SUBSTITUTION HAMILTONIANS WITH BOUNDED TRACE MAP ORBITS

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Abstract. We investigate discrete one-dimensional Schrödinger operators with aperiodic potentials generated by primitive invertible substitutions on a two-letter alphabet. We prove that the spectrum coincides with the set of energies having a bounded trace map orbit and show that it is a Cantor set of zero Lebesgue measure. This result confirms a suggestion arising from a study of Roberts and complements results obtained by Bovier-Ghez. As an application we present a class of models exhibiting purely singular continuous spectrum with probability one.

1. Introduction

Schrödinger operators with aperiodic potentials generated by substitution rules on finite alphabets have their origin in the mathematical modelling of quasicrystals, the one-dimensional Fibonacci chain being the most prominent example. The study of their spectral properties has been the objective of quite a number of papers, among which we want to mention [4, 5, 8, 9, 13, 14, 19, 20, 36, 37]. It turned out that there is an apparent tendency for these operators to have purely singular continuous zero-measure spectrum. By rather general principles, it follows for a large subclass of substitution-generated potentials, namely those being ergodic, that the absolutely continuous spectrum is always empty [27]. Moreover, there is a natural strategy for proving that the spectrum has zero Lebesgue measure. Namely, there is a set of energies A of zero Lebesgue measure [27] that is a priori contained in the spectrum Σ,

$$A \subseteq \Sigma. \tag{1}$$

To prove the zero-measure property, previous works by Sütő [37], Bellissard et al. [6, 4], and Bovier-Ghez [8] have established equality of these sets for certain models. The key role in this context is played by a dynamical system on some $\mathbb{R}^\nu$, the *trace map*, which is directly induced by the substitution generating the potentials of the operators under study. These authors have introduced two kinds of stable set B_∞ associated to such a dynamical system. Sütő and Bellissard et al. considered the set of energies giving rise to a bounded trace map orbit [36, 6] (compare also the earlier paper by Casdagli [10]). Their results apply in particular to the Fibonacci case. However, in a subsequent work [4], Bellissard et al. have studied another prominent example, the Hamiltonian generated by the period doubling substitution, and they have realized that the definition of the stable set has to be altered, as there are energies in the spectrum giving rise to unbounded trace map orbits [7]. Instead these authors considered the set of energies giving rise to non-escaping orbits, that is, orbits that are bounded (by a suitable constant) on a subsequence of iterates.

In either case, by (1) and Kotani [27], to establish the zero-measure property it suffices to prove the following inclusions,

$$\Sigma \subseteq B_\infty, \tag{2}$$

$$B_\infty \subseteq A. \tag{3}$$

The inclusion (2) can be proved by a strong convergence argument. However, one has to show at some point that certain sets related to the definition of the stable set are open — which is non-trivial and one of the major steps involved in the works mentioned above. It is this step where one has to include a careful study of the trace map, whereas in the proof of (3) one follows rather general lines and exploits the boundedness of the trace map orbit (resp., the boundedness of a subsequence thereof).

A proof of (2) and (3) implies that the sets A, Σ, B_∞ are equal and has the zero-measure property as an immediate corollary. However, if one can work with the stable set as defined in [6], another corollary can be deduced, namely, the following property:

BO: Every energy from the spectrum gives rise to a bounded trace map orbit.

Recent works have shown that property **BO** has very nice additional applications, for example,

1. power law growth of the resistance [21, 22],
2. α-continuity properties of the spectral measures which in turn yield bounds on the dynamics of the associated quantum systems [12, 23, 24], and

In the particular case of invertible substitutions on a two-letter alphabet, Roberts has studied the trace map dynamics in great detail [34]. He provided a classification of the orbits with respect to their escape properties. In fact, a closer look at his argument reveals that the non-escaping orbits are in fact bounded. While the paper [34] concentrates on the study of the trace map dynamics, the author suggests applying these results to the associated Hamiltonians. In particular, it is predicted that they should be useful when trying to follow the lines indicated above, that is, an attempt to prove (2) and (3) for these models.

Our motivation therefore stems from two sources. On the one hand, Roberts' results indeed allow for the general strategy to be pursued. In particular, we shall exhibit a class of substitution Hamiltonians with zero-measure spectrum and we shall make crucial use of Roberts' results on the trace map dynamics. On the other hand, we will also obtain property **BO** for the models under consideration, which is particularly nice in view of the possible applications indicated above. In addition, while our strategy is in certain respects similar to the one employed by Bovier-Ghez in [8], it improves upon essential aspects that led to multiple reformulations [9] which were caused by the desire to combine one-sided self-similarity properties with two-sided ergodicity. It will be seen from our proof that only two-sided ergodicity is indeed necessary.

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2. Models and results

In this section we define the operators we shall study in the sequel, indicate how the trace maps naturally arise from the substitution rules generating the potentials of those operators, and state the results to be proven in subsequent sections.

Consider a finite set $\mathcal{A} = \{a_1, \dots, a_s\}$, called the alphabet, the set $\mathcal{A}^*$ of finite sequences, called words, that can be built from elements of $\mathcal{A}$, the set $\mathcal{A}^{\mathbb{N}}$ of one-sided infinite sequences over $\mathcal{A}$, and the set $\mathcal{A}^{\mathbb{Z}}$ of two-sided infinite sequences over $\mathcal{A}$. A mapping $S : \mathcal{A} \to \mathcal{A}^*$ is called substitution. S can be extended to $\mathcal{A}^*$ and $\mathcal{A}^{\mathbb{N}}$ by $S(b_1 \dots b_n) = S(b_1) \dots S(b_n)$ and $S(b_1 b_2 \dots) = S(b_1)S(b_2) \dots$, respectively. A fixed point $u \in \mathcal{A}^{\mathbb{N}}$ of S is called substitution sequence. Given a substitution sequence u, we extend it to the left arbitrarily giving $\hat{u}$ and define

$$\Omega = \{\omega \in \mathcal{A}^{\mathbb{Z}} \; : \; \omega = \lim_{i\to\infty} T^{n_i}\hat{u}, \; n_i \to \infty\}, \tag{4}$$

where $(Tv)_n = v_{n+1}$ is the standard shift on $\mathcal{A}^{\mathbb{Z}}$, $\mathcal{A}$ is equipped with discrete topology, and the limit is taken with respect to pointwise convergence. Let $f : \mathcal{A} \to \mathbb{R}$ be given and consider the family $(H_\omega)_{\omega\in\Omega}$ of discrete one-dimensional Schrödinger operators in $l^2(\mathbb{Z})$ given by

$$(H_\omega \phi)(n) = \phi(n+1) + \phi(n-1) + V_\omega(n)\phi(n), \tag{5}$$

where

$$V_\omega(n) = f(\omega_n). \tag{6}$$

We only assume f to be non-constant. The actual numerical values the function f takes do not affect the qualitative results to be obtained. Our concern now is to study the spectral properties of the operators H_ω where we suppose that the potentials V_ω are aperiodic, for otherwise there is a well-established general theory treating the periodic case [33]. To fit such a family into the framework of ergodic operators we seek conditions on S such that the topological dynamical system (Ω, T) carries an ergodic measure. This in turn allows for the powerful Kotani theory [27] to be applied. To this end, consider the substitution matrix $M_S = (m_{ij})_{i,j=1,\dots,s}$ where m_{ij} is the number of a_j in $S(a_i)$. The substitution S is called primitive if there exists $k \in \mathbb{N}$ such that M_S^k is strictly positive. If S is primitive, then a substitution sequence always exists (at least for some power of S which is of course also primitive). Moreover, the system (Ω, T) is strictly ergodic [32], that is, there is a unique T-invariant measure μ which is necessarily ergodic (i.e., (Ω, T) is uniquely ergodic) and every $\omega \in \Omega$ has a dense T-orbit (i.e., the system is minimal). In particular, there is a closed set $\Sigma \subseteq \mathbb{R}$ such that $\sigma(H_\omega) = \Sigma$ for every $\omega \in \Omega$.

A standard formalism in the study of one-dimensional Schrödinger operators is the concept of transfer matrices, which is essentially a reformulation of the recurrence given by the eigenvalue equation

$$\phi(n+1) + \phi(n-1) + V_\omega(n)\phi(n) = E\phi(n), \tag{7}$$

namely,

$$M_{\omega,E}(n) = \begin{pmatrix} E - V_\omega(n) & -1 \\ 1 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} E - V_\omega(1) & -1 \\ 1 & 0 \end{pmatrix}. \tag{8}$$

If ϕ is a (not necessarily square-summable) solution of (7), then

$$\Phi(n) = M_{\omega,E}(n)\Phi(0), \tag{9}$$

where $\Phi(n) = (\phi(n+1), \phi(n))^T$. Hof has shown [19] that, in case S is primitive, for every E there exists a number $\gamma(E) \geq 0$, called Lyapunov exponent, such that, for every $\omega \in \Omega$, we have

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n} \ln \|M_{\omega,E}(n)\|. \tag{10}$$

Let $A = \{E \in \mathbb{R} : \gamma(E) = 0\}$. It is well known that A is a subset of the spectrum, that is, equation (1) holds. Moreover, it follows from Kotani [27] that the set A has zero Lebesgue measure,

$$|A| = 0. \tag{11}$$

It is useful to view the transfer matrix formalism as an energy-indexed representation of words occurring in the substitution sequence as matrices in $Sl(2,\mathbb{R})$. Fix some energy E and define the map $M_E : \mathcal{A} \to Sl(2,\mathbb{R})$ by

$$M_E(a_i) = \begin{pmatrix} E - f(a_i) & -1 \\ 1 & 0 \end{pmatrix}. \tag{12}$$

Extend this mapping to $\mathcal{A}^*$ by

$$M_E(b_1 \ldots b_n) = M_E(b_n) \times \cdots \times M_E(b_1). \tag{13}$$

The action of S may then be represented as

$$S(M_E(w)) = M_E(S(w)), \tag{14}$$

where w is a word in $\mathcal{A}^*$. Thus, a substitution rule induces a dynamical system on $SL(2,\mathbb{R})^{|\mathcal{A}|}$. In fact, by passing to the traces of those matrices, one obtains a dynamical system, the trace map, on some $\mathbb{R}^\nu$. This works for an arbitrary alphabet [1, 26, 8]. However, as we will focus on the case of a two-letter alphabet, we shall show how to derive the trace map only in this special case. Hence, consider $M_E : \mathcal{A} = \{a,b\}^* \to Sl(2,\mathbb{R})$ and define $x_E(n) = \frac{1}{2}\mathrm{tr}(M_E(S^n(a)))$, $y_E(n) = \frac{1}{2}\mathrm{tr}(M_E(S^n(b)))$, $z_E(n) = \frac{1}{2}\mathrm{tr}(M_E(S^n(ab)))$. The factor $\frac{1}{2}$ is included here to conform with Roberts' paper. Using

$$M^2 - \mathrm{tr}(M)M + I = 0 \tag{15}$$

for $M \in Sl(2,\mathbb{R})$, it is now easily seen that each of $x_E(n+1), y_E(n+1), z_E(n+1)$ can be expressed by a fixed polynomial expression in the variables $x_E(n), y_E(n), z_E(n)$. That is, there is a polynomial mapping F_S from $\mathbb{R}^3$ to itself such that, for every E, n,

$$F_S(x_E(n), y_E(n), z_E(n)) = (x_E(n+1), y_E(n+1), z_E(n+1)). \tag{16}$$

F_S is called the trace map associated to S. Thus one is led to study the dynamics of the mapping F_S on the initial point

$$\mathbf{x}_E = (x_E(0), y_E(0), z_E(0)) = \left(\tfrac{1}{2}\mathrm{tr}(M_E(a)), \tfrac{1}{2}\mathrm{tr}(M_E(b)), \tfrac{1}{2}\mathrm{tr}(M_E(ab))\right). \tag{17}$$

Example. Let us briefly discuss the Fibonacci case. S is given by $S(a) = b, S(b) = ba$. We therefore have

$$\begin{aligned}
x_E(n+1) &= \tfrac{1}{2}\mathrm{tr}(M_E(S^{n+1}(a))) = \tfrac{1}{2}\mathrm{tr}(M_E(S^n(b))) = y_E(n) \\
y_E(n+1) &= \tfrac{1}{2}\mathrm{tr}(M_E(S^{n+1}(b))) = \tfrac{1}{2}\mathrm{tr}(M_E(S^n(ba))) = \tfrac{1}{2}\mathrm{tr}(M_E(S^n(ab))) = z_E(n) \\
z_E(n+1) &= \tfrac{1}{2}\mathrm{tr}(M_E(S^{n+1}(ab))) = \tfrac{1}{2}\mathrm{tr}((M_E(S^n(b)))^2 M_E(S^n(a))) \\
&= \tfrac{1}{2}\mathrm{tr}(2y_E(n)M_E(S^n(b))M_E(S^n(a)) - M_E(S^n(a))) \\
&= 2y_E(n)z_E(n) - x_E(n).
\end{aligned}$$

Thus, the trace map associated to the Fibonacci substitution is given by $F_S(x,y,z) = (y, z, 2yz - x)$. Note that the Fibonacci trace map preserves the following quantity, known as the Fricke-Vogt invariant,

$$I(x,y,z) = x^2 + y^2 + z^2 - 2xyz - 1, \tag{18}$$

as can be checked by direct calculation. The proof of property **BO**, which is one of our main objectives, in the Fibonacci case makes crucial use of this invariant. Thus it is natural to consider the set of polynomial mappings preserving I, that is,

$$\mathcal{I} = \{F : \mathbb{R}^3 \to \mathbb{R}^3 \text{ polynomial} \, : \, I \circ F = I\}. \tag{19}$$

Every substitution S on a two-letter alphabet can naturally be considered as an element of $\mathrm{Hom}(F_2)$, F_2 being the free group with two generators. S is called invertible if it belongs to $\mathrm{Aut}(F_2)$. Now the set of substitutions having trace maps belonging to $\mathcal{I}$ is exactly the set of invertible substitutions. It is therefore natural to consider this class of substitutions when trying to establish **BO**. Let

$$B_\infty = \{E \in \mathbb{R} : \text{the orbit of } \mathbf{x}_E \text{ remains bounded}\}.$$

We shall prove the following theorem which has the natural additional assumption that S is primitive since our strategy relies heavily on ergodic properties of the induced operators. In particular, we want to employ the powerful Kotani result (11). In this sense, we achieve a natural application of Roberts' trace map results to substitution Hamiltonians.

Theorem 2.1. *Let S be a primitive invertible substitution on a two-letter alphabet such that the induced potentials V_ω are aperiodic. Then, we have $\Sigma = B_\infty = A$.*

As immediate consequences, we obtain the following corollaries.

Corollary 2.2. *Suppose S obeys the assumptions of Theorem 2.1. Then, for every $\omega \in \Omega$, the operator H_ω has purely singular spectrum supported on a Cantor set of zero Lebesgue measure.*

Remark. A Cantor set is by definition a closed nowhere dense set with no isolated points. Now the spectrum is of course closed and, by ergodicity, has no isolated points [11]. Finally, it is nowhere dense since it has zero Lebesgue measure by Theorem 2.1.

Corollary 2.3. *Suppose S obeys the assumptions of Theorem 2.1. Then property* **BO** *holds.*

In the introduction we have noted that substitution Hamiltonians exhibit an apparent tendency to have purely singular continuous spectrum. While no counterexample to this general hypothesis is known yet, the results in this direction obtained so far are not as general as the results on the Lebesgue measure of the spectrum. This is mainly due to the fact that the point spectrum behaves badly under strong convergence and, as opposed to the spectrum as a set, the point spectrum of an operator H_ω associated to some primitive substitution is a priori not independent of ω (see, however, [15] for uniform absence of eigenvalues in the Fibonacci case). On the other hand there are closed sets $\Sigma_{\mathrm{pp}}, \Sigma_{\mathrm{sc}}, \Sigma_{\mathrm{ac}}$ such that, for μ-a.e. $\omega \in \Omega$, we have $\sigma_{\mathrm{pp}}(H_\omega) = \Sigma_{\mathrm{pp}}, \sigma_{\mathrm{sc}}(H_\omega) = \Sigma_{\mathrm{sc}}, \sigma_{\mathrm{ac}}(H_\omega) = \Sigma_{\mathrm{ac}}$. Thus, the natural goal is to establish $\Sigma_{\mathrm{pp}} = \Sigma_{\mathrm{ac}} = \emptyset$. For any primitive substitution it follows from (11) that $\Sigma_{\mathrm{ac}} = \emptyset$ and, moreover, by Last-Simon [29], $\sigma_{\mathrm{ac}}(H_\omega) = \emptyset$ even holds for every $\omega \in \Omega$. Results of the type $\Sigma_{\mathrm{pp}} = \emptyset$ were shown in [13, 14]. Essentially, the results in these papers require the occurrence of a fourth power in a substitution sequence associated to a given primitive substitution S. Using Theorem 2.1, we can generalize this result, in the particular case under study, in the following way.

Theorem 2.4. *Let S be a primitive invertible substitution on a two-letter alphabet such that the induced potentials V_ω are aperiodic. Suppose that there exist $c \in \{a, b, ab\}$ and $k \in \mathbb{N}$ such that the word $S^k(ccc)$ occurs in a substitution sequence associated to S. Then, we have $\Sigma_{pp} = \emptyset$, that is,*

$$\mu(\{\omega \in \Omega : H_\omega \text{ has purely singular continuous spectrum}\}) = 1.$$

Remark. As an illustration, let us apply Theorem 2.4 to the generalized Fibonacci substitutions $S_l(a) = b, S_l(b) = b^l a$. Consider the words $S_l^n(b)$. They obviously converge in a suitable sense to a substitution sequence. Moreover, they obey the recursion $S_l^n(b) = (S_l^{n-1}(b))^l S_l^{n-2}(b)$. Let us first consider the case $l \geq 2$. We have

$$S_l^4(b) = \left((S_l^2(b))^l S_l(b)\right)^l (S_l(b))^l b.$$

Thus we can choose $c = b$ and $k = 1$. In the case $l = 1$, we have $S_1^6(b) = babbababbabbababbabab$ and we can choose $c = b$ and $k = 2$. Let us remark that related work can be found in [25].

3. Characterization of unbounded trace map orbits

In this section we provide a characterization for a trace map orbit to be unbounded in all cases relevant to primitive invertible substitutions. To a large extent, we summarize the essential part of Roberts [34]. However, below we will need slight extensions and variants of his results and we therefore include proofs of the properties to be exploited in the next section.

The set $\mathcal{I}$ in (19) naturally carries a group structure. In fact, $\mathcal{I}$ can be written as a semidirect product

$$\mathcal{I} = \mathcal{S} \otimes_s \mathcal{G}, \tag{20}$$

where $\mathcal{S} = \{\sigma_0 = Id, \sigma_1, \sigma_2, \sigma_3\} \simeq \mathcal{C}_2 \otimes \mathcal{C}_2$ is a normal subgroup, $\mathcal{C}_n$ denoting the cyclic group of order n [31] (see also [3]). The involutions σ_i, $i = 1, 2, 3$, are the pairwise sign changes, for example, $\sigma_1(x, y, z) = (x, -y, -z)$. Thus any element of $\mathcal{I}$ can uniquely be written as the composition of one of the σ_i and an element $F \in \mathcal{G}$. On the other hand, $\mathcal{G}$ corresponds to the trace maps corresponding to invertible substitutions on a two-letter alphabet. It can be generated by the involution p and the infinite-order mapping u [31], where

$$p(x, y, z) = (y, x, z), \; u(x, y, z) = (z, y, 2yz - x).$$

This can be seen by studying the correspondence between the group Φ_2 of automorphisms of F_2, the associated substitution matrices, and the associated trace maps. Note that the mappings $M_\bullet, F_\bullet$ are homomorphisms of groups [35], that is,

$$M_{S_1 S_2} = M_{S_1} \cdot M_{S_2}, \tag{21}$$

$$F_{S_1 S_2} = F_{S_1} \circ F_{S_2}, \tag{22}$$

where $\cdot$ (resp., $\circ$) denotes matrix product (resp., mapping composition) and the composition of substitutions is defined by $S_1 S_2(w) = S_2(S_1(w))$ for any $w \in \{a, b\}^*$. Now, Φ_2 is known to be generated by U, P, σ given in the table below [2, 30, 34].

S	M_S	F_S
$U : a \mapsto ab,\; b \mapsto b$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$u : (x, y, z) \mapsto (z, y, 2yz - x)$
$P : a \mapsto b,\; b \mapsto a$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$p : (x, y, z) \mapsto (y, x, z)$
$\sigma : a \mapsto a^{-1},\; b \mapsto b$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$s : (x, y, z) \mapsto (x, y, 2xy - z)$

Thus, by (21) and (22), the associated substitution matrices (resp., trace maps) generate the group of all substitution matrices (resp., trace maps) induced by elements from Φ_2. These two groups are $GL(2, \mathbb{Z})$ and $\mathcal{G}$. Moreover, two invertible substitutions induce the same trace map if and only if their substitution matrices differ at most by a change of sign [31]. Thus, $\mathcal{G}$ is isomorphic to $PGL(2, \mathbb{Z}) = GL(2, \mathbb{Z})/\{\pm I\}$. On the level of matrices and trace maps, however, the image of σ becomes redundant [35] and an arbitrary element does not have a canonical representation as a word in the generators. Roberts therefore considered the subgroup of orientation-preserving elements of $\mathcal{G}$, $\mathcal{G}_{OP} \simeq PSL(2,\mathbb{Z})$, and introduced another set of generators for this subgroup, namely

$$M_v = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; M_q = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \tag{23}$$

on the matrix level and

$$v : (x,y,z) \mapsto (y,x,2xy-z), \;\; q : (x,y,z) \mapsto (y,z,x) \tag{24}$$

on the trace map level, which is motivated by the well-known result that $PSL(2,\mathbb{Z})$ is the free product of a cyclic subgroup of order 2 with one of order 3 [28]. The above choices just correspond to the respective cyclic elements. As a consequence, every element of $\mathcal{G}_{OP}$ has a unique representation as a finite reduced word over the elements v, q, q^{-1}, where reduced means that each consecutive element of the word belongs to alternate subgroups and is different from the unit element. Corresponding to this representation, Roberts introduced the following classification.

Definition 3.1. *Let $Id \neq F \in \mathcal{G}_{OP}$ and consider the unique representation of F as a reduced word $w = w(q,v)$. Suppose w is of infinite order. Then w is called to be of*
Type I *if it begins with $q^{\pm 1}$ and ends with v,*
Type II *if it begins with v and ends with $q^{\pm 1}$,*
Type III *if it begins with q and ends with q,*
Type IV *if it begins with q^{-1} and ends with q^{-1},*
Type V *if it begins with v and ends with v, or begins with $q^{\pm 1}$ and ends with $q^{\mp 1}$.*

It is important to note that a word w of Type II–V is conjugate to some word w_1 of Type I, $w = w_c w_1 w_c^{-1}$. The word w_c is given by $q^{\mp 1}$ for Type II words, by q^{-1} for Type III words, and by q for Type IV words. The conjugacy in case of a Type V word does not have an explicit general form. Its existence, however, is obvious.

Let us look at the iteration of a Type I word w_1 on some initial point $\mathbf{x}$. In the limiting procedure, we obtain an infinite word w_∞ which is applied to $\mathbf{x}$. The basic building blocks of w_∞ are qv and $q^{-1}v$. It is therefore convenient to note

$$qv(x,y,z) = (x, 2xy - z, y), \; q^{-1}v(x,y,z) = (2xy - z, y, x) \tag{25}$$

and to trace the evolution of $\mathbf{x}$ under the successive application of such building blocks. Thus, let $b_i \in \{qv, q^{-1}v\}$ be chosen such that $w_\infty = \ldots b_3 b_2 b_1$ and define $\mathbf{x}_n = b_n \ldots b_1 \mathbf{x}$. We shall study unbounded orbits of w_1. If $\mathcal{O}(\mathbf{x}) = \{w_1^n(\mathbf{x}) : n \in \mathbb{N}\}$ is unbounded, then, with respect to the above definitions, the sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ is unbounded, too. Since q, q^{-1}, v preserve the invariant $I_0 = I(\mathbf{x})$, we have $I(\mathbf{x}_n) = I_0$ for every $n \in \mathbb{N}$. It is easily seen that, if

$$\|\mathbf{x}_n\|_2^2 > 2 + (1 + \sqrt{I_0})^2, \tag{26}$$

$\mathbf{x}_n$ has at least two coordinates whose modulus is greater that one [34], where $\|\cdot\|_2$ denotes the norm $\|(x,y,z)\|_2^2 = |x|^2 + |y|^2 + |z|^2$. Thus every $\mathbf{x}_n = (x_n, y_n, z_n)$ obeying (26) has to be of one of the following forms,

$$\begin{array}{llll} \text{(i)} & |x_n| > 1, & |y_n| > 1, & |z_n| > 1, \\ \text{(ii)} & |x_n| \leq 1, & |y_n| > 1, & |z_n| > 1, \\ \text{(iii)} & |x_n| > 1, & |y_n| \leq 1, & |z_n| > 1, \\ \text{(iv)} & |x_n| > 1, & |y_n| > 1, & |z_n| \leq 1. \end{array} \tag{27}$$

To show that unboundedness necessarily implies that the orbit diverges to infinity (an *escaping* orbit in [34]), we shall use another result from [34]. If, for some $n \in \mathbb{N}$, $\mathbf{x}_n$ obeys the following condition, called Condition E by Roberts,

$$|x_n| > 1,\ |y_n| > 1,\ |x_n y_n| > |z_n|, \tag{28}$$

then Condition E is obeyed by $\mathbf{x}_m$ for every $m \geq n$ and the orbit diverges super-exponentially in every coordinate [34], where here and in the following we mean by super-exponential divergence that the logarithm of the respective quantity diverges exponentially. In this connection, the following definition proves to be useful.

Definition 3.2. *An infinite-order element $w = w(q,v) \in \mathcal{G}_{OP}$ is called hyperbolic (parabolic) if the number of occurrences of q and the number of occurrences of q^{-1} in the reduced word w^n are (not) both infinite as $n \to \infty$.*

It will be seen that the dynamics of hyperbolic mappings behave very uniformly in the three coordinates since in this case unboundedness indeed implies that some $\mathbf{x}_n$ satisfies Condition E, thus the three coordinates either are all bounded or all super-exponentially increasing. Moreover, as we will show below, in the case of primitive invertible substitutions, the parabolic case does not occur. Therefore, the results on hyperbolic elements are particularly useful in our present context. In order to describe necessary and sufficient conditions for an unbounded orbit, let us recall from [34] the definition of the following regions.

Definition 3.3. *Define the following open subsets of $\mathbb{R}^3$.*

$$\begin{array}{lcl} T_1 & = & \{\mathbf{x} \in \mathbb{R}^3 \ :\ |x| > 1, |y| > 1, |xy| > |z|\}, \\ T_2 & = & \{\mathbf{x} \in \mathbb{R}^3 \ :\ |x| > 1, |y| > 1, |xy| < |z|\}, \\ T_q & = & \{\mathbf{x} \in \mathbb{R}^3 \ :\ |y| > 1, |z| > 1, |yz| > |x|\}, \\ T_{q^{-1}} & = & \{\mathbf{x} \in \mathbb{R}^3 \ :\ |x| > 1, |z| > 1, |xz| > |y|\}. \end{array}$$

Note in particular that $v(T_1) \subseteq T_2$, $q^{\pm 1}(T_2) \subseteq T_1$, and $T_{q^{\pm 1}} = q^{\mp 1}(T_1)$. Once it is shown that unboundedness implies Condition E, that is, entry into T_1, we can infer that Condition E is satisfied for all subsequent images under qv or $q^{-1}v$, that is, they remain in the region T_1, and we can thus immediately identify escape regions for words of Type I–IV. For words of Type V, the escape region does not have such an explicit general characterization, but again the important features to be employed in the next section are easily checked. We first treat words of Type I–IV.

Proposition 3.4. *Suppose $F \in \mathcal{G}_{OP}$ is of infinite order and hyperbolic. Consider the unique reduced word $w = w(q,v)$ representing F. Let $\mathbf{x} \in \mathbb{R}^3$ be given and consider the orbit of $\mathbf{x}$ under F, that is, $\mathcal{O}(\mathbf{x}) = \{F^n(\mathbf{x}) : n \in \mathbb{N}\}$. Let U be the open set T_1 if w is of Type I, T_2 if w is of Type II, T_q if w is of Type III, $T_{q^{-1}}$ if w is of Type IV. Then, the following holds,*

$$\mathcal{O}(\mathbf{x}) \text{ unbounded} \Leftrightarrow \exists n_0 \ :\ F^{n_0}(\mathbf{x}) \in U \Leftrightarrow \exists n_0 \ :\ \forall n \geq n_0 \ :\ F^n(\mathbf{x}) \in U.$$

Moreover, in either case, $\mathcal{O}(\mathbf{x})$ unbounded implies that $(F^n(\mathbf{x}))$ diverges super-exponentially in every coordinate.

Proof. Given Roberts' analysis as summarized above, it remains to be shown that, given some hyperbolic Type I word w_1 and some initial point $\mathbf{x}$ having an unbounded w_1-orbit, the associated vectors $\mathbf{x}_n$ as defined above must satisfy Conditions E for some index N. The assertion of the proposition then follows. Let $I_0 = I(\mathbf{x})$. Since the sequence $(w_1^l \mathbf{x})_{l \in \mathbb{N}}$ is unbounded, there exists $k \in \mathbb{N}$ such that

$$\|w_1^{k+1}\mathbf{x}\|_2 > \|w_1^k \mathbf{x}\|_2 \tag{29}$$

and all the $\mathbf{x}_n$ obtained during the application of w_1^2 to $w_1^k\mathbf{x}$ obey (26). In particular, all these $\mathbf{x}_n$ have a classification according to (27). By (29) there is an occurrence of v in w_1 which is applied to $\mathbf{x}_{n_0}$, for some appropriate choice of $n_0 \in \mathbb{N}$, such that $\|\mathbf{x}_{n_0+1}\|_2 = \|q^\varepsilon v \mathbf{x}_{n_0}\|_2 > \|\mathbf{x}_{n_0}\|_2$, where $\varepsilon \in \{-1, 1\}$ is the appropriate power of the q preceding this particular v in w_1. We consider the case $\varepsilon = 1$, the other case is similar. If the classification of $\mathbf{x}_{n_0}$ is (i), (iii), or (iv), then working similarly as in the proof of Proposition 3.4 in [34] shows that $\mathbf{x}_{n_0+1}$ obeys Condition E. Thus, it remains to consider the case of $\mathbf{x}_{n_0}$ having property (ii). By assumption, w_1 contains both q and q^{-1}. Thus, while applying w_1^2 to $w_1^k\mathbf{x}$, $\mathbf{x}_{n_0}$ will face a word of the form $q^{-1}v(qv)^j$, $j \ge 1$. Since $\mathbf{x}_{n_0}$ has property (ii), its first coordinate obeys $|x_{n_0}| \le 1$. By (25), we therefore have $|x_{n_0+j}| \le 1$. Thus, by assumption, $\mathbf{x}_{n_0+j}$ has property (ii). Now, either the application of $q^{-1}v$ to $\mathbf{x}_{n_0+j}$ increases the norm, in which case $\mathbf{x}_{n_0+j+1}$ obeys Condition E (check!), or we have $\|q^{-1}v\mathbf{x}_{n_0+j}\|_2 \le \|\mathbf{x}_{n_0+j}\|_2$. In this case we get $|z_{n_0+j+1}| \le 1$, and thus $|x_{n_0+j+1}| > 1$, $|y_{n_0+j+1}| > 1$. Hence, in this case $\mathbf{x}_{n_0+j+1}$ must obey Condition E as well. $\square$

Let us now turn to words of Type V. The appropriate choice of escape region is provided in the following proposition.

Proposition 3.5. *Suppose $F \in \mathcal{G}_{OP}$ is of infinite order and hyperbolic. Consider the unique reduced word $w = w(q,v)$ representing F. Suppose w is of Type V. Let $w_c = w_c(q,v)$ be the word that conjugates it to a Type I word w_1, that is, $w = w_c w_1 w_c^{-1}$. Let $\mathbf{x} \in \mathbb{R}^3$ be given and consider $\mathcal{O}(\mathbf{x})$. Define U to be the open set $w_c(T_1)$. Then, the following holds,*

$$\mathcal{O}(\mathbf{x}) \text{ unbounded} \Leftrightarrow \exists n_0 \ : \ F^{n_0}(\mathbf{x}) \in U \Leftrightarrow \exists n_0 \ : \ \forall n \ge n_0 \ : \ F^n(\mathbf{x}) \in U.$$

Again $\mathcal{O}(\mathbf{x})$ unbounded implies that $(F^n(\mathbf{x}))$ diverges super-exponentially in every coordinate. Moreover, we have $U \subseteq T_1 \cup T_2$.

Proof. Since w_c is a composition of homeomorphisms of $\mathbb{R}^3$, it follows that U is indeed open. Since $w^n = (w_c w_1 w_c^{-1})^n = w_c w_1^n w_c^{-1}$ the essential part of the dynamics is governed by w_1. Moreover, in an unbounded orbit, Condition E must be satisfied prior to some occurrence of v in w_1^n for n sufficiently large. Subsequent w_1's will then successively map into T_1 and have super-exponential growth in every coordinate. Our choice of U just reflects the appropriate choice of region which is entered after an application of w_c and thus after a complete application of w itself. Note that w_c^{-1} maps U back to T_1 and is preceded by the Type I word w_1. Thus, in essence we have a Type I dynamics, however, to obtain the actual iterate of w we have to map to U via w_c. Moreover, $U \subseteq T_1$ or $U \subseteq T_2$ depending on whether w_c's first symbol is $q^{\pm 1}$ or v. Putting this together, we see that the assertion follows from the result given in Proposition 3.4. □

These results in fact extend to orientation-reversing elements. There is a natural way to associate an orientation-preserving element to a given orientation-reversing element F, namely F^2.

Proposition 3.6. *Suppose $F \in \mathcal{G}_{OR}$ is such that F^2 is of infinite order and hyperbolic. Consider the unique reduced word $w = w(q, v)$ representing F^2. Let $\mathbf{x} \in \mathbb{R}^3$ be given and consider $\mathcal{O}(\mathbf{x}) = \{F^n(\mathbf{x}) : n \in \mathbb{N}\}$. Let U be the open set chosen according to the type of w as in the preceding two propositions. Then, the following holds,*

$$\mathcal{O}(\mathbf{x}) \text{ unbounded} \Leftrightarrow \exists n_0 \ : \ F^{n_0}(\mathbf{x}) \in U \Leftrightarrow \exists n_0 \ : \ \forall n \geq n_0 \ : \ F^n(\mathbf{x}) \in U.$$

Moreover, in either case, $\mathcal{O}(\mathbf{x})$ unbounded implies that $(F^n(\mathbf{x}))$ diverges super-exponentially in every coordinate.

Proof. The F-orbit of $\mathbf{x}$ is the union of the F^2-orbits of $\mathbf{x}$ and $F(\mathbf{x})$. If $\mathcal{O}(\mathbf{x})$ is unbounded, then at least one of these two F^2-orbits is unbounded and thus escaping at a super-exponential rate in each coordinate by Proposition 3.4 or Proposition 3.5. Moreover, this escaping orbit enters U, again by Proposition 3.4 or Proposition 3.5. Thus, $\mathcal{O}(\mathbf{x})$ enters U. On the other hand, the other F^2-orbit must also be unbounded and thus escaping, since the F-image of every bounded region is again bounded. Putting this together, we obtain that the F-orbit of $\mathbf{x}$ enters U, remains there, and diverges super-exponentially in each coordinate. □

Having described escape regions in the hyperbolic case, we shall now prove that this is sufficient to control the trace map dynamics for all primitive invertible substitutions.

Lemma 3.7. *Let S be a primitive invertible substitution and let F_S be the trace map associated to S. Then, F_S (resp., F_S^2 if F_S is orientation-reversing) is a hyperbolic infinite-order mapping.*

Proof. Let us consider the case $F_S \in \mathcal{G}_{OP}$; the orientation-reversing case can be treated similarly. It is easy to see that F_S must have infinite order since all the entries in M_S^n tend to $+\infty$ as $n \to \infty$, where $M_S \in PSL(2, \mathbb{Z})$ is the corresponding matrix. Thus, $\mathrm{tr}(M_S^n) \to \infty$ as $n \to \infty$. On the other hand, one verifies that the following equalities hold in $PSl(2, \mathbb{Z})$,

$$(M_q M_v)^n = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}, \ (M_{q^{-1}} M_v)^n = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}.$$

In particular, $\mathrm{tr}((M_q M_v)^n) = \mathrm{tr}((M_q M_v)^n) = 1$ for every n. Thus, a primitive S cannot induce a parabolic matrix. □

We end this section by a proposition summarizing the essential part of the above classifications.

Proposition 3.8. *Let S be a primitive invertible substitution and let F be a trace map associated to S. Then, there is an open set Esc_F such that, if the orbit of an initial point $\mathbf{x} \in \mathbb{R}^3$ under F is unbounded, it necessarily enters Esc_F. Moreover, the orbit remains in Esc_F and $(F^n(\mathbf{x}))$ diverges super-exponentially in every coordinate.*

Proof. The claim follows by combining the results contained in Propositions 3.4, 3.5, and 3.6 along with Lemma 3.7. □

4. Spectrum, bounded orbits, and vanishing Lyapunov exponents

As explained above, Theorem 2.1 follows once the inclusions (2) and (3) are established. If the substitution S obeys the assumptions of Theorem 2.1, then, as was discussed in the preceding section, the F_S-orbit of an initial point (x, y, z) either remains bounded or diverges super-exponentially in every coordinate.

Lemma 4.1. *Suppose S obeys the assumptions of Theorem 2.1. Then, inclusion* (2) *holds, that is, the sets Σ, B_∞ induced by S satisfy $\Sigma \subseteq B_\infty$.*

Proof. Let $F = F_S$ be the trace map arising from S and consider the orbit of the initial point $\mathbf{x}_E$ (cf. equation 17) for $E \in \mathbb{R}$. In the preceding section we have identified an open region Esc_F associated to F such that the orbit of $\mathbf{x}_E$ must enter Esc_F if it is unbounded (cf. Proposition 3.8). In this case every coordinate is diverging super-exponentially. Define

$$Esc_{F,m}(a) = \{E \in \mathbb{R} \, : \, |x_E(m)| > 1\},$$

$$Esc_{F,m}(b) = \{E \in \mathbb{R} \, : \, |y_E(m)| > 1\},$$

and

$$Esc_{F,m}(ab) = \{E \in \mathbb{R} \, : \, |z_E(m)| > 1\}.$$

Obviously, all the sets $Esc_{F,m}(a), Esc_{F,m}(b), Esc_{F,m}(ab)$ are open. Let us discuss the case where the representation of F is Type I; the other cases are similar. From the explicit form of Esc_F (cf. Section 3), we get

$$\begin{aligned}
B_\infty^c &\subseteq \bigcup_{k\in\mathbb{N}} \{E \, : \, F^k(\mathbf{x}_E) \in Esc_F\} \\
&= \bigcup_{k\in\mathbb{N}} \{E \, : \, F^k(\mathbf{x}_E) \in T_1\} \\
&= \bigcup_{k\in\mathbb{N}} Int\left(\{E \, : \, F^k(\mathbf{x}_E) \in T_1\}\right) \\
&= \bigcup_{k\in\mathbb{N}} \mathrm{Int}\left(\bigcap_{m\geq k} \{E \, : \, F^m(\mathbf{x}_E) \in T_1\}\right) \\
&\subseteq \bigcup_{k\in\mathbb{N}} \mathrm{Int}\left(\bigcap_{m\geq k} (Esc_{F,m}(a) \cap Esc_{F,m}(b))\right) \\
&\subseteq \bigcup_{k\in\mathbb{N}} \mathrm{Int}\left(\bigcap_{m\geq k} Esc_{F,m}(a)\right) \cap \bigcup_{k\in\mathbb{N}} \mathrm{Int}\left(\bigcap_{m\geq k} Esc_{F,m}(b)\right) \\
&\subseteq \Sigma^c,
\end{aligned}$$

where, in the last inclusion, we have used a result that essentially follows from a standard strong approximation argument. In a form suitable to our setting, we provide this result in Lemma 4.2 below. □

Lemma 4.2. *Let S be a primitive invertible substitution on a two-letter alphabet. Then, we have*

$$\bigcup_{k\in\mathbb{N}} \mathrm{Int}\left(\bigcap_{m\geq k} Esc_{F,m}(a)\right) \cup \bigcup_{k\in\mathbb{N}} \mathrm{Int}\left(\bigcap_{m\geq k} Esc_{F,m}(b)\right) \cup \bigcup_{k\in\mathbb{N}} \mathrm{Int}\left(\bigcap_{m\geq k} Esc_{F,m}(ab)\right) \subseteq \Sigma^c.$$

Proof. Choose $\omega \in \Omega$ arbitrarily and consider the operator H_ω. By primitivity of S we have

1. $\Sigma = \sigma(H_\omega)$,
2. for every C, the word $\omega_{-C} \dots \omega_C$ is contained in some $S^m(a)$,
3. for every C, the word $\omega_{-C} \dots \omega_C$ is contained in some $S^m(b)$, and
4. for every C, the word $\omega_{-C} \dots \omega_C$ is contained in some $S^m(ab)$.

Thus, by looking at Hamiltonians

$$(H_{x,m}\phi)(n) = \phi(n+1) + \phi(n-1) + V_{x,m}(n)\phi(n)$$

in $l^2(\mathbb{Z})$, where $x \in \{a, b, ab\}$, $V_{x,m}$ is periodic with period $|S^m(x)|$ taking the values $f(S^m(x))$ on its period (where, for some word $c_1 \dots c_k \in \{a,b\}^*$, $f(c_1 \dots c_k)$ just equals $f(c_1) \dots f(c_k)$), and the period interval located to satisfy the above local coincidence with V_ω, we see that, for every $x \in \{a, b, ab\}$, H_ω is the strong limit of $H_{x,m}$, $m \to \infty$. Thus, we obtain $\bigcup_{k\in\mathbb{N}} \mathrm{Int}(\bigcap_{m\geq k} \rho(H_{x,m})) \subseteq \Sigma^c$ [33]. The assertion now follows from $\rho(H_{x,m}) = Esc_{F,m}(x)$ [33]. □

Lemma 4.3. *Suppose S obeys the assumptions of Theorem 2.1. Then, inclusion (3) holds, that is, the sets B_∞, A induced by S satisfy $B_\infty \subseteq A$.*

Proof. Assume there exists $E \in B_\infty$ such that $\gamma(E) > 0$. Consider an arbitrary $\omega \in \Omega$. By Osceledec's theorem [11] there exists a solution ϕ_+ of (7) such that $\|\Phi_+(n)\|$ decays exponentially at $+\infty$ at the rate $\gamma(E)$, where $\Phi_+(n) = (\phi_+(n+1), \phi_+(n))^T$. Now, at least one of the words aa, bb necessarily occurs in the substitution sequence u. Let us look at the case where aa is a subword of u; the other case can be treated similarly. Since $E \in B_\infty$ there is a constant $C \geq 1$ such that, for every $k \in \mathbb{N}$, we have

$$|\mathrm{tr}(M_E(S^k(a)))| \leq C. \tag{30}$$

By $S(u) = u$ all the words $S^k(a)S^k(a)$, $k \in \mathbb{N}$, occur in u with a positive frequency. Pick n_0 such that, for every $n \geq n_0$ and every $m \in \mathbb{N}$, the solution ϕ_+ obeys

$$\|\Phi_+(n+m)\| \leq \exp(-\tfrac{1}{2}\gamma(E)m)\|\Phi_+(n)\|. \tag{31}$$

Choose k such that $\exp(-\frac{1}{2}\gamma(E)|S^k(a)|) < \frac{1}{2C}$. Now look for an occurrence of $f(S^k(a)S^k(a))$ in V_ω, that is, $f(S^k(a)S^k(a)) = V_\omega(l+1) \dots V_\omega(l + 2|S^k(a)|)$, such that $l \geq n_0$. Then, by (15), we have

$$\Phi_+(l + 2|S^k(a)|) - \mathrm{tr}(M_E(S^k(a)))\Phi_+(l + |S^k(a)|) + \Phi_+(l) = 0, \tag{32}$$

which in turn implies by (30)

$$\max(\|\Phi_+(l + |S^k(a)|)\|, \|\Phi_+(l + 2|S^k(a)|)\|) \geq \frac{1}{2C}\|\Phi_+(l)\|, \tag{33}$$

contradicting (31). □

Proof of Theorem 2.1. Lemma 4.1 and Lemma 4.3 together with (11) imply the following chain of inclusions,

$$\Sigma \subseteq B_\infty \subseteq A \subseteq \Sigma,$$

thus establishing equality of the three sets, as was claimed in Theorem 2.1. □

5. Singular continuous spectrum with probability one

The proof of Theorem 2.4 essentially follows the lines of [14]. However, by property **BO**, the three-block method used as a criterion to exclude eigenvalues in [14] can be replaced by the two-block method, both being variants of an idea originally due to Gordon [18]. The variants suitable to substitution potentials can be found in [36, 17] in two-block and three-block form, respectively. We shall only sketch the argument here and refer the reader to [14, 16] for further details. Define

$$C = \{\omega \in \Omega \ : \ \sigma_{\text{pp}}(H_\omega) = \emptyset\}.$$

Our goal is to prove, under the assumptions of Theorem 2.4,

$$\mu(C) = 1. \tag{34}$$

Proof of Theorem 2.4. Using ergodicity of μ and shift-invariance of C, it is easily seen that it suffices to exhibit Borel sets $G(n)$, $n \in \mathbb{N}$, such that these sets obey

$$\limsup G(n) \subseteq C \tag{35}$$

as well as

$$\limsup \mu(G(n)) > 0. \tag{36}$$

To this end we define

$$G(n) = \{\omega \in \Omega \ : \ \omega_1 \dots \omega_{2|S^{k+n}(c)|} \text{ is a subword of } S^{k+n}(ccc)\}.$$

It is easy to check that the $G(n)$ are Borel sets. Moreover, using **BO** and (15), one establishes (35) in a similar fashion as (33) was obtained (compare [15, 36]). Then one shows as in [14] that $\limsup \mu(G(n))$ can be bounded from below by a positive constant times the frequency of the word $S^k(ccc)$ in a substitution sequence associated to S. Using the fact that the frequency of every word which occurs in a fixed point of a primitive substitution is always positive [32], we infer (36), and hence (34), concluding the proof of the theorem. □

References

[1] Allouche, J.-P., Peyrière, J. : Sur une formule de récurrence sur les traces de produits de matrices associés a certaines substitutions, *C. R. Acad. Sci. Paris* **302**, 1135–1136 (1986)

[2] Baake, M., Grimm, U., Joseph, D. : Trace maps, invariants, and some of their applications, *Int. J. Mod. Phys. B* **7**, 1527–1550 (1993)

[3] Baake, M., Roberts, J. A. G. : Reversing symmetry group of $GL(2,\mathbb{Z})$ and $PGL(2,\mathbb{Z})$ matrices with connections to cat maps and trace maps, *J. Phys. A* **30**, 1549–1573 (1997)

[4] Bellissard, J., Bovier, A., Ghez, J.-M. : Spectral properties of a tight binding Hamiltonian with period doubling potential, *Commun. Math. Phys.* **135**, 379–399 (1991)

[5] Bellissard, J., Bovier, A., Ghez, J.-M. : Gap labelling theorems for one-dimensional discrete Schrödinger operators, *Rev. Math. Phys.* **4**, 1–37 (1992)

[6] Bellissard, J., Iochum, B., Scoppola, E., Testard, D. : Spectral properties of one-dimensional quasi-crystals, *Commun. Math. Phys.* **125**, 527–543 (1989)
[7] Bovier, A. : private communication
[8] Bovier, A., Ghez, J.-M. : Spectral properties of one-dimensional Schrödinger operators with potentials generated by substitutions, *Commun. Math. Phys.* **158**, 45–66 (1993)
[9] Bovier, A., Ghez, J.-M. : Erratum. Spectral properties of one-dimensional Schrödinger operators with potentials generated by substitutions, *Commun. Math. Phys.* **166**, 431–432 (1994)
[10] Casdagli, M. : Symbolic dynamics for the renormalization group of a quasiperiodic Schrödinger equation, *Commun. Math. Phys.* **107**, 295–318 (1986)
[11] Cycon, H. L., Froese, R., Kirsch, W., Simon, B. : *Schrödinger Operators, with Application to Quantum Mechanics and Global Geometry*, Springer, Berlin-Heidelberg-New York (1987)
[12] Damanik, D. : α-continuity properties of one-dimensional quasicrystals, *Commun. Math. Phys.* **192**, 169–182 (1998)
[13] Damanik, D. : Singular continuous spectrum for the period doubling Hamiltonian on a set of full measure, *Commun. Math. Phys.* **196**, 477–483 (1998)
[14] Damanik, D. : Singular continuous spectrum for a class of substitution Hamiltonians, *Lett. Math. Phys.* **46**, 303–311 (1998)
[15] Damanik, D., Lenz, D. : Uniform spectral properties of one-dimensional quasicrystals, I. Absence of eigenvalues, *Commun. Math. Phys.* **207** (1999), 687–696
[16] Damanik, D., Lenz, D. : Half-line eigenfunction estimates and singular continuous spectrum of zero Lebesgue measure, preprint
[17] Delyon, F., Petritis, D. : Absence of localization in a class of Schrödinger operators with quasiperiodic potential, *Commun. Math. Phys.* **103**, 441–444 (1986)
[18] Gordon, A. : On the point spectrum of the one-dimensional Schrödinger operator, *Usp. Math. Nauk* **31**, 257–258 (1976)
[19] Hof, A. : Some remarks on discrete aperiodic Schrödinger operators, *J. Stat. Phys.* **72**, 1353–1374 (1993)
[20] Hof, A., Knill, O., Simon, B. : Singular continuous spectrum for palindromic Schrödinger operators, *Commun. Math. Phys.* **174**, 149–159 (1995)
[21] Iochum, B., Testard, D. : Power law growth for the resistance in the Fibonacci model, *J. Stat. Phys.* **65**, 715–723 (1991)
[22] Iochum, B., Raymond, L., Testard, D. : Resistance of one-dimensional quasicrystals, *Physica A* **187**, 353–368 (1992)
[23] Jitomirskaya, S., Last, Y. : Dimensional Hausdorff properties of singular continuous spectra, *Phys. Rev. Lett.* **76**, 1765–1769 (1996)
[24] Jitomirskaya, S., Last, Y. : Power law subordinacy and singular spectra, II. Line operators, *Commun. Math. Phys.* **211** (2000), 643–658
[25] Kaminaga, M. : Absence of point spectrum for a class of discrete Schrödinger operators with quasiperiodic potential, *Forum Math.* **8**, 63–69 (1996)
[26] Kolár, M., Nori, F. : Trace maps of general substitutional sequences, *Phys. Rev. B* **42**, 1062–1065 (1990)
[27] Kotani, S. : Jacobi matrices with random potentials taking finitely many values, *Rev. Math. Phys.* **1**, 129–133 (1989)
[28] Kurosh, A.G. : *The Theory of Groups*, Vol. 2, Chelsea, New York (1960)
[29] Last, Y., Simon, B. : Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators, *Invent. Math.* **135**, 329–367 (1999)
[30] Magnus, W., Karrass, A., Solitar, D. : *Combinatorial Group Theory*, Dover, New York (1976)
[31] Peyrière, J., Wen, Z.-Y., Wen, Z.-X. : Polynomes associés aux endomorphismes de groupes libres, *Enseign. Math.* **39**, 153–175 (1993)
[32] Queffélec, M. : *Substitution Dynamical Systems - Spectral Analysis*, Lecture Notes in Mathematics, Vol. 1284, Springer, Berlin-Heidelberg-New York (1987)
[33] Reed, M., Simon, B. : *Methods of Modern Mathematical Physics, Vol. IV: Analysis of Operators*, Academic Press, New York (1977)
[34] Roberts, J.A.G. : Escaping orbits in trace maps, *Physica A* **228**, 295–325 (1996)
[35] Roberts, J.A.G., Baake, M. : Trace maps as 3D reversible dynamical systems with an invariant, *J. Stat. Phys.* **74**, 829–888 (1994)
[36] Sütő, A. : The spectrum of a quasiperiodic Schrödinger operator, *Commun. Math. Phys.* **111**, 409–415 (1987)

[37] Sütő, A. : Singular continuous spectrum on a Cantor set of zero Lebesgue measure for the Fibonacci Hamiltonian, *J. Stat. Phys.* **56**, 525–531 (1989)