Equivariant Sternberg Theorem

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Abstract

Smooth vector fields with linear symmetries and anti-symmetries are considered. We prove that provided symmetry group is compact then a smooth conjugacy in Sternberg Theorem can be chosen symmetric.

Introduction

The well known Sternberg Theorem (see [St]) asserts that if two local smooth vector fields are formally conjugate at a hyperbolic singularity then they are smoothly conjugate. This result reduces local classification and normalization problems to the formal ones.

In the last years there is a splash of interest to systems with symmetries and anti-symmetries (*reversible systems*) (for references see [LR1]). This activity faces, in particular, a similar "equivariant" problem: could one provide a conjugacy (smooth, formal) of two symmetric or anti-symmetric vector fields via a transformation keeping the property? The linear and the formal aspects of the problem were considered in [LR2, Ga]. But, as far as the authors know, a related version of Smooth Conjugacy Sternberg Theorem has not been published yet, though some recent papers make a more or less explicit use of this result (see [AfM, CHa]). Moreover, in [Bo] the following problem is posed: does a smooth vector field anti-commuting with a linear involution R (*R-reversible*) admits a smooth R-symmetric normalization.

The aim of the present paper is to prove an equivariant version of Sternberg Theorem (Theorem 2.4 below). As a corollary this gives the positive answer to the mentioned question in [Bo], and contains simultaneously the main results of [Bo] and [S2]. Actually, we prove a more general result ("modulo central manifold"). First it was stated in [B1] for local diffeomorphisms and then in [S1] for vector fields (see also [BrK1, IL]).

We use the so-called *deformation method* or *homotopy trick* going back to J.Mather (see [M, AVG, BrK2]). This method reduces a nonlinear problem on equivalence of two vector fields (or more generally, of two local mappings near a common singularity) to that on solving a linear functional equation (*cohomology equation*). We give here a complete proof of its solvability in C^∞ case. A similar equation related to conjugacy of local diffeomorphisms was considered first in [B2].

1 Definitions

By a *local diffeomorphism* (*local vector field*) we mean in what follows either a germ of diffeomorphisms (vector fields) at a point or a representative of this germ defined in a neighborhood of the point. Sometimes, we omit for briefness the word "local".

Let $\mathfrak{G}$ be a group of local diffeomorphisms $\Phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $\sigma : \mathfrak{G} \to \mathbb{R}$ be a *multiplicative character* of $\mathfrak{G}$, *i.e.*, a homomorphism into the multiplicative group $\mathbb{R}^*$.

Definition. *A local vector field ξ is said to be $(\mathfrak{G}, \sigma)$-equivariant if*

$$G_*\xi = \sigma(G)\xi \quad (G \in \mathfrak{G}).$$

In this definition, $G_*\xi$ denotes a usual action of the local diffeomorphism G on the vector field ξ:

$$(G_*\xi)(x) = DG(x)\xi(G^{-1}x),$$

where $DG(x)$ is the Jacobi matrix.

A local transformation H keeps the property of ξ to be $(\mathfrak{G}, \sigma)$-equivariant if it commutes with every element $G \in \mathfrak{G}$, *i.e.*, $GH = HG$. Such a local diffeomorphism is said to be $\mathfrak{G}$-*equivariant.*

Similar definition may be applied to the *formal* category. Namely, let $\mathfrak{G}$ be a group of *formal* transformations of $\mathbb{R}^n$ and $\sigma : \mathfrak{G} \to \mathbb{R}$ be a multiplicative character. A formal vector field $\hat{\xi}$ is said to be $(\mathfrak{G}, \sigma)$-*equivariant* if

$$G_*\hat{\xi} = \sigma(G)\hat{\xi} \quad (G \in \mathfrak{G}).$$

A formal diffeomorphism $\hat{T}$ is called $\mathfrak{G}$-*equivariant* if it commutes with every $G \in \mathfrak{G}$.

In what follows, we assume that $\mathfrak{G}$ is *a compact group of linear operators in* $\mathbb{R}^n$ and σ *is continuous.* Hence, $\mathfrak{G}$ can be equally regarded as a group of local diffeomorphisms or as of formal ones. Without loss of generality $\mathfrak{G}$ is supposed to be a closed subgroup of the group $O(n, \mathbb{R})$ of all orthogonal transformaions. Thus, every element $G \in \mathfrak{G}$ keeps the standard inner product in $\mathbb{R}^n$. And besides, the continuity of σ implies

$$\sigma(G) = \pm 1 \quad (G \in \mathfrak{G}).$$

Let ξ be a local (or a formal) $(\mathfrak{G}, \sigma)$-equivariant vector field, $\xi(0) = 0$, and

$$\Lambda \equiv D\xi(0)$$

be its *linearization* at the origin. Then Λ is a $(\mathfrak{G}, \sigma)$-equivariant transformation of $\mathbb{R}^n$:

$$G\Lambda G^{-1} = \sigma(G)\Lambda \quad (G \in \mathfrak{G}).$$

Since $G \in \mathfrak{G}$ keeps the standard inner product, the conjugate operator Λ^* is $(\mathfrak{G}, \sigma)$-equivariant as well.

2 Main results

Recall that two local (formal) vector fields ξ and η are said to be C^k (formally) conjugate if there exists a C^k (formal) diffeomorphism Φ such that

$$\Phi_* \xi \;=\; \eta.$$

2.1. Formal normal form. The following assertion can be proved via usual formal power series technique (cf. [B3, Ga]).

Theorem 2.1. *Let $\hat{\xi}$ be a $(\mathfrak{G}, \sigma)$-equivariant formal vector field,*

$$\hat{\xi}(0) \;=\; 0, \quad D\hat{\xi}(0) \;=\; \Lambda.$$

Then there exists a $\mathfrak{G}$-equivariant formal diffeomorphism Φ bringing $\hat{\xi}$ to the normal form

$$\Phi_* \hat{\xi} \;=\; \Lambda \;+\; \tau,$$

where the formal series τ contains no terms of order less than 2 and commutes with Λ^*.

Let

$$\mathbb{R}^n \;=\; E_s \,\oplus\, E_u \,\oplus\, E_c$$

be a Λ-invariant splitting such that all the real parts of the eigenvalues of the restriction $\Lambda_s \,\equiv\, \Lambda|E_s$ ($\Lambda_u \,\equiv\, \Lambda|E_u$; $\Lambda_c \,\equiv\, \Lambda|E_c$) are negative (positive; equal to zero). If a formal vector field $\hat{\xi} \;=\; \Lambda \,+\, \ldots$ is in the normal form then the subspaces $E_s \oplus E_c$ and $E_u \oplus E_c$ are invariant with respect to the formal phase flow of $\hat{\xi}$.

Now we pass to the C^∞ case making use of the "equivariant" Borel Lemma:

Let Γ be a formal $\mathfrak{G}$-equivariant diffeomorphism. Then there is a local $\mathfrak{G}$-equivariant diffeomorphism $\Phi \,\in\, C^\infty$ such that its Taylor series $\hat{\Phi}$ at the origin is equal to Γ.

Similarly, *given a $(\mathfrak{G}, \sigma)$-equivariant formal vector field τ there is a local $(\mathfrak{G}, \sigma)$-equivariant vector field $\xi \,\in\, C^\infty$ such that its Taylor series $\hat{\xi}$ at the origin is equal to τ.* Moreover, *if the subspaces $E_s \oplus E_c$ and $E_u \oplus E_c$ are invariant with respect to the formal phase flow of τ then ξ may be chosen with the same property.*

Since the group $\mathfrak{G}$ consits of orthogonal transformations one can prove all assertions of the equivariant Borel Lemma in a usual way (see, for example, [H]).

As a result we get

Corollary 2.2. *Given a local $(\mathfrak{G}, \sigma)$-equivariant C^∞ vector field η,*

$$\eta(0) \;=\; 0, \quad D\eta(0) \;=\; \Lambda,$$

there exist a local $\mathfrak{G}$-equivariant C^∞ diffeomorphism Φ and a local $(\mathfrak{G}, \sigma)$-equivariant C^∞ vector field ξ such that

a) *the formal Taylor series $\hat{\xi}$ at the origin is in the normal form;*

b) *the subspaces $E_u \oplus E_c$ and $E_s \oplus E_c$ are invariant with respect to the phase flow of ξ;*

c) *the formal Taylor series $\widehat{\Phi_* \xi}$ and $\hat{\eta}$ at the origin coincide.*

Speaking otherwise, the vector fields ξ and η "are conjugate up to a flat remainder", *i.e.*, all derivatives of the difference $\Phi_* \xi - \eta$ vanish at the origin. Usually, the main functional problem is "to kill" the flat remainder.

2.2. Smooth conjugacy. Fix a local $(\mathfrak{G}, \sigma)$-equivariant C^∞ vector field ξ, $\xi(0) = 0$, $D\xi(0) = \Lambda$, such that the subspaces $E_u \oplus E_c$ and $E_s \oplus E_c$ are invariant with respect to the phase flow of ξ.

Denote by δ_u and Δ_u (respectively, by $-\Delta_s$ and $-\delta_s$) the minimum and the maximum of the real parts of the eigenvalues of the restriction $\Lambda | E_u$ (respectively, $\Lambda | E_s$). Given an integer k put

$$\rho(k, \Lambda) = \max \left\{ \frac{\Delta_s + k\Delta_u}{\delta_s}, \frac{\Delta_u + k\Delta_s}{\delta_u} \right\} + k + 1.$$

Theorem 2.3. *Let η be a local $(\mathfrak{G}, \sigma)$-equivariant C^K vector field, $\eta(0) = 0$, $K \geq \rho(k, \Lambda)$ and all derivatives of the difference $\xi - \eta$ up to the order K vanish on E_c. Then there is a local $\mathfrak{G}$-equivariant C^k diffeomorphism H which conjugates ξ and η, i.e. $H_* \xi = \eta$.*

In particular, if $K = \infty$ then $k = \infty$ as well.

This theorem is the main result of the paper. Its proof is given in Sections 3 – 5.

It follows from the conditions of Theorem 2.3 that $D\eta(0) = D\xi(0) = \Lambda$ and that the subspace E_c is invariant with respect to the phase flow of η. It follows from the conclusion of Theorem 2.3 that η has local invariant submanifolds M_1 and M_2 which are C^k diffeomorphic to the subspaces $E_u \oplus E_c$ and $E_s \oplus E_c$ respectively.

2.3. Hyperbolic case. Recall that ξ said to be *hyperbolic* at the origin if $E_c = 0$, *i.e.*, Λ has no eigenvalues in the imaginary axis.

Two local C^∞ vector fields ξ and η are said to be *formally $\mathfrak{G}$-conjugate* if there exists a formal $\mathfrak{G}$-equivariant transformation Γ such that

$$\Gamma_* \hat{\xi} = \hat{\eta}.$$

Theorem 2.4. *Let ξ and η be local $(\mathfrak{G}, \sigma)$-equivariant C^∞ vector fields having a hyperbolic singularity at the origin. If they are formally $\mathfrak{G}$-conjugate then there is a local $\mathfrak{G}$-equivariant C^∞ diffeomorphism H wich conjugates ξ and η.*

Proof. By Corollary 2.2 one can suppose that the subspaces $E_u \oplus E_c$ and $E_s \oplus E_c$ are ξ-invariant. Let Γ be a formal $\mathfrak{G}$-equivariant transformation conjugating $\hat{\xi}$ and $\hat{\eta}$, and let T be a local C^∞ diffeomorphism with the Taylor series $\hat{T} = \Gamma$. Then the vector field $\zeta = T_* \eta$ is $(\mathfrak{G}, \sigma)$-equivariant and the difference $\zeta - \xi$ is flat at the origin. According to Theorem 2.3 ($E_c = 0$), the vector fields ζ and ξ are conjugate via a $\mathfrak{G}$-equivariant local C^∞ diffeomorphism H. Hence,

$$(HT)_* \eta = \xi$$

and HT is $\mathfrak{G}$-equivariant as well. $\square$

Theorem 2.4 itself is Equivariant Sternberg Theorem. Together with Corollary 2.2 it provides a construction of equivariant smooth normal forms.

Samovol [S2] considers the special case of Theorem 2.4 when $\mathfrak{G}$ is formed by powers of a given operator J of a finite order (J^m = identity for some m), and $\sigma(G) = 1$ for all $G \in \mathfrak{G}$. He shows also that the assumption of compactness for $\mathfrak{G}$ is essential.

As another special case we derive from Theorem 2.1 the following answer to the question posed in [Bo].

Given a linear transformation $U \in GL(n, \mathbb{R})$, a vector field ξ is called *U-reversible* if

$$U_* \xi = -\xi.$$

Corollary 2.5. *Let U be an orthogonal transformation, ξ and η be local hyperbolic U-reversible C^∞ vector fields. Assume that there is a formal transformation Γ commuting with U such that $\Gamma_* \hat{\xi} = \hat{\eta}$. Then one can find a local C^∞ diffeomorphism H which commutes with U and conjugates ξ and η.*

Proof. Put

$$\mathfrak{G} \equiv \overline{\bigcup_{m \in \mathbb{Z}} \{U^m\}}, \qquad \mathfrak{G}_0 \equiv \overline{\bigcup_{k \in \mathbb{Z}} \{U^{2k}\}}, \qquad \mathcal{A} \equiv \overline{\bigcup_{k \in \mathbb{Z}} \{U^{2k+1}\}}.$$

Since U is orthogonal then $\mathfrak{G}$ is a compact subgroup of $GL(n, \mathbb{R})$, $\mathfrak{G}_0$ is a normal subgroup of $\mathfrak{G}$ and $\mathfrak{G} = \mathfrak{G}_0 \cup \mathcal{A}$. One can easily see that given $G \in \mathfrak{G}_0$, $H \in \mathcal{A}$ we get $G_* \xi = \xi$, $H_* \xi = -\xi$. Since $\xi \not\equiv 0$ then $\mathfrak{G}_0$ and $\mathcal{A}$ do not intersect. Besides,

$$G \in \mathfrak{G}_0, \quad H \in \mathcal{A} \Rightarrow GH \in \mathcal{A}, \quad HG \in \mathcal{A}$$

and

$$H_1, H_2 \in \mathcal{A} \Rightarrow H_1 H_2 \in \mathfrak{G}_0.$$

Therefore, by putting

$$\sigma(G) = 1 \quad (G \in \mathfrak{G}_0), \quad \sigma(H) = -1 \quad (H \in \mathcal{A})$$

we get a continuous character σ of the compact subgroup $\mathfrak{G} \subset O(n)$.

The vector fields ξ and η turn out to be $(\mathfrak{G}, \sigma)$-equivariant, and the formal conjugacy Γ — $\mathfrak{G}$-equivariant. It remains to apply Theorem 2.4. $\square$

3 Deformation method.

As we have mentioned above, the deformation method reduces a conjugacy problem to that of solving a multidimensional linear equation.

Given vector fields ξ and η consider the *deformation* $h(x, \varepsilon) \equiv \varepsilon \eta(x) + (1 - \varepsilon) \xi(x)$ and the vector field

$$\Xi \equiv h(x, \varepsilon) \frac{\partial}{\partial x} + 0 \cdot \frac{\partial}{\partial \varepsilon}$$

on $\mathbb{R}^{n+1}$. Take another vector field

$$\Phi(x,\varepsilon) \equiv \varphi(x,\varepsilon)\frac{\partial}{\partial x} + 1\cdot\frac{\partial}{\partial\varepsilon}$$

Let

$$Q^t(x,\varepsilon) = (q^t(x,\varepsilon),\varepsilon), \quad P^t(x,\eta) = (p^t(x,\varepsilon),\varepsilon + t)$$

be the pase flows of Ξ and Φ respectively. Put $H(x) = p^1(x,0)$.

Lemma 3.1. *Let the vector fields Φ and Ξ commute, i.e.*

$$[\,\Phi,\Xi\,] = 0. \tag{1}$$

Then H conjugates ξ and η, i.e., $H_*\xi = \eta$.

Proof. Note that the phase flows of ξ and η are, respectively, $q_0^t(x) \equiv q^t(x,0)$ and $q_1^t(x) \equiv q^t(x,1)$. Since the phase flows P^t and Q^t commute then

$$Q^{t_2}(P^{t_1}(x,\varepsilon)) = (q^{t_2}(p^{t_1}(x,\varepsilon),\varepsilon+t_1),\varepsilon+t_1) = (p^{t_1}(q^{t_2}(x,\varepsilon),\varepsilon),\varepsilon+t_1) = P^{t_1}(Q^{t_2}(x,\varepsilon)),$$

or

$$q^{t_2}(p^{t_1}(x,\varepsilon),\varepsilon+t_1) = p^{t_1}(q^{t_2}(x,\varepsilon),\varepsilon).$$

For $\varepsilon = 0$, $t_1 = 1$, $t_2 = t$ this gives

$$q^t(p^1(x,0),1) = p^1(q^t(x,0),0).$$

For $H(x) = p^1(x,0)$ we get

$$q^t(H(x),1) = H(q^t(x,0)),$$

or

$$q_1^t(H(x)) = H(q_0^t(x)),$$

i.e., H is a conjugacy between ξ and η. □

Condition (1) is equivalent to the equation

$$D_1\varphi(x,\varepsilon)\cdot h(x,\varepsilon) = D_1 h(x,\varepsilon)\cdot\varphi(x,\varepsilon) + \eta(x) - \xi(x), \tag{2}$$

where

$$D_1 \equiv \frac{\partial}{\partial x}.$$

This equation is called the *cohomology equation.*

4 $\mathfrak{G}$-equivariant conjugacy

Thus, we have shown that the deformation method reduces a conjugacy problem to that of finding a smooth solution $\varphi(x,\varepsilon)$ of the linear non-homogeneous partial differential equation (2), and then to integrating the system $\dot{x} = \varphi(x,\varepsilon)$, $\dot{\varepsilon} = 1$. The problem of solvability of the cohomology equation is considered in Section 5. Here we assume that a local C^k solution φ exists, and prove $\mathfrak{G}$-equivariant conjugacy between ξ and η.

Lemma 4.1. *If φ is a C^k solution of equation* (2) *then for every $U \in \mathfrak{G}$ the map*

$$(U_*^{-1}\varphi)(x,\varepsilon) \equiv U^{-1}\varphi(Ux,\varepsilon),$$

is a solution of equation (2) *as well.*

Proof. Denote $\tilde{\varphi} = U_*^{-1}\varphi$. By virtue of $(\mathfrak{G},\sigma)$-equivariance of ξ and η, the following equalities are true:

$$\begin{aligned} Uh(x,\varepsilon) &= \sigma(U)h(Ux,\varepsilon); \\ UD_1h(x,\varepsilon) &= \sigma(U)D_1h(Ux,\varepsilon)U; \\ U(\eta-\xi)(x,\varepsilon) &= \sigma(U)(\eta-\xi)(Ux,\varepsilon). \end{aligned}$$

Hence,

$$\begin{aligned} D_1\tilde{\varphi}(x,\varepsilon)h(x,\varepsilon) &= U^{-1}D_1\varphi(Ux,\varepsilon)Uh(x,\varepsilon) \\ &= \sigma(U)U^{-1}D_1\varphi(Ux,\varepsilon)h(Ux,\varepsilon) \\ &= \sigma(U)U^{-1}[D_1h(Ux,\varepsilon)\varphi(Ux,\varepsilon)+(\eta-\xi)(Ux,\varepsilon)] \\ &= U^{-1}[\sigma(U)D_1h(Ux,\varepsilon)]\varphi(Ux,\varepsilon)+U^{-1}[\sigma(U)(\eta-\xi)(Ux,\varepsilon)] \\ &= U^{-1}UD_1h(x,\varepsilon)U^{-1}\varphi(Ux,\varepsilon)+U^{-1}U(\eta-\xi)(x,\varepsilon) \\ &= D_1h(x,\varepsilon)\tilde{\varphi}(x,\varepsilon)+(\eta-\xi)(x,\varepsilon). \end{aligned}$$

Therefore, $\tilde{\varphi}$ is a solution of the cohomology equation. □

It is well known that since $\mathfrak{G}$ is compact then one can find a Haar measure μ on $\mathfrak{G}$ which is right invariant with respect to $\mathfrak{G}$. We can suppose without loss of generality that $\mu(\mathfrak{G}) = 1$. Let φ be a C^k solution of equation (2). Define a C^k function $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ by

$$\psi(x,\varepsilon) = \int_{\mathfrak{G}} U^{-1}\varphi(Ux,\varepsilon)d\mu(U) \qquad (x \in \mathbb{R}^n,\ \varepsilon \in \mathbb{R}). \tag{3}$$

Lemma 4.2. *The function ψ is a $\mathfrak{G}$-equivariant solution of the cohomology equation* (2).

Proof. By linearity of the cohomology equation, it follows from Lemma 5.1 that ψ is a solution. Since μ is right invariant, for any $V \in \mathfrak{G}$ we have

$$V^1\Psi(Vx,\varepsilon) = V^{-1}\int_{\mathfrak{G}} U^{-1}\varphi(UVx,\varepsilon)d\mu(U) =$$

$$= \int_{\mathfrak{G}}(UV)^{-1}\varphi(UVx,\varepsilon)d\mu(U) = \int_{\mathfrak{G}} W^{-1}\varphi(Wx,\varepsilon)d\mu(W) = \psi(x,\varepsilon).$$

Here we put $W = UV$. As a result we get

$$V^{-1}\psi(Vx,\varepsilon) = \psi(x,\varepsilon),$$

i.e., ψ is $\mathfrak{G}$ - equivariant. $\square$

Proof of Theorem 2.3. Take a local C^k solution φ of equation (2). Then the function ψ given by formula (3) is a local $\mathfrak{G}$-equivariant C^k solution of (2). Let $F^t(x,\varepsilon) \equiv (f^t(x,\varepsilon), \varepsilon + t)$ be the phase flow of the vector field

$$\Psi(x,\varepsilon) = \psi(x,\varepsilon)\frac{\partial}{\partial x} + 1 \cdot \frac{\partial}{\partial \varepsilon}.$$

Put $H(x) = f^1(x,0)$. Since $[\Psi, \Xi] = 0$ then, by Lemma 3.1, the local C^k diffeomorphism H conjugates ξ and η. Show that H is $\mathfrak{G}$-equivariant. Choose $U \in \mathfrak{G}$ and denote $\tilde{f}^t(x,\varepsilon) \equiv U^{-1}f^t(Ux,\varepsilon)$. Then

$$\left.\frac{d\tilde{f}^t(x,\varepsilon)}{dt}\right|_{t=0} = U^{-1}\left.\frac{df^t(Ux,\varepsilon)}{dt}\right|_{t=0} = U^{-1}\psi(Ux,\varepsilon) = \psi(x,\varepsilon).$$

Hence $\tilde{F}^t(x,\varepsilon) \equiv (\tilde{f}^t(x,\varepsilon), \varepsilon + t)$ is also a phase flow of the vector field Ψ. It follows from Uniqueness Theorem that $\tilde{F}^t = F^t$ and $H(x) = U^{-1}H(Ux)$. We conclude that H is $\mathfrak{G}$-equivariant completing the proof of Theorem 2.3. $\square$

5 Solution of the cohomology equation

Thus it remains to show that the family of linear non-homogeneous partial differential equations

$$D_1\varphi(x,\varepsilon)h(x,\varepsilon) = A(x,\varepsilon)\varphi(x,\varepsilon) + \gamma(x) \qquad (x \in \mathbb{R}^n,\ \varepsilon \in I = [0,1]) \tag{4}$$

has a local solution of the corresponding smoothness class. We consider here in detail the C^∞ case. It should be noted that the method of proof applied can be easily transfered to a finitely smooth case (with some natural modifications in definitions of semi-norms and in estimates).

We suppose in what follows that the functions h, A and γ are all C^∞ smooth. Since the problem under consideration is local, we assume without loss of generality that the functions A and γ have compact supports, and the vector field h takes the form

$$h(x,\varepsilon) = \Lambda x + g(x,\varepsilon),$$

where the function g has a finite support as well. Remind that the space $\mathbb{R}^n$ is splitted into the direct sum of Λ-invariant linear subspaces

$$\mathbb{R}^n = E_s \oplus E_u \oplus E_c$$

in such a way that all the real parts of the eigenvalues of the operator $\Lambda_s \equiv \Lambda|E_s$ ($\Lambda_u \equiv \Lambda|E_u$; $\Lambda_c \equiv \Lambda|E_c$) are negative (positive; equal to zero). Accept the notation

$$x \equiv (u,v,w) \in E_s \oplus E_u \oplus E_c.$$

Denote by $F^t(x,\varepsilon)$ the phase flow of the vector field

$$h(x,\varepsilon) \;=\; (\Lambda x \;+\; g(x,\varepsilon))\frac{\partial}{\partial x},$$

and assume that the subspaces $E_u \oplus E_c$ and $E_s \oplus E_c$ are F^t - invariant for all $\varepsilon \in I$.

Lemma 5.1. *Let all the derivatives of a C^∞ vector function γ vanish on E_c, i.e.,*

$$D^k\gamma(0,0,w) \;=\; 0 \qquad (k \;=\; 0,1,\;\ldots\,).$$

Then the equation (4) *has a C^∞ solution $\varphi(x,\varepsilon)$ defined in a neighborhood of the interval $I \subset \mathbb{R}^{n+1}$.*

Proof. One can choose a number $\alpha > 0$ such that

$$\begin{aligned} \|\mathrm{pr}_{E_s} F^t(x,\varepsilon)\| &\leq \|u\|e^{-\alpha t} \qquad (t>0),\\ \|\mathrm{pr}_{E_u} F^t(x,\varepsilon)\| &\leq \|v\|e^{\alpha t} \qquad (t<0). \end{aligned}$$

Show that for each $k = 1,2,\ldots$ there is a number $C_k > 0$ such that

$$\|D_1^i F^t(x,\varepsilon)\| \leq C_k e^{C_k t} \qquad (1 \leq i \leq k;\; x \in \mathbb{R}^n,\; \varepsilon \in I,\; t > 0).$$

Consider first the case $k = 1$. Observe that

$$\frac{d(D_1F^t(x,\varepsilon))}{dt} = \Big(\Lambda + D_1 g(F^t(x,\varepsilon),\varepsilon)\Big)\, D_1F^t(x,\varepsilon).$$

Therefore

$$D_1F^t(x,\varepsilon) = \left. D_1F^t(x,\varepsilon)\right|_{t=0} + \int_0^t (\Lambda + D_1 g(F^s(x,\varepsilon),\varepsilon))\, D_1F^s(x,\varepsilon)ds.$$

Hence for all $t > 0$

$$\|D_1F^t(x,\varepsilon)\| \leq B + C\int_0^t \|D_1F^s(x,\varepsilon)\|ds.$$

By Gronwall inequality, this gives

$$\|D_1F^t(x,\varepsilon)\| \leq Be^{Ct} \leq C_1e^{C_1 t} \qquad (t>0).$$

The same arguments work in a general case. Taking into account that

$$\frac{d(D_1^kF^t(x,\varepsilon))}{dt} = \Big(\Lambda + D_1 g(F^t(x,\varepsilon),\varepsilon)\Big)\, D_1^kF^t(x,\varepsilon) + R(x,\varepsilon,t),$$

where the function R is polynomial with respect to the functions $D_1^iF^t(x,\varepsilon)$ $(i = 1,\ldots,k-1)$, we conclude that

$$\|D_1^kF^t(x,\varepsilon)\| \leq De^{Mt} + C\int_0^t \|D_1^kF^s(x,\varepsilon)\|ds.$$

Apply again Gronwall inequality and get the desired estimate.

Prove that the function γ can be decomposed into the sum of two C^∞ functions which are flat, respectively, on $E_u \oplus E_c$ and on $E_s \oplus E_c$, *i.e.*

$$\gamma = \gamma_+ + \gamma_-,$$

and

$$D^k\gamma_+(0,v,w) = D^k\gamma_-(u,0,w) = 0 \qquad (k = 1,2,\ldots;\ (u,v,w) \in \mathbb{R}^n).$$

In fact, by Whitney Theorem, one can find a C^∞ function γ_+ such that

$$D^k\gamma_+(0,v,w) = 0, \quad D^k\gamma_+(u,0,w) = D^k\gamma(u,0,w) \qquad (k = 1,2,\ldots;\ (u,v,w) \in \mathbb{R}^n).$$

Put $\gamma_- = \gamma - \gamma_+$. Then

$$D^k\gamma_-(u,0,w) = 0 \qquad (k = 1,2,\ldots;\ (u,v,w) \in \mathbb{R}^n).$$

Note that since the cohomology equation has an affine structure, suffice it to find solutions φ_+ and φ_- of the equations

$$D_1\varphi_\pm(x,\varepsilon)h(x,\varepsilon) = A(x,\varepsilon)\varphi_\pm(x,\varepsilon) + \gamma_\pm(x).$$

Then a solution we are looking for is given by

$$\varphi = \varphi_+ + \varphi_-.$$

Show that the equation

$$D_1\varphi_+(x,\varepsilon)h(x,\varepsilon) = A(x,\varepsilon)\varphi_+(x,\varepsilon) + \gamma_+(x) \tag{5}$$

has a C^∞ solution. Define a linear operator $\mathcal{L}$ by

$$(\mathcal{L}\tau)(x,\varepsilon) = -\int_0^\infty \tau(F^t(x,\varepsilon),\varepsilon)dt.$$

Let $\mathfrak{M}_+ \subset C^\infty(\mathbb{R}^n \times I, \mathbb{R}^n)$ be the set of functions which are flat on $E_u \oplus E_c$, and let $\mathfrak{N}_+ \subset \mathfrak{M}_+$ be the subset of functions having compact supports. For a positive integer k and a positive ν put

$$\|\varphi\|_{k\nu} = \sup_{\mathbb{R}^n\times I}\max_{j=0,\ldots,k}\frac{\|D_1^j\varphi(x,\varepsilon)\|}{\|u\|^\nu}, \quad x = (u,v,w).$$

By $\mathfrak{M}^k_{+\nu}$ ($\mathfrak{N}^k_{+\nu}$) denote the set of C^∞ functions (having compact supports) such that $\|\varphi\|_{k\nu} < \infty$.

Let us prove *that the operator $\mathcal{L}$ transfers $\mathfrak{N}_+$ into $\mathfrak{M}_+$*. In fact, let $\tau \in \mathfrak{N}_+$, $\varphi = \mathcal{L}\tau$. Put

$$\tau_t(x,\varepsilon) = \tau(F^t(x,\varepsilon),\varepsilon).$$

Then

$$\begin{aligned}\|D_1^k\tau_t(x,\varepsilon)\| &\le \|D_1^k\tau(F^t(x,\varepsilon),\varepsilon)\|\|D_1F^t(x,\varepsilon)\|^k \\ &\quad + \sum_{j=1}^{k-1}\sum_p E_{jp}\|D_1^j\tau(F^t(x,\varepsilon),\varepsilon)\|\|D_1^{p_1}F^t(x,\varepsilon)\|\cdots\|D_1^{p_j}F^t(x,\varepsilon)\|,\end{aligned}$$

where $p = (p_1, \ldots, p_j)$, $p_1 + \ldots p_s = k$ and E_{jp} are some constants. Since $\tau \in \mathfrak{N}_+$ then $\tau \in \mathfrak{N}^k_{+\nu}$ for all k and ν. The prvevious estimate gives

$$\begin{aligned}\frac{\|D_1^k \tau_t(x,\varepsilon)\|}{\|u\|^\nu} &\leq \frac{\|D_1^k \tau(F^t(x,\varepsilon),\varepsilon)\|}{\|\mathrm{pr}_{E_s} F^t(x,\varepsilon)\|^\nu} \cdot \frac{\|\mathrm{pr}_{E_s} F^t(x,\varepsilon)\|^\nu}{\|u\|^\nu} \|D_1 F^t(x,\varepsilon)\|^k \\ &+ \sum_{j=1}^{k-1} \sum_p E_{jp} \frac{\|D_1^j \tau(F^t(x,\varepsilon),\varepsilon)\|}{\|\mathrm{pr}_{E_s} F^t(x,\varepsilon)\|^\nu} \cdot \frac{\|\mathrm{pr}_{E_s} F^t(x,\varepsilon)\|^\nu}{\|u\|^\nu} \\ &\times \|D_1^{p_1} F^t(x,\varepsilon)\| \cdots \|D_1^{p_j} F^t(x,\varepsilon)\|.\end{aligned}$$

Hence

$$\begin{aligned}\|\tau_t\|_{k\nu} &\leq \|\tau\|_{k\nu} e^{-\alpha\nu t} C_k^k e^{C_k k t} + \sum_p \sum_{j=1}^{k-1} E_{jp} \|\tau\|_{k\nu} e^{-\alpha\nu t} C_k^{k-1} e^{C_k(k-1)t} \\ &\leq R_k \|\tau\|_{k\nu} e^{(-\alpha\nu + C_k k)t}.\end{aligned}$$

Since

$$\|D_1^k \tau_t(x,\varepsilon)\| \leq \|\tau_t\|_{k\nu} \|u\|^\nu$$

we have

$$\|D_1^k \varphi(x,\varepsilon)\| \leq R_k \|\tau\|_{k\nu} \|u\|^\nu \int_0^\infty e^{(-\alpha\nu + C_k k)t} dt.$$

Choose ν_k in such a way that if $\nu \geq \nu_k$ then $-\alpha\nu + C_k k < 0$. Then the integral converges and

$$\|D_1^k \varphi(x,\varepsilon)\| \leq R_k \|\tau\|_{k\nu} \|u\|^\nu \frac{1}{|-\alpha\nu + C_k k|}.$$

We conclude that $\varphi \in \mathfrak{M}^k_{+\nu}$ for all k and $\nu \geq \nu_k$. Hence $\varphi \in C^\infty$ and $D^k\varphi(0,v,w,\varepsilon) = 0$ $(k = 0,1,\ldots;\ (v,w,\varepsilon) \in E_u \oplus E_c \times I)$. Therefore $\varphi \in \mathfrak{M}_+$.

Now consider the parial differential equation

$$D_1 \varphi(x,\varepsilon) h(x,\varepsilon) = \tau(x,\varepsilon). \tag{6}$$

Chek that the formula

$$\varphi = \mathcal{L}\tau$$

gives a solution of equation (6).

In fact,

$$\begin{aligned}\varphi(F^s(x,\varepsilon),\varepsilon) &= -\int_0^\infty \tau(F^{t+s}(x,\varepsilon),\varepsilon)dt = -\int_s^\infty \tau(F^t(x,\varepsilon),\varepsilon)dt \\ &= -\int_0^\infty \tau(F^t(x,\varepsilon),\varepsilon)dt + \int_0^s \tau(F^t(x,\varepsilon),\varepsilon)dt \\ &= \varphi(x,\varepsilon) + \int_0^s \tau(F^t(x,\varepsilon),\varepsilon)dt.\end{aligned}$$

Differentiate both parts of the equality with respect to s, and put $s=0$. We get

$$D_1\varphi(F^s(x,\varepsilon),\varepsilon)\cdot \left.\frac{dF^s(x,\varepsilon)}{ds}\right|_{s=0} = \tau(F^s(x,\varepsilon),\varepsilon)|_{s=0},$$

or

$$D_1\varphi(x,\varepsilon)h(x,\varepsilon) = \tau(x,\varepsilon).$$

Now define a linear operator $\mathcal{T}$ by the formula

$$(\mathcal{T}\varphi)(x,\varepsilon) = A(x,\varepsilon)\varphi(x,\varepsilon) + \gamma_+(x).$$

Thus we reduce our problem to solving the equation

$$\varphi = \mathcal{LT}\varphi. \tag{7}$$

Prove that one can find a C^∞ solution which is flat on $E_u \oplus E_c$. Since $\gamma_+ \in \mathfrak{N}^k_{+\nu}$ for every k and ν, and the function A has a compact support, the operator $\mathcal{T}$ transfers $\mathfrak{M}^k_{+\nu}$ into $\mathfrak{N}^k_{+\nu}$, and besides,

$$\|\mathcal{T}\varphi\|_{k\nu} \le P_k\|\varphi\|_{k\nu} + \|\gamma_+\|_{k\nu}$$

for some constants P_k. Therefore,

$$\|\mathcal{LT}\varphi\|_{k\nu} \le \frac{Q_k}{|-\alpha\nu + C_k k|}(\|\varphi\|_{k\nu} + \|\gamma_+\|_{k\nu})$$

for some constants Q_k. Clearly, $\mathcal{LT}$ transfers $\mathfrak{M}_+$ to $\mathfrak{M}_+$. In the space $C^\infty(\mathbb{R}^n \times I, \mathbb{R}^n)$ equiped with the compact-open topology, we are going to choose a set which is convex and compact with respect to this topology, and such that the operator $\mathcal{LT}$ maps it into itself. To this end, for each $k = 1, 2, \ldots$ take a number ν_k in such a way that

$$\frac{Q_k}{|-\alpha\nu + C_k k|} \le \frac{1}{2}.$$

Denote

$$a_k = \frac{Q_k}{|-\alpha\nu + C_k k|}\|\gamma_+\|_{k\nu_k}.$$

Put

$$M_k = 3a_k.$$

Set

$$\mathcal{M}_k = \{\varphi \in \mathfrak{M}_+ :\ \|\varphi\|_{k\nu_k} \le M_k\} \qquad (k = 1, 2, \ldots),$$

and

$$\mathcal{M} = \bigcap_k \mathcal{M}_k.$$

The set $\mathcal{M}$ is a convex compact subset of $\mathfrak{M}_+$. Therefore $\mathcal{LT}(\mathcal{M}) \subset \mathcal{M}$, and by Tihonov Fixed Point Theorem [T] the set $\mathcal{M}$ contains a fixed point φ_0 of the operator $\mathcal{LT}$. Clearly φ_0 is a C^∞ solution of equation (7).

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