# DYNAMICAL ZETA FUNCTION FOR SEVERAL STRICTLY CONVEX OBSTACLES

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ABSTRACT. The behavior of the dynamical zeta function $Z_D(s)$ related to several strictly convex disjoint obstacles is similar to that of the inverse $Q(s) = \frac{1}{\zeta(s)}$ of the Riemann zeta function $\zeta(s)$. Let $\Pi(s)$ be the series obtained from $Z_D(s)$ summing only over primitive periodic rays. In this paper we examine the analytic singularities of $Z_D(s)$ and $\Pi(s)$ close to the line $\Re s = s_2$, where $s_2$ is the abscissa of absolute convergence of the series obtained by the second iterations of the primitive periodic rays. We show that at least one of the functions $Z_D(s)$, $\Pi(s)$ has a singularity at $s = s_2$.

## 1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be an open and connected domain with $C^\infty$ boundary $\partial\Omega$ having the form $\Omega = \mathbb{R}^n \setminus K$, where

$$K = \cup_{j=1}^{Q} K_j, \;\; K_i \cap K_j = \emptyset, \text{ for } i \neq j \tag{1.1}$$

and $K_j$ are strictly convex compact obstacles for $j = 1, \ldots, Q$, $Q \geq 3$. Throughout this paper we suppose that $K$ satisfies the following condition introduced by Ikawa ([6]):

$(H)$ The convex hull of every two connected components of $K$ does not have common points with any other connected component of $K$.

Consider the reflecting rays in $\overline{\Omega}$ (see [6] and Chapter 2 in [18] for a precise definition). Under the condition $(H)$ every periodic ray is ordinary reflecting, that is $\gamma$ has no tangent segments. Given a *periodic reflecting ray* $\gamma$ in $\overline{\Omega}$ with $m_\gamma$ reflections, we denote by $T_\gamma$ the primitive period (length) of $\gamma$, by $d_\gamma = lT_\gamma$, $l \in \mathbb{N}$, the period of $\gamma$ and by $P_\gamma$ the linear Poincaré map related to $\gamma$. Setting $|\det(I - P_\gamma)| = |I - P_\gamma|$, it is easy to prove (see [17], Appendix) that there exist constants $b_1 > 0, \;\; b_2 > 0, \;\; B_0 > 0$ so that

$$B_0 e^{2b_1 d_\gamma} \leq |I - P_\gamma| \leq e^{2b_2 d_\gamma}. \tag{1.2}$$

Denote by $\Xi$ the set of all reflecting periodic rays in $\overline{\Omega}$ and set

$$d_0 = \min \;\; \text{dist}_{i \neq j} \;\; (K_i, K_j), \quad D_0 = \max \;\; \text{dist}_{i \neq j}(K_i, K_j).$$

For the counting function of the lengths of periodic rays there exists a constant $a_0 > 0$ such that

$$\sharp\{\gamma \in \Xi : d_\gamma \leq q\} \leq e^{a_0 q} \tag{1.3}$$

(see [6], [21] and Chapter 2 in [18]). In this note we examine the dynamical zeta function

$$Z_D(s) = \sum_{\gamma \in \Xi} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} e^{-sd_\gamma}, \; s \in \mathbb{C}, \tag{1.4}$$

where the summation is over all periodic rays $\gamma \in \Xi$. This zeta function is related to the trace formula for the unitary group associated to the Dirichlet problem for the wave equation

$$\begin{cases} (\partial_t^2 - \Delta_x)u = 0 \text{ in } \mathbb{R} \times \Omega, \\ u = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ u(0,x) = f_1(x), \partial_t u(0,x) = f_2(x). \end{cases} \tag{1.5}$$

The form of $Z_D(s)$ is obtained by the Laplace transformation of the distribution

$$\sum_{\gamma \in \Xi} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} \delta(t - d_\gamma) \tag{1.6}$$

which in turn is the sum of the principal singularities of $u(t) \in \mathcal{D}'(\mathbb{R}^+)$ given by

$$u(t) = \sum_{\lambda_j} e^{it\lambda_j}, \ t > 0.$$

Here $\lambda_j \in \mathbb{C}$ are the poles of the scattering matrix related to the problem (1.5) and the summation is over all poles counted with their multiplicities. We refer to [7], [8], [17], [20] for a more detailed description of this link and to [1], [27], [13], [5], [18], [20] for the trace formulas leading to (1.6).

By using (1.2) and (1.3), it is easy to see that there exists $s_1 \in \mathbb{R}$ called abscissa of absolute convergence such that for $\Re s > s_1$ the series (1.4) is absolutely convergent. Despite many works in physical literature and the numerical analysis results concerning n-disk problems (see for example [3], [28], [12], [27] and the references cited there), to our best knowledge in the general case the problem of the existence of at *least one singularity* of $Z_D(s)$ is still open. The existence of analytic singularity has been proved by Ikawa [9] in the case when $K$ is the union of several balls with radius $r \leq r_0$, provided $r_0 > 0$ sufficiently small. Recently, Stoyanov [24] generalized the result of Ikawa for several obstacles satisfying some geometrical conditions and having diameters less than $r_0$. It was proved in [17] that $Z_D(s)$ has no singularities on the line $\Re s = s_1$. In fact we have a stronger result and following the recent works of Stoyanov (see [22], [25]), we know that there exists $\delta_0 > 0$ such that $Z_D(s)$ is analytic for $\Re s > s_1 - \delta_0$ (see also [10] for the special case $s_1 > 0$.) This means that $Z_D(s)$ is analytic in a domain around $\Re s = s_1$ and this phenomenon of cancellations is typical for dynamical zeta functions (see [4], [15], [22], [23], [25]). On the other hand, since $Z_D(s)$ is a Dirichlet series with real coefficients changing their signs, the situation is very similar to that for the inverse $Q(s) = \frac{1}{\zeta(s)}$ of the classical Riemann zeta function $\zeta(s)$. It is well known that $Q(s)$ is analytic on the line $\Re s = 1$ and $Q(s)$ has non-real singularities on the *critical line* $\Re s = 1/2$. Moreover, we have the representation

$$\log \zeta(s) = \sum_{m=1}^{\infty} \sum_{p \in \mathbf{P}} \frac{1}{m} \frac{1}{p^{ms}}, \ \Re s > 1, \tag{1.7}$$

where $\mathbf{P}$ denotes the set of prime numbers. Consequently, the analytic behavior of $\log \zeta(s)$ for $1/2 < \Re s \leq 1$ is characterized by the continuation of the function

$$\pi(s) = \sum_{p \in \mathbf{P}} \frac{1}{p^s}, \ \Re s > 1$$

and the critical line $\Re s = 1/2$ is related to $m = 2$ in the representation (1.7).

Denote by $\mathcal{P}$ the set of all primitive periodic rays. In this note we examine the analytic singularities of $Z_D(s)$ close to the line $\Re s = s_2$, where $s_2 < s_1$ is the abscissa of the absolute convergence of the series $\Pi_2(s)$ obtained from $Z_D(s)$ when we sum only over the rays $2\gamma$, $\gamma \in \mathcal{P}$, that is over the second iteration of primitive rays (see Section 4 for a precise definition). We show that the line $\Re s = s_2$ plays a role in the investigation of the singularities of $Z_D(s)$. Similarly to $\pi(s)$, introduce the function

$$\Pi(s) = \sum_{\gamma\in\mathcal{P}} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} e^{-sT_\gamma},\ \Re s > s_1,$$

where the summation is over the primitive rays $\gamma \in \mathcal{P}$. Next let $h_\Pi < s_1$ be the abscissa of holomorphy of $\Pi(s)$ given by

$$h_\Pi = \inf\{t \in \mathbb{R} :\ \Pi(s)\ \text{is analytic for}\ \Re s > t\}.$$

Our main result is the following

**Theorem 1.** *At least one of the functions $Z_D(s), \Pi(s)$ has a singularity at $s = s_2$ and the difference $Z_D(s) - \Pi(s)$ is analytic for*

$$s \in \{z \in \mathbb{C} :\ \Re z > s_2\}.$$

*Moreover, if $s_2 \neq h_\Pi$, then $Z_D(s)$ has a singularity at $z$ with $\Re z > \max\{s_2, h_\Pi\} - \epsilon_1$, where $\epsilon_1 > 0$ is sufficiently small.*

Following the same way, we may show that if we consider the series obtained by summing over all iterations of the primitive rays of order $(2m-1)$, the corresponding function will be singular at $s = s_{2m}$ if $Z_D(s)$ is analytic at $s = s_{2m}$. Here $s_k$ is the abscissa of absolute convergence of the series obtained by summing over all iterations of order $k \geq 2$ and we show that $s_1 - h_t < s_k < s_{k-1}$, $h_t > 0$ being the topological entropy of the billiard flow (see Proposition 2). Thus if $Z_D(s)$ is analytic for $\Re s > s_1 - h_t$, for any fixed $M \geq 2$ one obtains a singularity of the sum of series related to the iterations $m \leq M$. This corollary yields some information for the numerical analysis since in the numerical experiences one treats series with finite number iterations.

The existence of a singularity $z_0$ of $\Pi(s)$ such that $\Re z_0 > s_2 - \epsilon_0$, $\epsilon_0 > 0$, $\Im z_0 \neq 0$, is an interesting open problem, but it seems that the difficulty of this problem could be compared with that of the existence or the absence of singularities of $\pi(s)$ for $1/2 < \Re s < 1$. If fact, the dynamics of the periodic orbits is chaotic and the random change of signs of the coefficients in (1.4) plays some essential role. We *conjecture* that in general $Z_D(s)$ is not singular at $s_2$ and Theorem 1 shows that in this case $\Pi(s)$ must be singular at $s_2$. It is expected that there exist non-real singularities $z$ of $\Pi(s)$ with $\Re z$ arbitrary close to line of holomorphy $\Re s = h_\Pi$ of $\Pi(s)$. This will lead to singularities of $Z_D(s)$. In fact we have two possibilities:

$$(a)\ s_2 \neq h_\Pi,\ \ (b)\ s_2 = h_\Pi.$$

Our analysis in Section 4 implies that in the case $(a)$ the function $Z_D(s)$ must be singular either at $s = s_2$ ($s_2 > h_\Pi$) or at a point $z$ close to the line $\Re s = h_\Pi$ ($s_2 < h_\Pi$) and we obtain a solution of the so called modified Lax-Phillips conjecture (see [8], [9], [24]). In the case $(b)$ we have a phenomenon similar to the famous Riemann conjecture for $\zeta(s)$ and the maximal domain $\Re s > t$, where $\Pi(s)$ is analytic, is determined by the line $\Re s = s_2$. Finally, notice that the existence of non-real singularity of the function $Z_D(s)$ has been established by Ikawa [9] for sufficiently small disjoint spheres and

by Stoyanov [24] for sufficiently small obstacles under some conditions. It is not clear if the singularities found in these articles lie in the domain $\Re s > s_2$ and we will discuss this problem in Section 4.

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## 2. Symbolic dynamics

We will write $Z_D(s)$ as a Selberg zeta function using the argument of Section 5, [17]. First assume $n = 3$ and let $\lambda_{\gamma,i}$, $i = 1, 2$, $|\lambda_{\gamma,i}| > 1$, be the eigenvalues of the Poincaré map $P_\gamma$ of the ray $\gamma \in \mathcal{P}$. Set

$$\delta_\gamma = -\frac{1}{2}\log(\lambda_{\gamma,1}\lambda_{\gamma,2}),\ \nu_\gamma = -\log\lambda_{\gamma,1},\ \mu_\gamma = -\log\lambda_{\gamma,2}.$$

The product $\lambda_{\gamma,1}\lambda_{\gamma,2}$ and the sum $\lambda_{\gamma,1} + \lambda_{\gamma,2}$ are positive and $\delta_\gamma < 0$. Given $\gamma \in \mathcal{P}$, introduce

$$r_\gamma = \begin{cases} 0 & \text{if } m_\gamma = 2k, \\ 1 & \text{if } m_\gamma = 2k+1. \end{cases}$$

Then for $\Re s \gg s_1$ we have

$$Z_D(s) = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{m=1}^{\infty}\sum_{\gamma\in\mathcal{P}} T_\gamma(-1)^{mr_\gamma} e^{m(-sT_\gamma+\delta_\gamma+k\nu_\gamma+p\mu_\gamma)}. \tag{2.1}$$

We refer to [17] for the details of the proof of this representation. For $n = 2$ we have a simpler formula since there is only one eigenvalue $\lambda_\gamma > 1$ and we get

$$Z_D(s) = \sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\sum_{\gamma\in\mathcal{P}} T_\gamma(-1)^{mr_\gamma} e^{m(-sT_\gamma+\delta_\gamma+k\nu_\gamma)}, \tag{2.2}$$

where $\delta_\gamma = -\frac{1}{2}\log\lambda_\gamma$, $\nu_\gamma = 2\delta_\gamma$. Consider the leading term of $Z_D(s)$ obtained for $k = p = 0$ (resp. $k = 0$ for $n = 2$) and having the form

$$Z(s) = -\frac{d}{ds}Z_0(s),\ \ Z_0(s) = \sum_{m=1}^{\infty}\frac{1}{m}\sum_{\gamma\in\mathcal{P}}(-1)^{mr_\gamma} e^{m(-sT_\gamma+\delta_\gamma)}.$$

We will write $Z_0(s)$ by using a symbolic model. Let us recall some notations concerning the symbolic dynamics. Given a $Q \times Q$ matrix $A(i,j)_{i,j=1,...,Q}$ such that

$$A(i,j) = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

introduce the spaces

$$\Sigma_A = \{\xi = \{\xi_i\}_{i=-\infty}^{\infty} :\ \xi_i \in \{1,\dots,Q\},\ A(\xi_i,\xi_{i+1}) = 1\},$$

$$\Sigma_A^+ = \{\xi = (\xi_0,\xi_1,\dots) :\ A(\xi_i,\xi_{i+1}) = 1,\ \forall i \geq 0\}.$$

Let $\sigma_A$ be the shift on $\Sigma_A$, $\Sigma_A^+$ given, respectively, by

$$(\sigma_A\xi)_i = \xi_{i+1},\ \forall i \in \mathbb{Z},\ \ (\sigma_A\xi)_i = \xi_{i+1},\ \forall i \geq 0.$$

For every $\xi \in \Sigma_A$ there exists an unique ray $\gamma(\xi)$ with successive reflection points on

$$...,\partial K_{j-1},\partial K_j,\partial K_{j+1},...$$

(see [6], [18]). Let $P_j(\xi)$ be the $j^{\text{th}}$ reflection point of $\gamma(\xi)$ and let $f(\xi) = \|P_0(\xi) - P_1(\xi)\|$. If $\gamma = \gamma(\xi) \in \mathcal{P}$ has $m$ reflections and primitive period $T_\gamma$, then

$$T_\gamma = f(\xi) + f(\sigma_A\xi) + \cdots + f(\sigma_A^{m-1}\xi) = S_m f(\xi).$$

Also (see [8], [9]), there exists a function $g(\xi)$ such that

$$\delta_\gamma = g(\xi) + g(\sigma_A\xi) + \cdots + g(\sigma_A^{m-1}\xi) = S_m g(\xi).$$

For $\Re s$ large we may write $Z_0(s)$ as follows

$$Z_0(s) = \sum_{m=2}^{\infty} \frac{(-1)^m}{m} \sum_{\sigma_A^m\xi=\xi} e^{S_m(-sf(\xi)+g(\xi))}.$$

Given a continuous function $F(\xi) \in C(\Sigma_A)$, introduce

$$\text{var}_n F = \sup_{\xi,\eta\in\Sigma_A} \{|F(\xi) - F(\eta)| : \xi_i = \eta_i \text{ for } |i| \le n\}$$

and for $0 < \theta < 1$ consider the norms

$$|F|_\theta = \sup_n \frac{\text{var}_n F}{\theta^n},\ \|F\|_\infty = \sup_{\xi\in\Sigma_A} |F(\xi)|,\ \|F\|_\theta = \|F\|_\infty + |F|_\theta.$$

Let $\mathcal{F}_\theta(\Sigma_A) \subset C(\Sigma_A)$, $\mathcal{F}_\theta(\Sigma_A^+) \subset C(\Sigma_A^+)$ be Banach spaces with norm $\|.\|_\theta$. It follows from the exponential instability of the billiard ball map that with some constant $0 < \theta < 1$, depending on the geometry of $K$, we have $f(\xi)$, $g(\xi) \in \mathcal{F}_\theta(\Sigma_A)$ (see for more details [8], [9], [17], [22], [24] ). We introduce the suspended flow $\sigma^f$ over the space

$$\Sigma_A^f = \{(\xi, t) : \xi \in \Sigma_A,\ 0 \le t \le f(\xi)\}$$

with the identification $(\xi, f(\xi)) \sim (\sigma_a(\xi), 0)$ (see [16]) and notice that the topological entropy $h_t > 0$ of the suspended flow $\sigma^f$ over $\Sigma_A^f$ is given by

$$h_t = \sup_{\mu\in\mathcal{M}} \frac{h_\mu(\sigma_A)}{\int_{\Sigma_A} f d\mu}.$$

Finally, recall that the pressure $P(F)$ of a function $F \in C(\Sigma_A)$ is given by

$$P(F) = \sup_{m\in\mathcal{M}} \Big(h_\mu(\sigma_A) + \int_{\Sigma_A} F d\mu\Big),$$

where $h_\mu(\sigma_A)$ is the measure entropy of $\sigma_A$ and the sup is taken over the set $\mathcal{M}$ of all probabilistic measures on $\Sigma_A$ invariant with respect to $\sigma_A$.

## 3. Summation over the iterated periodic rays

Given an integer $k \ge 1$, let $s_k \in \mathbb{R}$ be determined by the equality

$$P(-s_k k f + k g) = 0.$$

It is well known (see [16]) that for every function $\varphi(\xi) \in \mathcal{F}_\theta(\Sigma_A)$ there exists $h$, $\psi \in \mathcal{F}_{\theta^{1/2}}(\Sigma_A)$ so that

$$\varphi(\xi) = h(\xi) + \psi(\sigma_A(\xi)) - \psi(\xi)$$

and the function $h(\xi)$ depends only on the coordinates $(\xi_0, \xi_1, \dots)$. If this case we will write $\varphi \sim h$. Obviously, if $F \sim \tilde{F}$, we have $P(F) = P(\tilde{F})$. Passing to functions $f \sim \tilde{f}$, $g \sim \tilde{g}$, we get

$$Z_0(s) = \sum_{m=2}^{\infty} \frac{(-1)^m}{m} \sum_{\sigma_A^m \xi = \xi} e^{S_m(-s\tilde{f}(\xi) + \tilde{g}(\xi))}.$$

It follows easily from the results in [16] that $s_k$ is the abscissa of absolute convergence of the series

$$P_k(s) = \frac{1}{k} \sum_{\gamma \in \mathcal{P}} (-1)^{km_\gamma} e^{-ksT_\gamma + k\delta_\gamma}.$$

To show that the sequence $s_k$ is decreasing, we will prove the following

**Lemma 1.** *For all $k \geq 1$ we have $s_{k+1} < s_k$.*

**Proof.** The pressure of the function $-s_k kf + kg$ is zero, so we may find a function $h \in \mathcal{F}_{\theta^{1/2}}(\Sigma_A^+)$ so that $h \sim -s_k kf + kg$, $P(h) = 0$ and we may choose $h$ (see for more details [16]) so that

$$\sum_{\sigma_A \eta = \xi} e^{h(\eta)} = 1, \ \forall \xi \in \Sigma_A^+.$$

This implies $h(\eta) \leq \alpha_k < 0$ for all $\eta \in \Sigma_A^+$ and

$$k \int_{\Sigma_A} (-s_k f + g) d\mu \leq \alpha_k$$

for each $\mu \in \mathcal{M}$. It is clear that

$$h_\mu(\sigma) + \int_{\Sigma_A} (-s_k(k+1)f + (k+1)g) d\mu$$

$$\leq \sup_{\mu \in \mathcal{M}} \Big[ h_\mu(\sigma) + \int_{\Sigma_A} (-s_k kf + kg) d\mu \Big] + \frac{\alpha_k}{k} = \frac{\alpha_k}{k} < 0, \ \forall \mu \in \mathcal{M}.$$

This implies

$$P\Big(-s_k(k+1)f + (k+1)g\Big) = \sup_{\mu \in \mathcal{M}} \Big[ h_\mu(\sigma) + \int_{\Sigma_A} (-s_k(k+1)f + (k+1)g) d\mu \Big] \leq \frac{\alpha_k}{k}.$$

On the other hand, $P\Big(-s_{k+1}(k+1)f + (k+1)g\Big) = 0$ and since the function

$$\mathbb{R} \ni s \longrightarrow P\Big(-s(k+1)f + (k+1)g\Big)$$

is strictly decreasing, we get $s_{k+1} < s_k$.

To study the convergence of the series over the iterated rays we need the following

**Proposition 1.** *For every $k \geq 1$ there exists $\epsilon > 0$, depending on $k$, such that the series*

$$\sum_{m=k+1}^{\infty} P_m(s) = \sum_{m=k+1}^{\infty} \sum_{\gamma \in \mathcal{P}} \frac{(-1)^{mr_\gamma}}{m} e^{m(-sT_\gamma + \delta_\gamma)}$$

*is absolutely convergent for $\Re s \geq s_k - \frac{\epsilon}{k+1}$.*

**Proof.** As in the proof of Lemma 1, we choose $h$ so that $h \sim -ks_k f + kg$, $h(\eta) < 0$, for all $\eta \in \Sigma_A^+$. First, assume that $s_k < 0$. We choose $\epsilon > 0$ small enough in order to arrange the inequality

$$\sup_{\eta\in\Sigma_A^+} h(\eta) = \alpha_k \leq (k+1)k\epsilon s_k \|f\|_\infty.$$

Let $\eta \in \Sigma_A^+$ correspond to a primitive periodic ray $\gamma \in \mathcal{P}$ with $m$ reflections as it has been explained in Section 2. We obtain

$$S_m(-ks_k f + kg)(\eta) = -ks_k T_\gamma + k\delta_\gamma.$$

On the other hand, it is clear that $T_\gamma \leq m\|f\|_\infty$ and we get

$$S_m h(\eta) \leq m(k+1)k\epsilon s_k \|f\|_\infty \leq (k+1)k\epsilon s_k T_\gamma.$$

From the equality $S_m(-ks_k f + kg)(\eta) = S_m h(\eta)$, we deduce

$$-s_k T_\gamma + \delta_\gamma \leq (k+1)\epsilon s_k T_\gamma,\ \forall \gamma \in \mathcal{P}.$$

Now let $0 \leq u \leq \frac{\epsilon}{k+1}$. Then

$$\begin{aligned} -s_k(1+u)T_\gamma + \delta_\gamma &\leq (k+1)\epsilon s_k T_\gamma - s_k u T_\gamma \\ &\leq \Big((k+1)\epsilon - \frac{\epsilon}{k+1}\Big) s_k T_\gamma \leq \epsilon s_k T_\gamma \end{aligned}$$

and we get the lower bound

$$1 > 1 - e^{-s_k(1+u)T_\gamma + \delta_\gamma} \geq 1 - e^{\epsilon s_k T_\gamma} \geq 1 - e^{2s_k\epsilon d_0} = \frac{1}{C_{\epsilon,k}} > 0.$$

Thus for $0 \leq u \leq \frac{\epsilon}{k+1}$ the series

$$\sum_{m=k+1}^{\infty} e^{m(-s_k(1+u)T_\gamma + \delta_\gamma)} = \frac{e^{(k+1)(-s_k(1+u)T_\gamma+\delta_\gamma)}}{1 - e^{-s_k(1+u)T_\gamma+\delta_\gamma}} \leq C_{\epsilon,k} e^{(k+1)(-s_k(1+u)T_\gamma+\delta_\gamma)} \tag{3.1}$$

is convergent.

Next we obtain

$$\begin{aligned} &-(k+1)s_k(1+u)T_\gamma + (k+1)\delta_\gamma \\ &\leq -s_k k T_\gamma + k\delta_\gamma + (k+1)\epsilon s_k T_\gamma - (k+1)u s_k T_\gamma \leq -s_k(1-\epsilon)kT_\gamma + k\delta_\gamma. \end{aligned}$$

Since $s_k$ is the abscissa of absolute convergence of the series of $k$ iterated rays, we deduce

$$\sum_{\gamma\in\mathcal{P}} e^{-s_k(1-\epsilon)kT_\gamma + k\delta_\gamma} < \infty.$$

Thus we conclude that

$$\sum_{m=k+1}^{\infty} \sum_{\gamma\in\mathcal{P}} e^{m(-s_k(1+u)T_\gamma+\delta_\gamma)} < \infty$$

and the series

$$\sum_{m=k+1}^{\infty} \sum_{\gamma\in\mathcal{P}} \frac{(-1)^{mr_\gamma}}{m} e^{m(-sT_\gamma+\delta_\gamma)}$$

is absolutely convergent for $\Re s \geq s_k - \frac{\epsilon}{k+1}$.

Passing to the case $s_k > 0$, choose $\epsilon > 0$ to arrange the inequalities

$$\sup_{\eta\in\Sigma_A^+} h(\eta) \leq -(k+1)k\epsilon s_k \|f\|_\infty,$$

$$-s_kT_\gamma + \delta_\gamma \leq -(k+1)\epsilon s_kT_\gamma,\ \forall T_\gamma \in \mathcal{P}.$$

For $0 \leq u \leq \frac{\epsilon}{k+1}$ we deduce

$$-s_k(1-u)T_\gamma + \delta_\gamma \leq -(k+1)\epsilon s_kT_\gamma + s_kuT_\gamma \leq -\epsilon s_kT_\gamma$$

which yields

$$\sum_{m=k+1}^{\infty} e^{m(-s_k(1-u)T_\gamma+\delta_\gamma)} \leq C_{\epsilon,k}e^{(k+1)(-s_k(1-u)T_\gamma+\delta_\gamma)}. \tag{3.2}$$

On the other hand,

$$-(k+1)s_k(1-u)T_\gamma + (k+1)\delta_\gamma \leq -s_k(1+\epsilon)kT_\gamma + k\delta_\gamma$$

and this leads to

$$\sum_{m=k+1}^{\infty}\sum_{\gamma\in\mathcal{P}} e^{m(-s_k(1-u)T_\gamma+\delta_\gamma)} < \infty\,.$$

Finally, in the case $s_k = 0$, we arrange

$$\sup_{\eta\in\Sigma_A^+} h(\eta) \leq -(k+1)k\epsilon\|f\|_\infty,$$

$$\delta_\gamma \leq -(k+1)\epsilon T_\gamma,\ \forall T_\gamma \in \mathcal{P}.$$

Repeating the above argument, we establish for $0 \leq u \leq \frac{\epsilon}{k+1}$ the convergence of the series

$$\sum_{m=k+1}^{\infty}\sum_{\gamma\in\mathcal{P}} e^{m(uT_\gamma+\delta_\gamma)} < \infty\,,$$

and this completes the proof.

To compare $s_k$ and $s_1$, consider the measure $\nu \in \mathcal{M}$ for which we have

$$P(-s_1f+g) = h_\nu(\sigma_A) + \int_{\Sigma_A}(-s_1f+g)d\nu = 0.$$

This measure is called equilibrium state of $-s_1f+g$ (see [16]). Then we obtain

$$P\Big(-k(s_1 - \frac{k-1}{k}h_t)f + kg)\Big) \geq h_\nu(\sigma_A) + k\int_{\Sigma_A}(-s_1f+g)d\nu + (k-1)h_t\int_{\Sigma_A} fd\nu$$

$$= (k-1)[h_t\int_{\Sigma_A} fd\nu - h_\nu(\sigma_A)] \geq 0.$$

Comparing this with $P(-ks_kf+kg) = 0$, we deduce

$$s_k \geq s_1 - \frac{k-1}{k}h_t. \tag{3.3}$$

Thus we have proved the following

**Proposition 2.** *The sequence $s_k$ is convergent and $\lim_{k\to\infty} s_k \geq s_1 - h_t$.*

It is interesting to note that the abscissa $c_0$ of simple convergence of the Dirichlet series $Z_0(s)$ satisfies the estimate

$$c_0 \geq s_1 - h_t,$$

but it is difficult to compare $c_0$ with $s_k$.

## 4. Singularities on the line $\Re s = s_2$

Consider the Dirichlet series

$$P_2(s) = \frac{1}{2}\sum_{\gamma\in\mathcal{P}} e^{-2sT_\gamma+2\delta_\gamma}$$

with positive coefficients. According to a classical result, this series has an analytic singularity at $s = s_2$. On the other hand, Proposition 1 implies that the sum over all iterated rays $k\gamma, \gamma \in \mathcal{P}, k \geq 3$, given by

$$\sum_{k=3}^{\infty} P_k(s),$$

is analytic for $\Re s \geq s_2 - \epsilon/3$ for some $\epsilon > 0$. It is clear that the singularities of $Z_0(s)$ for $\Re s > s_2$ are related to those of the series obtained by summing only over the primitive rays

$$P_1(s) = \sum_{\gamma\in\mathcal{P}} (-1)^{r_\gamma} e^{-sT_\gamma+\delta_\gamma}.$$

Let $h_p$ be the abscissa of *holomorphy* of the Dirichlet series $P_1(s)$. More precisely, $h_p$ is the *minimal* real number $t$ such that $P_1(s)$ is analytic for $\Re s > t$. We have three possibilities:

$$(i)\ h_p > s_2, \quad (ii)\ h_p = s_2, \quad (iii)\ h_p < s_2.$$

In the case (i), the function $P_1(s)$, and hence $Z_0(s)$, has either a singularity on the line $\Re s = h_p$ or there exists a sequence of singularities $z_j$ with $\Re z_j \to h_p$, $|\Im z_j| \to \infty$. In the case (iii), the function $P_2(s)$ produces a singularity of $Z_0(s)$ at $s = s_2$. In the case $(ii)$ we must examine the singularities of the sum $P_1(s) + P_2(s)$. Of course, if $P_1(s)$ is analytic at $s = s_2$, we have the same situation as in the case (iii). Thus a cancellation of the singularities of $P_1(s) + P_2(s)$ at the point $s_2$ is possible only if $P_1(s)$ is singular at $s = s_2$. Thus we have the following

**Theorem 2.** *At least one of the functions $Z_0(s)$, $P_1(s)$ has a singularity at $s = s_2$. Moreover, the difference $Z_0(s) - P_1(s)$ is analytic for*

$$s \in \{z \in \mathbb{C} : \Re z > s_2\}.$$

We may compare the functions $Z_0(s)$ and $Z_D(s)$. As it was shown in [8], [17], [24] there exists $\mu_1 > 0$ such that $Z_D(s)-Z_0(s)$ is analytic for $\Re s > s_1-\mu_1$. The number $\mu_1$ depends on the geometry of obstacles (see Appendix in [17] and [24]). In some cases we may show that $s_2 > s_1 - \mu_1$. For example this is true if $n = 2$ and $s_2 < 0$. Nevertheless, it is more natural to deal with the function $\Pi(s)$ introduced in Section 1. As above, let $h_\Pi$ be the abscissa of the holomorphy of the Dirichlet series $\Pi(s)$ introduced in Section 1. We consider again three cases:

$$(i)\ h_\Pi > s_2, \ (ii)\ h_\Pi = s_2, \ ,(iii)\ h_\Pi < s_2.$$

For $m \geq 2$ and $n = 3$ the analysis of the series

$$\Pi_m(s) = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{\gamma\in\mathcal{P}} \frac{1}{m}(-1)^{mr_\gamma} e^{m(-sT_\gamma+\delta_\gamma+k\nu_\gamma+p\mu_\gamma)}, \ \Re s > s_1$$

is completely similar to that of $P_m(s)$. In fact the abscissa of absolute convergence of $\Pi_m(s)$ coincides with that of $P_m(s)$ and we may apply Proposition 1 for the series

$$\sum_{m=k+1}^{\infty}\Pi_m(s)=\sum_{m=k+1}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{\gamma\in\mathcal{P}}\frac{1}{m}(-1)^{mr_\gamma}e^{m(-sT_\gamma+\delta_\gamma+k\nu_\gamma+p\mu_\gamma)}.$$

The case $n=2$ is treated in a similar way and repeating the argument of the proof of Theorem 2, we obtain Theorem 1.

Following the same way, we may consider the function

$$\mathbf{\Pi_3}(s)=\Pi(s)+\Pi_2(s)+\Pi_3(s)=\sum_{\gamma\in\Xi_3}(-1)^{m_\gamma}T_\gamma|I-P_\gamma|^{-1/2}e^{-sd_\gamma},\ \Re s>s_1,$$

where the summation is over all rays $\gamma\in\Xi_3\subset\Xi$ which are either primitive or they are obtained by two or three iterations of primitive periodic rays. Then at least one of the functions $Z_D(s)$, $\mathbf{\Pi_3}(s)$ has a singularity at $s=s_4$ and it is possible to iterate this argument.

It is interesting to mention that for all $k\in\mathbb{N}$ we have

$$s_k>b_0=\sup_{\gamma\in\mathcal{P}}\frac{\delta_\gamma}{T_\gamma}. \tag{4.1}$$

In [17] it was established that $b_0<0$ so we need to check (4.1) only for $s_k<0$. In this case the argument of the proof of Proposition 1 shows that

$$-s_kT_\gamma+\delta_\gamma\leq\epsilon_kT_\gamma,\ \forall\gamma\in\mathcal{P}$$

with some $\epsilon_k<0$ and we obtain (4.1). The number $b_0$ has been introduced in [17]. This number is related to the sequence of poles

$$s_{m,\gamma}=\frac{\delta_\gamma}{T_\gamma}+\frac{2m\pi}{T_\gamma}i,\ \ r_{m,\gamma}=\frac{\delta_\gamma}{T_\gamma}+\frac{(2m+1)\pi}{T_\gamma}i,\ m\in\mathbb{Z}$$

obtained from the series formed by all iteration of a *fixed* periodic primitive ray $\gamma$.

For several strictly convex small obstacles Ikawa [9] and Stoyanov [24] established the existence of a complex singularity

$$z_0=\alpha+i\frac{\pi}{d_1},\ \ \alpha\in\mathbb{R}$$

of $Z_D(s)$ with $d_1$ sufficiently close to $D_0$. Following the analysis in Section 7, [24], we conclude that $s_1-b_K\leq\alpha<s_1$ with

$$b_K\geq\frac{1}{D_0}\ln\Big(1+\frac{\kappa_{\min}}{\nu_0}D_0\Big).$$

Here $\kappa_{\min}>0$ is the minimal normal curvature of $\partial K$ and $\nu_0>0$ is a constant depending on the geometry of $K$. For obstacles having sufficiently small diameters, we may arrange the inequality $b_K\geq h_t$. Indeed, it is sufficient to have

$$h_\mu(\sigma_A)\leq\frac{d_0}{D_0}\ln\Big(1+\frac{\kappa_{\min}}{\nu_0}D_0\Big)\leq b_K\int_{\Sigma_A}f\,d\mu$$

for every $\sigma_A$ invariant measure $\mu \in \mathcal{M}$. If the diameters of the obstacles are sufficiently small, then $\kappa_{\min}$ is large enough, and we get $b_K \geq h_t$. Combining this with (3.3), we obtain immediately

$$s_1 - b_K \leq s_1 - h_t < s_k, \ \forall k \in \mathbb{N}.$$

Consequently, the line $\Re s = s_k$ lies in the domain where we have complex singularities and this agrees with the conjecture that we have complex singularities of $Z_D(s)$ close to the line $\Re s = h_\Pi$ or close to the line $\Re s = s_2$.

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