)$.

Let G be a separable locally compact group and let V be a continuous unitary representation of G on a Hilbert space $\mathcal{H}$. Denote by

$$\mathcal{M} := \{V_g \mid g \in G\}''$$

the von Neumann algebra generated by the representation V and let

$$\mathcal{M}' = (V, V) := \{M' \in \mathcal{B}(\mathcal{H}) \mid M' V_g = V_g M' \ , \ g \in G\}$$

be the von Neumann algebra of operators commuting with the representation V. If $\mathcal{A}$ is an abelian von Neumann subalgebra of $\mathcal{M}'$, then there exists a compact, separable Hausdorff space Z, a regular Borel measure dz on Z and a unitary transformation onto a direct integral Hilbert space

$$F \colon \mathcal{H} \longrightarrow \int_Z^{\oplus} \mathcal{H}(z)\, dz, \tag{A.1}$$

such that

$$F\mathcal{A}F^{-1} = \{\, M_f \mid f \in L_\infty(Z, dz) \,\}$$

(M_f being the multiplication operator with f). The von Neumann algebra $F\mathcal{A}'F^{-1}$ consists of all decomposable operators w.r.t. the direct integral (A.1), i.e., if $D \in F\mathcal{A}'F^{-1}$ then we can write

$$D = \int_Z^{\oplus} D(z)\, dz.$$

In particular, $V_g \in \mathcal{M} \subset \mathcal{A}'$, and therefore

$$FV_gF^{-1} = \int_Z^{\oplus} V_g(z)\, dz,$$

where $V(z)$ is a unitary representation of G on $H(z)$ a.e. (see [**W92**, Section 14.8 ff.]). There are several natural choices for the abelian von Neumann algebra $\mathcal{A}$:

(i) If $\mathcal{A} = \mathcal{M} \cap \mathcal{M}'$ is the centre of $\mathcal{M}$, then, for a.e. $z \in Z$, the von Neumann algebra generated by the representations $V(z)$ are factors, i.e.,

$$\mathcal{M}(z) \cap \mathcal{M}(z)' := \{V_g(z) \mid g \in G\}'' \cap \{V_g(z) \mid g \in G\}' = \mathbb{C}\mathbb{1}_{\mathcal{H}(z)}.$$

This choice is due to von Neumann.

(ii) If $\mathcal{A}$ is maximal abelian in $\mathcal{M}'$, i.e., $\mathcal{A} = \mathcal{A}' \cap \mathcal{M}'$, then the components $V(z)$ of the direct integral decomposition of V are irreducible a.e. This choice is due to Mautner and was used in Eqs. (3.2) and (3.3) of Section 3.

Finally, we mention a class of groups, where the previous decomposition results become particularly simple. A group G is of type I if all its unitary continuous representations V are of type I, i.e., each V is quasi-equivalent to some multiplicity free representation. Compact or abelian groups are examples of type I groups. If G is of type I, then the dual $\widehat{G}$ (i.e., the set of all equivalence classes of continuous unitary irreducible representations of G) becomes a nice measure space ("smooth" in the terminology of [**M76**, Chapter 2]). In this case one can take $\widehat{G}$ as the measure space Z in the Mautner decomposition (ii). For discrete groups the previous definition of Type I is equivalent to the one given in Section 5 (cf. [**Th64**]).

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References

[Br81] R. Brooks, *The fundamental group and the spectrum of the Laplacian*, Comment. Math. Helv. **56** (1981), 581–598.

[BSu92] J. Brüning and T. Sunada, *On the spectrum of periodic elliptic operators*, Nagoya Math. J. **126** (1992), 159–171.

[DaH87] E. B. Davies and E. M. II Harrell, *Conformally flat Riemannian metrics, Schrödinger operators, and semiclassical approximation*, J. Diff. Equations **66** (1987), 165–188.

[Da96] E. B. Davies, *Spectral theory and differential operators*, Cambridge University Press, Cambridge, 1996.

[Gr01] M. J. Gruber, *Noncommutative Bloch theory*, J. Math. Phys. **42** (2001), no. 6, 2438–2465.

[HeP03] R. Hempel and O. Post, *Spectral gaps for periodic elliptic operators with high contrast: An overview*, Progress in Analysis, Proceedings of the 3rd International ISAAC Congress Berlin 2001, Vol. 1 (2003), 577–587.

[K95] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[LP05] F. Lledó and O. Post, *Existence of spectral gaps, covering manifolds and residually finite groups*, arXiv:math-ph/0503005 (2005).

[M76] G. W. Mackey, *The theory of unitary group representations*, University of Chicago Press, Chicago, 1976.

[Mo72] C. C. Moore, *Groups with finite dimensional irreducible representations*, Trans. Amer. Math. Soc. **166** (1972), 401–410.

[P03] O. Post, *Periodic manifolds with spectral gaps*, J. Diff. Equations **187** (2003), 23–45.

[RS80] M. Reed and B. Simon, *Methods of modern mathematical physics I: Functional analysis*, Academic Press, New York, 1980.

[RS78] M. Reed and B. Simon, *Methods of modern mathematical physics IV: Analysis of operators*, Academic Press, New York, 1978.

[Sk87] M. M. Skriganov, *Geometric and arithmetic methods in the spectral theory of multidimensional periodic operators*, Proc. Steklov Inst. Math. (1987), no. 171, 1–121, (translation).

[Su88] T. Sunada, *Fundamental groups and Laplacians*, Geometry and analysis on manifolds, Lecture Notes Mathematics 1339 (Katata/Kyoto, 1987), Springer, Berlin, 1988, pp. 248–277.

[Th64] E. Thoma, *Über unitäre Darstellungen abzählbarer, diskreter Gruppen*, Math. Ann. **153** (1964), 111–138.

[W92] N. R. Wallach, *Real reductive groups. II*, Pure and Applied Mathematics, vol. 132, Academic Press, Boston, 1992.

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