SLOW MOTION AND METASTABILITY FOR A NON LOCAL EVOLUTION EQUATION

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Abstract. In this paper we consider a non local evolution equation in one dimension, which describes the dynamics of a ferromagnetic system in the mean field approximation. In the presence of a small external magnetic field, this equation admits two stationary homogeneous solutions, which represent the stable and metastable phases of the physical system. We prove the existence of an invariant, one dimensional manifold connecting the stable and metastable phases. This is the unstable manifold of a distinguished, spatially non homogeneous, stationary solution of the evolution equation, called the critical droplet. We show that the points on the manifold are droplets longer or shorter than the critical one, and that their motion is very slow in agreement with the theory of metastable patterns. The existence of the critical droplet was firstly proved in [7], but no uniqueness result was guaranteed by that approach. However we give here a different proof, which is also supplied with a local uniqueness result. Finally, a detailed description of the spatial structure of the critical droplet is given, which will be also a key tool to study the global structure of the unstable manifold.

1. Introduction

There are many examples in physics of metastable states like supercooled liquids, supersaturated vapors and solutions, or ferromagnets with magnetization opposite to the field. The phenomenon of metastability occurs in thermodynamic systems close to a first order phase transition. A system is initially prepared in an equilibrium pure phase, and then the thermodynamic parameters are changed to values for which there is a different equilibrium pure phase. Under suitable conditions, the system does not undergo a phase transition, but it remains in an apparent equilibrium, the *metastable state*, which is very similar to the initial one. This state persists until some (even slight) disturbance leads the system to the stable equilibrium. For instance we can prepare a glass of water in an environment at temperature slightly below zero degrees centigrade. The glass looks apparently in an equilibrium liquid phase, and this state may last for a long time; but eventually an irreversible process takes place and the glass of water suddenly freezes.

A first theoretical explanation of metastability goes back to the classical van der Waals Maxwell theory of liquid-vapor phase transition. In this theory the notion of metastable states emerges in a natural way, as describing homogeneous, almost equilibrium phases which however have higher free energy with respect to the corresponding stable equilibria.

As a matter of fact, the appearance of metastable states is a common feature of all the microscopic *mean field theories* of phase transition. In fact, due to the *mean*

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field approximation, the range of the interaction between the particles coincides with the macroscopic size of the system, and this fact implies the non convexity of the free energy, thus allowing metastable branches in the phase diagram.

However, the mean field approximation is unrealistic, and in fact it produces nonphysical features like thermodynamic instabilities and the (already mentioned) non convexity of the free energy.

A more realistic mean field approximation, called the *local mean field limit*, has been introduced in the 60's by Kac, Uhlenbeck, and Hemmer, [15]. They consider interactions, usually called *Kac potentials*, that depend on a scaling parameter $\gamma > 0$, studying the limit $\gamma \downarrow 0$ where the range of the interaction becomes infinite. This program has been carried out by Kac, Uhlenbeck and Hemmer, [15], in some particular models, and then by Lebowitz and Penrose in a more general class of systems, [16]. These results prove that the phase diagram converges, for any temperature, to the van der Waals phase diagram, comprehensive of the Maxwell equal area rule.

Because of the dynamical nature of the phenomenon, a fully satisfactory description of metastability can be found only in the framework of non equilibrium statistical mechanics. In this setting the first rigorous approach to metastability goes back to Lebowitz and Penrose, [17], who give a general method for describing metastable states. Moreover, they apply this method to systems with Kac potentials, giving a rigorous justification (for what concerns the static properties) of the van der Waals description of metastable states.

In more recent years, [8], Ising spin systems with Glauber dynamics and Kac potentials have been introduced in order to analyze non equilibrium phenomena like phase separation and interface dynamics. In this model the mean field free energy density is given by

$$F(s) = -\frac{1}{2}s^2 - hs - \beta^{-1} i(s), \quad s \in [-1,1], \tag{1.1}$$

$$i(s) = -\frac{1+s}{2}\log\frac{1+s}{2} - \frac{1-s}{2}\log\frac{1-s}{2}, \tag{1.2}$$

where β is the inverse temperature, s the average magnetization, and h an external positive magnetic field. The quadratic term $-s^2/2$ is the internal energy density, $-hs$ the energy density due to the field h, and $i(s)$ the entropy. The critical points of $F(s)$ are the solutions of the mean field equation:

$$s = \tanh\{\beta[s+h]\}. \tag{1.3}$$

Given $\beta > 1$, there is an $h_\beta > 0$ such that for $h \in [0, h_\beta]$ Eq. (1.3) has three and only three different roots, denoted by

$$m^-_{\beta,h} < m^0_{\beta,h} \le 0 < m^+_{\beta,h}. \tag{1.4}$$

For $h > 0$, $|m^-_{\beta,h}| < m^+_{\beta,h}$ and $m^0_{\beta,h} < 0$; for $h = 0$, $m^0_{\beta,h} = 0$ and $m^+_{\beta,h} = -m_{\beta,h} \doteq m_\beta$. The two phases $\pm m_\beta$ are thermodynamically stable at $h = 0$, while $m^0_{\beta,h} = 0$ is unstable. For $h > 0$, $m^+_{\beta,h}$ is the only stable phase, $m^0_{\beta,h}$ is still unstable, while $m^-_{\beta,h}$ becomes metastable.

However these are static considerations, the dynamics of persistence and decay of the metastable state can be investigated only going back to the stochastic evolution of the underlying spin system, by characterizing the tunneling from the metastable to the stable phase.

According to general heuristic arguments we expect the transition occurs through the *nucleation* of a sufficiently large droplet of the stable phase, which will start to grow undergoing an irreversible process leading to the stable phase everywhere. On the contrary, small droplets will have a tendency to shrink. This arguing leads to believe that the transition occurs through the formation of a well defined *critical droplet*, which breaks the spatial homogeneity in the metastable state.

A specific feature of stochastic dynamics with Kac potentials is its almost deterministic behavior for small values of the scaling parameter γ (i.e. when the range of the interaction is large). In fact in [8] it is shown that in the continuum limit the Ising spin system with Glauber dynamics and Kac potentials gives rise to a local magnetization density $m = m(t,x)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, which evolves according to the non local evolution equation:

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta[J * m + h]\}, \tag{1.5}$$

where J is a non negative, even function which is related to the (long range) coupling of the spin-spin interaction, and $J * m$ denotes the convolution between J and m. In agreement with (1.1) we also assume that J is normalized so that

$$\int dx\, J(x) = 1, \tag{1.6}$$

which implies the homogeneous stationary solutions of (1.5) to coincide with those of (1.3).

This paper is concerned with the metastable behavior of the deterministic evolution equation (1.5) in dimension $d = 1$. We prove the existence of an invariant manifold connecting the stable and metastable phases. This manifold is the union of two heteroclinic orbits, connecting the above phases with a distinguished, non homogeneous, stationary solution of (1.5), which here represents the critical droplet.

Our ultimate purpose is the complete description of the tunneling from the metastable to the stable phase for the underlying stochastic spin dynamics. According to the pathwise approach to metastability in the case of reversible dynamics, [4, 13, 18], the present analysis will play a central role in determining the typical path of the tunneling transition, which will be the subject of future works.

In [7] it is proved the existence, for $\beta > 1$ and h small enough, of the critical droplet, also called the *bump*, i.e. a spatially non homogeneous, symmetric solution q of the non local, one dimensional equation:

$$q(x) = \tanh\{\beta[(J * q)(x) + h]\}, \qquad x \in \mathbb{R}, \tag{1.7}$$

with asymptotic conditions

$$\lim_{|x|\to\infty} q(x) = m^-_{\beta,h}, \tag{1.8}$$

which is therefore a stationary solution of (1.5). When h is very small the region where q is close to the stable phase is of order $|\log h|$, while the length of the transition layers from this region to the one where the metastable phase dominates is of order 1. The layers may be described approximately as standing waves of (1.5) with $h = 0$. These are translations and/or reflections of the *instanton* $\bar m(x)$, a strictly increasing and antisymmetric function solving:

$$\bar m(x) = \tanh\{\beta(J * \bar m)(x)\}, \qquad x \in \mathbb{R}, \tag{1.9}$$

with asymptotic conditions

$$\lim_{x\to\pm\infty} \bar{m}(x) = \pm m_\beta. \tag{1.10}$$

All the translations of q are stationary solutions, and it is not known whether these are the only (spatially non homogeneous) solutions of (1.7) with asymptotic conditions (1.8). In [7] the existence of the bump is proved by applying the Newton method to find the zeros of the map defined (on the space of continuous and symmetric functions) by the r.h.s. of (1.5). Consequence of this approach is the lack of any uniqueness result.

We partially fill up this gap here by giving a different proof which is also supplied with a local uniqueness result (in the space of symmetric functions, see below). We also give a detailed description of the spatial structure of the bump, by showing it is a strictly decreasing function for $x > 0$, converging exponentially fast to the metastable phase as $|x| \to +\infty$.

The existence of an invariant, one dimensional, unstable manifold $\mathcal{W}$ through q, follows in a standard way (see for instance [14]) from the existence, proved in [6], of an isolated, simple, positive eigenvalue of the operator obtained by linearizing (1.7) around the critical droplet q. Nevertheless we give in Sect. 6 an explicit construction needed to establish the other results. This manifold consists of the union of two branches $\mathcal{W}_\pm$. The points on $\mathcal{W}_-$ (resp. $\mathcal{W}_+$) are symmetric functions, non increasing for $x > 0$, strictly smaller (resp. larger) than q. We thus refer to the points on $\mathcal{W}_-$ (resp. $\mathcal{W}_+$) as the sub-critical (resp. super-critical) droplets. In the branch $\mathcal{W}_-$ where the length of the droplets is shorter than that of q, the evolution shrinks it further, while it grows if it is larger. This shrinking (resp. growing) process goes on indefinitely and we actually prove the branch $\mathcal{W}_-$ (resp. $\mathcal{W}_+$) connects the bump with the metastable (resp. stable) phase. We prove this last result by using the comparison theorem which holds for Eq. (1.5). We construct suitable upper solutions for the sub-critical droplet and lower solutions for the super-critical one that converge to the metastable, respectively stable, phase as t diverges. This method does not allow to catch the actual speed of convergence to the metastable and stable phases, which will be the subject of a future work.

However we have a detailed description of large part of the relaxation process. The situation is in fact similar to that of the invariant manifolds for metastable patterns in solutions of the singularly perturbed Ginzburg-Landau equation, [2, 3, 12]. We show that, for h very small, a large part, $\overline{\mathcal{W}}$, of the unstable manifold $\mathcal{W}$ consists of droplets with two well-separated transition layers, whose patterns are described approximately by the standing waves. The dynamics on $\overline{\mathcal{W}}$ is then reduced to the motion of the layer locations, $\pm\xi$, which can be described, to high accuracy, by the ordinary differential equation:

$$\dot{\xi} = -\mu K e^{-2\alpha\xi} + 2m_\beta\, \mu\, h, \tag{1.11}$$

where the positive coefficients K, α, and μ depend on J and β. Thus the velocity turns out to be the sum of two terms. The first one is an attractive term, due to the interaction between the layers, in agreement to the analogous effect proved for the singularly perturbed Ginzburg-Landau equation. The second one is a constant, positive drift, which is linear in the small magnetic field h. It coincides with the first order expansion of the velocity for the traveling wave solutions of (1.5), see (2.10) in the next section. Since the motion is very slow (for large ξ), following [12] we refer to $\overline{\mathcal{W}}$ as a *slow motion manifold*. The velocity field has a zero at

$(2\alpha)^{-1}\log[K(2m_\beta h)^{-1}]$ which gives, at the lower order, (one half of) the length of the critical droplet.

Our approach is based on the geometric method suggested by G. Fusco and J. Hale, [12], and developed by J. Carr and R. Pego, [2, 3], for the singularly perturbed Ginzburg-Landau equation (see also the more recent paper [11] by J.-P. Eckmann and J. Rougemont). By restricting our analysis to the space of symmetric functions (which is an invariant set for the dynamics), we first construct an "approximately invariant" manifold $\mathcal{M}$ of quasi-stationary states $n_\xi(x) \approx \bar{m}(\xi - |x|)$, for $\xi > 0$ in a suitable interval and h small. The motion near $\mathcal{M}$ is described in terms of coordinates along and transversal to $\mathcal{M}$ (the Fermi coordinates). This manifold contains the essential dynamics: due to a strong transverse stability, the solutions near $\mathcal{M}$ are squeezed into a *thin channel* which contains $\mathcal{M}$ and where the layer motion is described with high accuracy by (1.11). We next prove that there exists an unique stationary solution of Eq. (1.5) near $\mathcal{M}$, which is then identified with the critical droplet q and it turns out to be strictly contained in the thin channel. The slow motion manifold $\overline{\mathcal{W}}$ is thus defined by that part of $\mathcal{W}$ inside the channel.

In the next section we state the main definitions and results and we give an outline of the paper.

2. Definitions and results

We first state the assumptions on the interaction J appearing in (1.5). The function $J \in C^3(\mathbb{R})$ is symmetric, non negative, and satisfying (1.6). Moreover $\sup\{x \in \mathbb{R} : J(x) > 0\} = 1$ and $J'(x) < 0$ for $x \in (0,1)$.

In the whole paper we consider the evolution defined by (1.5) as an equation in the space $L_\infty(\mathbb{R};[-1,1])$, which we rewrite as:

$$\frac{dm}{dt} = f(m), \tag{2.1}$$

where

$$f(m) \doteq -m + \tanh\{\beta[J * m + h]\}. \tag{2.2}$$

We observe that the Cauchy problem has a unique solution in $L_\infty(\mathbb{R})$ because the map f is uniformly Lipschitz continuous. Moreover, since $|\tanh z| < 1$ for all $z \in \mathbb{R}$, the set $L_\infty(\mathbb{R};[-1,1])$ is invariant for the dynamics. Analogously there exists a unique solution of the Cauchy problem in $C_0(\mathbb{R})$, the space of continuous and bounded functions on $\mathbb{R}$, and the set $C(\mathbb{R},[-1,1])$ is left invariant. We denote by $S_t(m)$ the flow solution with initial datum m, so that S_t defines a semi-group on $L_\infty(\mathbb{R};[-1,1])$ for which $C(\mathbb{R},[-1,1])$ is an invariant (closed) subspace.

Since J is a symmetric function, by uniqueness, the evolution preserves the parity of the initial datum. In particular the space $C^{\mathrm{sym}}(\mathbb{R},[-1,1])$ of symmetric, continuous functions with range $[-1,1]$ is an invariant set.

In the sequel we will often need to study the dependence of $S_t(m)$ on the initial datum m. A first estimate is:

$$\|S_t(m+u) - S_t(m)\|_\infty \le e^{k_1 t}\|u\|_\infty, \tag{2.3}$$

where $k_1 > 0$ is the Lipschitz coefficient of f, i.e. for any $m, u \in L_\infty(\mathbb{R})$,

$$\|f(m+u) - f(m)\|_\infty \le k_1\|u\|_\infty. \tag{2.4}$$

For a more refined bound, let L_m be the linear operator defined by

$$L_m u = -u + p_m J * u, \qquad p_m \doteq \frac{\beta}{\cosh^2\{\beta J * m\}}. \tag{2.5}$$

Then L_{m+h} is the derivative $Df|_m$ of $f(m)$ at m, so that:

$$f(m+u)-f(m)-L_{m+h}u = \beta^2 (J*u)^2 \int_0^1 ds\, (1-s) \tanh''\{\beta[J*(m+su)+h]\}, \tag{2.6}$$

from which it follows that there exists $k_2 > 0$ such that:

$$\|f(m+u) - f(m) - L_{m+h}u\|_\infty \le k_2 \|u\|_\infty^2. \tag{2.7}$$

Stationary solutions. **(**[5, 6, 7, 9, 10]**).** Given $\beta > 1$, there exists a unique (modulo translations) solution $\bar m(x)$ of (1.9) with asymptotes (1.10), which we call the *instanton*. It is a C^∞, strictly increasing, antisymmetric function. Moreover, letting $\alpha > 0$ be such that

$$\beta(1-m_\beta^2)\int dz\, J(z) e^{-\alpha z} = 1, \tag{2.8}$$

there are $a > 0$, $\alpha_0 > \alpha$, and $c > 0$ so that, for all $x \ge 0$,

$$\left|\bar m(x) - (m_\beta - a e^{-\alpha x})\right| + \left|\bar m'(x) - \alpha a e^{-\alpha x}\right| + \left|\bar m''(x) + \alpha^2 a e^{-\alpha x}\right| \le c e^{-\alpha_0 x}, \tag{2.9}$$

where $\bar m'$ and $\bar m''$ are respectively the first and second derivatives of $\bar m$.

For small $h > 0$, the existence of traveling waves solutions of (1.7), connecting the stable and metastable phases $m_{\beta,h}^\pm$, can be obtained by using a perturbative argument around the instanton. These are solutions of the form $\tilde m_h(x - v(h)t)$, where $\tilde m_h(x)$ is a strictly increasing function of x with asymptotes $m_{\beta,h}^\pm$ at $\pm\infty$ and $\|\tilde m_h - \bar m\|_\infty = \mathrm{O}(h)$. The velocity of the front $v(h)$ is a negative, strictly decreasing function of h, such that:

$$\lim_{h\to 0^+} \frac{v(h)}{h} = -2 m_\beta\, \mu, \qquad \mu \doteq \left[\int dy\, \frac{\bar m'(y)^2}{\beta(1-\bar m(y)^2)}\right]^{-1}. \tag{2.10}$$

On a physical background, μ is the linear transport coefficient which represents the *mobility* of an interface separating the stable phases, see e.g. [1] and [19] for details. We refer to [5] for stability and other properties of fronts.

When $h > 0$ is small it is natural to conjecture the existence of the critical droplet in a neighborhood of the symmetric function

$$\bar m_\xi(x) \doteq \bar m(\xi - |x|). \tag{2.11}$$

This is proved in [7], where the Newton method is used to find the zeros of the map $f(m)$ on $C^{\mathrm{sym}}(\mathbb{R}, [-1,1])$ (a good starting point of the Newton map being chosen in such a neighborhood). The result is summarized in the following theorem.

Theorem 2.1. **(**[7]**)** *Given $\beta > 1$, there is $h_0 > 0$ and, for any $h \in (0, h_0]$, there is $q \in C^{\mathrm{sym}}(\mathbb{R}; [-1,1])$ which solves* (1.7) *with asymptotes as in* (1.8)*. Moreover there is $\xi^* = \xi^*(h)$ such that:*

$$\lim_{h\downarrow 0} \|q - \bar m_{\xi^*}\|_\infty = 0 \tag{2.12}$$

and

$$\lim_{h\downarrow 0} e^{2\alpha \xi^*(h)} h = (2m_\beta)^{-1} K, \tag{2.13}$$

where

$$K \doteq \frac{a}{\beta(1-m_\beta^2)} \int dx \, \frac{e^{\alpha x}\bar{m}'(x)(m_\beta^2 - \bar{m}^2(x))}{1-\bar{m}^2(x)}, \tag{2.14}$$

with α *defined by* (2.8) *and* a *as in* (2.9).

The quasi-invariant manifold. We now focus on the dynamics (2.1) for $h > 0$ small, when restricted to the class $L_\infty^{\mathrm{sym}}(\mathbb{R};[-1,1])$. We are interested in the behavior of solutions which consist of two well separated layers, i.e. which belong to a neighborhood of $\{\bar{m}_\xi; \xi > \ell\}$ ($\bar{m}_\xi$ defined in (2.11)) for ℓ large enough. According to (2.13), we may expect the interesting part of the relaxation process is caught by restricting our analysis to patterns with layers at a distance no too larger than $(2\alpha)^{-1}|\log h|$. In fact layers far apart from each other interact too weakly and the effect of the magnetic field predominates (one could prove they simply go away with an almost constant velocity, each one eventually approaching a traveling wave). With this in mind, for any $\ell > 1$, $h < e^{-2\alpha\ell}$, and $\kappa > 0$, we introduce the interval

$$\Gamma_{\ell,\kappa} \doteq \{\xi \,:\, \ell < \xi < (2\alpha)^{-1}|\log h| + \kappa\}. \tag{2.15}$$

Observe that for $\xi \in \Gamma_{\ell,\kappa}$ one has:

$$h < e^{2\alpha\kappa}e^{-2\alpha\xi}, \tag{2.16}$$

i.e. $h = O(e^{-2\alpha\xi})$.

Theorem 2.2. *Given* $\beta > 1$, *there is* $\ell_0 > 1$ *such that, for any* $h < e^{-2\alpha\ell_0}$, *there exists a* C^1*–map:*

$$\Gamma_{\ell_0,+\infty} \ni \xi \longmapsto n_\xi \in C^{\mathrm{sym}}(\mathbb{R};[-1,1]) \tag{2.17}$$

with the following properties. For some $\delta_0 > 0$ *and any* $\kappa > 0$, *there is* $c_0 = c_0(\kappa) > 0$ *such that, for any* $\xi \in \Gamma_{\ell_0,\kappa}$,

$$\|n_\xi(x) - \bar{m}_\xi(x)\|_\infty \;\leq\; c_0 e^{-\alpha\xi}, \tag{2.18}$$

$$\|\partial_\xi n_\xi(x) - \bar{m}_\xi'(x)\|_\infty \;\leq\; c_0 e^{-\alpha\xi}, \tag{2.19}$$

$$\|f(n_\xi) - V(\xi)\,\partial_\xi n_\xi\|_\infty \;\leq\; c_0 e^{-(2\alpha+\delta_0)\xi}, \tag{2.20}$$

with

$$V(\xi) \doteq -\mu K e^{-2\alpha\xi} + 2m_\beta\,\mu\,h \tag{2.21}$$

and α, μ, *and* K *as in* (2.8), (2.10), *and* (2.14) *respectively.*

This theorem will be proved in Sect. 8 where the function n_ξ is explicitly given in Definition 8.2. For any $\ell \geq \ell_0$ and $\kappa > 0$ we refer to

$$\mathcal{M}_{\ell,\kappa} \doteq \{n_\xi : \xi \in \Gamma_{\ell,\kappa}\} \tag{2.22}$$

as the *quasi-invariant manifold*: by (2.20), $f(n_\xi)$, $n_\xi \in \mathcal{M}_{\ell,\kappa}$, is approximately tangent to $\mathcal{M}_{\ell,\kappa}$ and very small.

In the sequel we shall need a good control on the spectral properties of the linear operator $Df|_{n_\xi} = L_{n_\xi+h}$, where L_m is defined in (2.5).

In [6] several properties about the spectrum of the linear operator L_m are proved when m is close to [a slightly modification of] the reflected instanton $\bar{m}_\xi(x)$ (ξ large). We shall use these results to prove Theorem 2.2 and Proposition 2.3 below, details are given in Sect. 8.

Proposition 2.3. *Let n_ξ, $\xi \in \Gamma_{\ell_0,\kappa}$, be as in Theorem 2.2 and set $L_\xi \doteq L_{n_\xi+h} = Df|_{n_\xi}$, $p_\xi \doteq p_{n_\xi+h}$. For any $\kappa > 0$ there are $\ell_1 \ge \ell_0$, $c_1 > 1$, and $\omega_1 > 0$ such that, for any $h < e^{-2\alpha\ell_1}$ and $\xi \in \Gamma_{\ell_1,\kappa}$, the following holds.*

1) There exist $\lambda_\xi > 0$ and strictly positive functions $v_\xi, v_\xi^* \in C_0^{\rm sym}(\mathbb{R})$ so that:

$$L_\xi v_\xi = \lambda_\xi v_\xi, \qquad v_\xi^* L_\xi = \lambda_\xi v_\xi^*, \qquad v_\xi^* = p_\xi v_\xi. \tag{2.23}$$

Moreover

$$\left\| e^{L_\xi t} \right\|_\infty \le c_1 e^{\lambda_\xi t} \qquad \forall\, t \ge 0. \tag{2.24}$$

2) We have:

$$c_1^{-1} e^{-2\alpha\xi} \le \lambda_\xi \le c_1 e^{-2\alpha\xi} \tag{2.25}$$

and, for $\alpha' = \alpha'(\xi) \doteq \alpha - c_1 e^{-2\alpha\xi}$,

$$v_\xi(x) \le c_1 e^{-\alpha'|\xi - x|} \qquad \forall\, x \in \mathbb{R}_+. \tag{2.26}$$

Moreover, calling

$$\tilde m(x) \doteq \sqrt{\mu}\, \bar m(x), \qquad \tilde m_z(x) \doteq \tilde m(z - |x|), \quad z > 0, \tag{2.27}$$

(μ as in (2.10)) we also have:

$$\left| v_\xi(x) - \tilde m_\xi'(x) \right| \le c_1 e^{-2\alpha\xi + \alpha|\xi - x|} \xi^4 \qquad \text{for} \quad |\xi - x| \le \xi/2. \tag{2.28}$$

3) Assume v_ξ and v_ξ^* are normalized in such a way that

$$\int_0^\infty dx\, \frac{v_\xi(x)^2}{p_\xi(x)} \equiv \int_0^\infty dx\, v_\xi^*(x) v_\xi(x) = 1 \tag{2.29}$$

and define the linear functional π_ξ on $L_\infty^{\rm sym}(\mathbb{R})$ as

$$\pi_\xi(\psi) \doteq \int_0^\infty dx\, v_\xi^*(x)\psi(x). \tag{2.30}$$

We have, for any $\psi \in L_\infty^{\rm sym}(\mathbb{R})$,

$$|\pi_\xi(\psi)| \le c_1 \|\psi\|_\infty, \tag{2.31}$$

and, for any $w \in L_\infty^{\rm sym}(\mathbb{R})$ such that $\pi_\xi(w) = 0$,

$$\left\| e^{L_\xi t} w \right\|_\infty \le c_1 e^{-\omega_1 t} \|w\|_\infty \quad \forall\, t \ge 0. \tag{2.32}$$

In particular the inverse L_ξ^{-1} exists and, for some constant $\hat c_1 = \hat c_1(\kappa)$,

$$\|L_\xi^{-1}\|_\infty \le \hat c_1 \lambda_\xi^{-1}, \tag{2.33}$$

while, if $w \in L_\infty^{\rm sym}(\mathbb{R})$ is such that $\pi_\xi(w) = 0$, then

$$L_\xi^{-1} w = -\int_0^\infty dt\, e^{L_\xi t} w, \qquad \|L_\xi^{-1} w\|_\infty \le \frac{c_1}{\omega_1} \|w\|_\infty. \tag{2.34}$$

Fermi coordinates. We introduce tubular coordinates in a neighborhood of $\mathcal{M}_{\ell,\kappa}$ in the following way. The estimates (2.19) and (2.28) suggest that for ξ large enough the vectors v_ξ and $\partial_\xi n_\xi$ are almost parallel. Therefore we can use the projection (2.30) to define a transverse direction to $\mathcal{M}_{\ell,\kappa}$. To this end we shall need the following lemma, proved in Sect. 8.

Lemma 2.4. *There are $\delta_1 > 0$ and $\alpha_1 > \alpha$ and, for any $\kappa > 0$, there are $\ell_2 \geq \ell_1$ and $c_2 > 0$ such that, for any $h < e^{-2\alpha\ell_2}$ and $\xi \in \Gamma_{\ell_2,\kappa}$, the following holds.*

1) Recalling (2.27),

$$\begin{aligned} \left|\partial_\xi v_\xi(x) - \tilde{m}''_\xi(x)\right| &\leq c_2 e^{-\alpha\xi}\xi^4 \qquad \text{for} \quad |\xi - x| \leq \xi/2, && (2.35)\\ |\partial_\xi v_\xi(x)| &\leq c_2\xi^2 v_\xi(x) \qquad \forall\, x \in \mathbb{R}, && (2.36)\\ \left|\frac{d\lambda_\xi}{d\xi}\right| &\leq c_2 e^{-\alpha_1\xi}. && (2.37) \end{aligned}$$

Moreover, for any $\psi \in L^{\mathrm{sym}}_\infty(\mathbb{R})$,

$$|\partial_\xi \pi_\xi(\psi)| \leq c_2\|\psi\|_\infty, \tag{2.38}$$

where

$$\partial_\xi \pi_\xi(\psi) \doteq \int_0^\infty dx\, \partial_\xi v^*_\xi(x)\psi(x). \tag{2.39}$$

2) We have:

$$\begin{aligned} \|\partial_\xi n_\xi - \pi_\xi(\partial_\xi n_\xi)v_\xi\|_\infty &\leq c_2 e^{-\delta_1\xi}, && (2.40)\\ \left|\pi_\xi(\partial_\xi n_\xi)^{-1} - \sqrt{\mu}\right| &\leq c_2 e^{-\delta_1\xi}, && (2.41)\\ |\partial_\xi \pi_\xi(\partial_\xi n_\xi)| &\leq c_2 e^{-\delta_1\xi}. && (2.42) \end{aligned}$$

Given $\varepsilon > 0$, $\kappa > 0$, $\ell \geq \ell_1$, and $h < e^{-2\alpha\ell}$, let

$$\mathcal{B}_{\ell,\kappa,\varepsilon} := \left\{ m \in L^{\mathrm{sym}}_\infty(\mathbb{R}) \;:\; \inf_{\xi\in\Gamma_{\ell,\kappa}} \|m - n_\xi\|_\infty < \varepsilon \right\}. \tag{2.43}$$

A standard application of the Implicit Function Theorem implies the following result.

Theorem 2.5. *For any $\kappa > 0$ let ℓ_2 be as in Lemma 2.4. Then there are $\varepsilon_0 > 0$, $\bar{c} \geq 1$, $\bar{\kappa} > \kappa$, and $\ell_3 > \bar{\ell} \geq \ell_2$ such that, for any $h < e^{-2\alpha\ell_3}$, there exists a C^1–map $\Xi : \mathcal{B}_{\ell_3,\kappa,\varepsilon_0} \to \Gamma_{\bar{\ell},\bar{\kappa}}$ such that, for $\xi = \Xi(m)$,*

$$\begin{aligned} \pi_\xi(m - n_\xi) &= 0, && (2.44)\\ \|m - n_\xi\|_\infty &\leq \bar{c}\inf\{\|m - n_\zeta\|_\infty : \zeta \in \Gamma_{\ell_3,\kappa}\}. && (2.45) \end{aligned}$$

Moreover, if $\xi_0 \in \Gamma_{\ell_3,\kappa}$ and $\|m - n_{\xi_0}\|_\infty < \varepsilon_0$, then:

$$|\xi - \xi_0| \leq \bar{c}\|m - n_{\xi_0}\|_\infty. \tag{2.46}$$

The proof, which is sketched at the end of Sect. 8, is a standard application of the Contraction Mapping Principle, see also the analogous result in [2]. As a corollary we get the existence of tubular coordinates. Let

$$\bar{\mathcal{S}}_{\ell,\kappa,\varepsilon} \doteq \{(\xi,\varphi) \in \Gamma_{\ell,\kappa} \times L^{\mathrm{sym}}_\infty(\mathbb{R}) \;:\; \pi_\xi(\varphi) = 0,\, \|\varphi\|_\infty < \varepsilon\}. \tag{2.47}$$

For $(\xi,\varphi) \in \bar{\mathcal{S}}_{\ell,\kappa,\varepsilon}$ define $M(\xi,\varphi) \doteq n_\xi + \varphi$ and $\mathcal{S}_{\ell,\kappa,\varepsilon} \doteq M(\bar{\mathcal{S}}_{\ell,\kappa,\varepsilon})$. Then:

Corollary 2.6. *For any $\kappa > 0$, let ε_0 and ℓ_3 be as in Theorem 2.6. Then, for any $h < e^{-2\alpha\ell_3}$, the map $M : \bar{\mathcal{S}}_{\ell_3,\kappa,\varepsilon_0} \to \mathcal{S}_{\ell_3,\kappa,\varepsilon_0}$ is differentiable, one to one, and $\Xi(M(\xi,\varphi)) = \xi$. Moreover $\mathcal{S}_{\ell_3,\kappa,\varepsilon_0}$ is open in $L^{\mathrm{sym}}_\infty(\mathbb{R})$.*

Corollary 2.6 thus gives the existence of tubular coordinates in a neighborhood of the manifold $\mathcal{M}_{\ell,\kappa}$. We point out that the size of the above neighborhood is not vanishing as $h \downarrow 0$.

Equations of motion. Consider a solution of (2.1) $S_t(m)$, $t \in [0,T]$, $T>0$, which lies in the tubular neighborhood $\mathcal{S}_{\ell_3,\kappa,\varepsilon_0}$ with ℓ_3 and ε_0 as in Corollary 2.6. Then we can decompose $S_t(m) = n_{\xi(t)} + \varphi(t)$. By differentiating the identity $\pi_\xi(\varphi) = 0$, from (2.1) we obtain the equations of motion in the Fermi coordinates (ξ,φ):

$$\begin{aligned} \left[\pi_\xi\left(\partial_\xi n_\xi\right) - \partial_\xi \pi_\xi(\varphi)\right]\dot\xi &= \pi_\xi\left(f(n_\xi+\varphi)\right), &(2.48)\\ \dot\varphi &= f(n_\xi+\varphi) - \partial_\xi n_\xi \dot\xi, &(2.49)\end{aligned}$$

where $(\dot\xi,\dot\varphi)$ denotes the time derivative of (ξ,φ). Observe that, from Lemma 2.4, by choosing ε small enough (depending on κ), the coefficient of $\dot\xi$ in (2.48) is non zero for any (ξ,φ) in the tubular neighborhood $\mathcal{S}_{\ell_3,\kappa,\varepsilon}$.

Given $\kappa>0$ and $\ell>\ell_3$ such that $e^{-3\alpha\ell/2} < \varepsilon_0$, we introduce the *thin channels* $\mathcal{Z}_{\ell,\kappa,Q}$, $Q \ge 1$, defined by:

$$\mathcal{Z}_{\ell,\kappa,Q} \doteq \left\{ m = n_\xi + \varphi \in \mathcal{S}_{\ell_3,\kappa,\varepsilon} \,:\, \xi \in \Gamma_{\ell,\kappa},\ \|\varphi\|_\infty \le Q^{-1} e^{-3\alpha\xi/2} \right\}. \tag{2.50}$$

Then:

Theorem 2.7. *For any $\kappa>0$ there are $\varepsilon_1 \in (0,\varepsilon_0]$, $\ell_4 \ge \ell_3$, $Q \ge 1$, and $\nu>0$, such that, for any $h < e^{-2\alpha\ell_4}$, the following holds. If $m \in \mathcal{S}_{\ell_4,\kappa,\varepsilon_1}$ then, as long as $S_t(m) \in \mathcal{S}_{\ell_4,\kappa,\varepsilon_0}$, one has:*

$$\|\varphi(t)\|_\infty \le e^{-3\alpha\xi(t)/2} + \left(Q\|\varphi(0)\|_\infty - e^{-3\alpha\xi(0)/2}\right) e^{-\nu t}, \tag{2.51}$$

where $S_t(m) = n_{\xi(t)} + \varphi(t)$.

We may assume ε_1 so small and ℓ_4 so large that $(1+Q)\varepsilon_1 < \varepsilon_0$ and $e^{-3\alpha\ell_4/2} < \varepsilon_1$. By (2.51) we thus see that the tube $\mathcal{S}_{\ell_4,\kappa,\varepsilon_1}$ is exponentially attracted towards the channel $\mathcal{Z}_{\ell_4,\kappa,1}$. Moreover, for $m \in \mathcal{Z}_{\ell,\kappa,Q}$ there are only two possibilities: either $S_t(m)$ stays forever in $\mathcal{Z}_{\ell,\kappa,1}$, or there is a finite time t^* for which $\xi(t^*)$ belongs to the boundary of $\Gamma_{\ell_4,\kappa}$. The dynamics in the channel $\mathcal{Z}_{\ell_4,\kappa,1}$ is essentially reduced to the motion of the coordinate $\xi(t)$, for which we have an explicit formula:

Theorem 2.8. *For any $\kappa>0$ there is $\delta^*>0$ such that, for any $\ell>\ell_3$ sufficiently large, as long as $S_t(m) = n_{\xi(t)} + \varphi(t) \in \mathcal{Z}_{\ell,\kappa,1}$, one has:*

$$\dot\xi = V(\xi) + O\left(e^{-(2\alpha+\delta^*)\xi}\right), \tag{2.52}$$

with $V(\xi)$ as in (2.21).

Theorems 2.7 and 2.8 will be proved in Sect. 3.

The critical droplet. Our next result concerns the existence and uniqueness of the critical droplet in a neighborhood of $\mathcal{M}_{\ell,\kappa}$. As already mentioned Theorem 2.1 does not provide any uniqueness result: the following theorem is proved in Sect. 4.

Theorem 2.9. *There is κ_0 such that the following holds. For any $\kappa \ge \kappa_0$ there are $\varepsilon_2 \in (0,\varepsilon_0]$, $\ell_5 \ge \ell_3$, and $h_0 < e^{-2\alpha\ell_5}$ such that, for any $h \in (0,h_0]$, the equation $f(m)=0$ in $\mathcal{S}_{\ell_5,\kappa,\varepsilon_2}$ has a unique solution $m = n_{\bar\xi} + \bar\varphi$. Moreover:*

$$\lim_{h\downarrow 0} e^{2\alpha\bar\xi} h = (2m_\beta)^{-1} K, \qquad \lim_{h\downarrow 0} e^{2\alpha\bar\xi}\|\bar\varphi\|_\infty = 0, \tag{2.53}$$

with K as in (2.14).

From (2.53), (2.18), (2.12), and (2.13) this solution to $f(m) = 0$ has to coincide with the critical droplet of Theorem 2.1, i.e. $q = n_{\bar{\xi}} + \bar{\varphi}$.

We have a detailed information on the spatial structure of the critical droplet, which is the content of the following proposition, proved in Sect. 5.

Proposition 2.10. *Given $\beta > 1$, there is $h^* \in (0, h_0]$ (h_0 as in Theorem 2.1) such that for any $h \in (0, h^*]$ the bump $q(x)$ is a strictly decreasing function on $\mathbb{R}_+$ (actually $q'(x) < 0$ for all $x > 0$). Moreover, letting $\gamma > 0$ be such that*

$$\beta\left[1 - (m^-_{\beta,h})^2\right] \int dz\, J(z) e^{-\gamma z} = 1, \tag{2.54}$$

and ξ_q be the (unique) positive zero of $q(x)$, there are $A > 0$, $\delta > 0$, and $C > 0$ so that, for all $x \geq \xi_q$,

$$\begin{aligned} &\left|q(x) - (m^-_{\beta,h} + A e^{-\gamma(x-\xi_q)})\right| + \left|q'(x) + \gamma A e^{-\gamma(x-\xi_q)}\right| \\ &\qquad + \left|q''(x) - \gamma^2 A e^{-\gamma(x-\xi_q)}\right| \leq C e^{-(\gamma+\delta)(x-\xi_q)}. \end{aligned} \tag{2.55}$$

Finally, as $h \downarrow 0$, A, δ, and C remain strictly positive and bounded, while $\xi_q \to +\infty$.

The invariant manifold. The qualitative behavior of the dynamics around the bump follows from the spectral properties of the linear operator $L \doteq Df|_q = L_{q+h}$. Since q satisfies (1.7) we compute, for any $\psi \in L^{\mathrm{sym}}_\infty(\mathbb{R})$,

$$L\psi = -\psi + pJ * \psi, \tag{2.56}$$

where

$$p(x) \doteq \beta\left[1 - q(x)^2\right]. \tag{2.57}$$

Given $\zeta \in \mathbb{R}$, we introduce the normed spaces

$$X_\zeta \doteq \{w : \mathbb{R} \to \mathbb{R} \text{ measurable and symmetric } : \|w\|_\zeta < \infty\}, \tag{2.58}$$

where

$$\|w\|_\zeta \doteq \sup_{x \in \mathbb{R}_+} e^{\zeta x} |w(x)|.$$

The following proposition follows from the results proved in [6, 7] and for the reader convenience we give detailed references on where the proofs can be found in Sect. 8.

Proposition 2.11. *Given $\beta > 1$ let h^* be as in Proposition 2.10. Then there are constants $C_0 > 1$ and $C_1 > 0$ such that for any $h \in (0, h^*]$ the following holds.*

1) There are $\lambda > 0$ and strictly positive functions $v, v^* \in C^{\mathrm{sym}}_0(\mathbb{R})$ so that:

$$Lv = \lambda v, \qquad v^* L = \lambda v^*, \tag{2.59}$$

$$v^*(x) = p(x) v(x) \qquad \forall\, x \in \mathbb{R}, \tag{2.60}$$

and

$$\frac{h}{C_0} \leq \lambda \leq C_0 h. \tag{2.61}$$

2) There is a unique $\gamma_v > \gamma > 0$, (γ as in (2.54)) such that:

$$\beta\left(1 - (m^-_{\beta,h})^2\right) \int dz\, J(z) e^{-\gamma_v z} = 1 + \lambda, \tag{2.62}$$

and there is $M_v > 0$ for which

$$\lim_{x \to +\infty} e^{\gamma_v x} v(x) = M_v. \tag{2.63}$$

Moreover, for any $\zeta \le \gamma_v$ *and* $t \ge 0$,

$$\left\|e^{Lt}w\right\|_\zeta \le C_1 e^{\lambda t}\|w\|_\zeta \qquad \forall\, w \in X_\zeta. \tag{2.64}$$

3) Assume v *and* v^* *are normalized in such a way that*

$$\int_0^\infty dx\,\frac{v(x)^2}{p(x)} \equiv \int_0^\infty dx\, v^*(x)v(x) = 1, \tag{2.65}$$

and define the linear functional π *on* X_ζ, $\zeta < \gamma_v$, *as:*

$$\pi(\psi) \doteq \int_0^\infty dx\, v^*(x)\psi(x). \tag{2.66}$$

Then there is $\omega > 0$ *so that, for any* $w \in L_\infty^{\rm sym}(\mathbb{R})$ *such that* $\pi(w) = 0$ *and* $t \ge 0$,

$$\left\|e^{Lt}w\right\|_\infty \le C_1 e^{-\omega t}\left\|w\right\|_\infty. \tag{2.67}$$

Moreover

$$\frac{1}{C_0} \le \omega \le C_0. \tag{2.68}$$

4) Finally, recalling (2.27),

$$\lim_{h\downarrow 0}\|v - \tilde m'_{\xi^*}\|_\infty = 0, \tag{2.69}$$

with ξ^* *as in* (2.13).

As we are going to see, from the fact that the linearized operator around q has only one positive eigenvalue, it follows the existence, for any h small enough, of two distinct, one dimensional manifolds $\mathcal{W}_\pm \subset C^{\rm sym}(\mathbb{R};[-1,1])$. We give the precise statement in the next theorem, proved in Sect. 6.

Theorem 2.12. *Given* $\beta > 1$ *let* h^* *be as in Proposition 2.10. Then, for any* $h \in (0,h^*]$, *there are two distinct, one dimensional manifolds* $\mathcal{W}_\pm \subset C^{\rm sym}(\mathbb{R};[-1,1])$,

$$\mathcal{W}_\pm = \left\{m_s^\pm \,:\, s\in\mathbb{R}\right\}. \tag{2.70}$$

For any $s \in \mathbb{R}$ *and* $t \ge 0$,

$$S_t\left(m_s^\pm\right) = m_{s+t}^\pm \tag{2.71}$$

and

$$\lim_{s\to-\infty}\|m_s^\pm - q\|_\infty = 0. \tag{2.72}$$

Moreover, given μ *as in* (2.27), *we have:*

$$\lim_{s\to-\infty} e^{-\lambda s}\left\|\frac{dm_s^\pm}{ds} \mp \lambda e^{\lambda s}\frac{1}{\sqrt{\mu}}v\right\|_\infty = 0. \tag{2.73}$$

Finally, for any $s \in \mathbb{R}$, *the (symmetric) functions* $m_s^\pm$ *are non increasing on* $\mathbb{R}_+$ *and satisfy:*

$$m_{\beta,h}^- \le m_s^-(x) \le q(x) \le m_s^+(x) \le m_{\beta,h}^+ \qquad \forall\, x\in\mathbb{R}. \tag{2.74}$$

Thus the one dimensional manifolds $\mathcal{W}_\pm$ originates at $s = -\infty$ from q, (2.72), and are time invariant, (2.71). Each one of them is therefore described by a single orbit of S_t with time going from $-\infty$ to $+\infty$. The two orbits are denoted by $m_s^\pm$ and the parameter s is identified with time. Of course the origin of time is arbitrary and this can be exploited to fix up the constants in such a way that (2.73) holds, we refer to Sect. 6 for details on this point.

By integrating (2.73) from $-\infty$ to s we get

$$m_s^\pm \approx q \pm e^{\lambda s} \frac{1}{\sqrt{\mu}} v. \tag{2.75}$$

Therefore, since from (2.69) we have

$$\lim_{h \downarrow 0} \sup_{x \le 0} \left| v(x) - \sqrt{\mu}\, \bar m'(x+\xi^*) \right| = 0, \tag{2.76}$$

by (2.12) and (2.69), for h small and $x \le 0$,

$$m_s^\pm(x) \approx \bar m(x+\xi^*) \pm e^{\lambda s} \bar m'(x+\xi^*) \approx \bar m(x+\xi^* \pm e^{\lambda s}). \tag{2.77}$$

By symmetry the result extends to $x \ge 0$. Thus the points in a neighborhood of q that are in $\mathcal{W}_+$ are "droplets longer" than q while those in $\mathcal{W}_-$ are shorter. Their length changes at the exponential rate λ, which is therefore the Lyapunov exponent at q, with $\mathcal{W}_\pm$ the corresponding unstable manifolds. Since $\lambda \approx h$, for h small, there is a *dormant instability*, in the sense that for small h (which is the case of interest in metastability) even though ultimately unstable, the bump q seems in fact stable for very long times (of order h^{-1}).

By (2.53) the critical droplet q belongs to the interior of the channel $\mathcal{Z}_{\ell_4,\kappa,Q}$, for κ large and h small enough. Then, according to Theorem 2.7, a large part $\overline{\mathcal{W}}$ of the invariant manifold $\mathcal{W}$ is contained in $\mathcal{Z}_{\ell_4,\kappa,1}$: it thus consists of droplets with two well-separated transition layers whose locations evolve according to Eq. (2.52).

Our last result concerns the global structure of the invariant manifold $\mathcal{W}$. More precisely we show that the sub-critical and super-critical droplets will eventually go to the metastable and stable phase respectively: this is the content of the next theorem proved in Sect. 7.

Theorem 2.13. *Given $\beta > 1$ there is $h^\dagger \in (0, h^*]$ (h^* as in Proposition 2.10) such that, for any $h \in (0, h^\dagger]$,*

$$\lim_{s \to +\infty} \left\| m_s^- - m_{\beta,h}^- \right\|_\infty = 0, \tag{2.78}$$

$$\lim_{s \to +\infty} m_s^+(x) = m_{\beta,h}^+ \qquad \forall\, x \in \mathbb{R}. \tag{2.79}$$

A notation warning: In Sects. 3, 4, and 8 we shall denote by $C = C(\kappa)$ a generic positive function of κ whose numerical value may change from line to line.

3. Local attractiveness

We start with a preliminary lemma.

Lemma 3.1. *For any $\kappa > 0$ let ℓ_3 and ε_0 be as in Theorem 2.5. Then, there exists $G = G(\kappa)$ such that, for any $h < e^{-2\alpha\ell_3}$, $N > 0$, and $m = n_\xi + \varphi \in \mathcal{S}_{\ell_3,\kappa,\varepsilon_0}$, one has:*

$$\|S_t(m) - n_\xi\|_\infty \le \left(c_1 e^{-\omega_1 t} + N\|\varphi\|_\infty\right) \|\varphi\|_\infty + N e^{-2\alpha\xi} \qquad \forall\, t < T_{m,N}, \tag{3.1}$$

with c_1 and ω_1 as in Proposition 2.3 and

$$T_{m,N} \doteq \min\left\{ e^{2\alpha\xi}; \frac{N}{G[1 + N^2(e^{-2\alpha\xi} + \|\varphi\|_\infty^2)]} \right\}. \tag{3.2}$$

Proof. The function $\psi_t \doteq S_t(m) - n_\xi$ solves

$$\frac{d\psi_t}{dt} = L_\xi \psi_t + [f(n_\xi + \psi_t) - L_\xi \psi_t]\,,$$

which implies, since $\psi_0 = \varphi$,

$$\psi_t = e^{L_\xi t}\varphi + \int_0^t ds\, e^{L_\xi (t-s)} [f(n_\xi + \psi_s) - L_\xi \psi_s]\,.$$

By (2.24) we then get:

$$\begin{aligned}\|\psi_t - e^{L_\xi t}\varphi\|_\infty &\le c_1 \int_0^t ds\, e^{\lambda_\xi (t-s)} \|f(n_\xi + \psi_s) - L_\xi \psi_s\|_\infty \\ &\le c_1 \int_0^t ds\, e^{\lambda_\xi (t-s)} \left(\|f(n_\xi)\|_\infty + k_2 \|\psi_s\|_\infty^2\right),\end{aligned}$$

where in the last inequality we used (2.7) (recall that $L_\xi \doteq L_{n_\xi + h}$). By (2.32), since $\pi_\xi(\varphi) = 0$, $\|\psi_s\|_\infty \le c_1 \|\varphi\|_\infty + \|\psi_s - e^{L_\xi s}\varphi\|_\infty$. Moreover, by (2.16), (2.20) and (2.21), $\|f(n_\xi)\|_\infty \le C e^{-2\alpha\xi}$. Finally, by (2.25), if $t \le e^{2\alpha\xi}$ then $\lambda_\xi t \le c_1$. We conclude that there exists $G = G(\kappa) > 0$ such that:

$$\|\psi_t - e^{L_\xi t}\varphi\|_\infty \le G \int_0^t ds\, \left(e^{-2\alpha\xi} + \|\varphi\|_\infty^2 + \|\psi_s - e^{L_\xi t}\varphi\|_\infty^2\right) \qquad \forall\, t \le e^{2\alpha\xi}. \quad (3.3)$$

Given $N > 0$, fix $\tau < T_{m,N}$ and let $T \le \tau$ be the first time when the inequality

$$\|\psi_t - e^{L_\xi t}\varphi\|_\infty \le N(e^{-2\alpha\xi} + \|\varphi\|_\infty^2) \quad (3.4)$$

becomes an equality. Then, by (3.3),

$$\begin{aligned}N(e^{-2\alpha\xi} + \|\varphi\|_\infty^2) &\le GT\left[e^{-2\alpha\xi} + \|\varphi\|_\infty^2 + N^2\left(e^{-2\alpha\xi} + \|\varphi\|_\infty^2\right)^2\right] \\ &\le N\left(e^{-2\alpha\xi} + \|\varphi\|_\infty^2\right)\frac{\tau}{T_{m,N}} < N\left(e^{-2\alpha\xi} + \|\varphi\|_\infty^2\right).\end{aligned}$$

We have thus reached a contradiction and hence (3.4) is valid for all $t \le \tau$. By (2.32) and (3.4) the inequality in (3.1) holds for any $t \le \tau$. The lemma is thus proved. □

Proof of Theorem 2.7. We fix the parameter N of Lemma 3.1 such that

$$4\bar{c}c_1 e^{3\alpha\bar{c}\varepsilon_0/2} e^{-\omega_1 T_N} = 1, \qquad T_N \doteq \frac{N}{2G}, \quad (3.5)$$

with c_1, ω_1 as in Proposition 2.3, $\bar{c}$, ε_0 as in Theorem 2.5, and G as in Lemma 3.1. We next define:

$$Q \doteq \frac{1}{2}\left(1 + e^{\omega_1 T_N}\right) = \frac{1}{2} + 2\bar{c}c_1 e^{3\alpha\bar{c}\varepsilon_0/2}, \qquad b \doteq \frac{1}{2}e^{-3\alpha\bar{c}\varepsilon_0/2}. \quad (3.6)$$

(note that $Q \ge 1$ and $b < 1/2$). We finally choose $\ell_4 > \ell_3$ large enough and $\varepsilon_1 \in (0, \varepsilon_0]$ small enough that:

$$N^2(e^{-2\alpha\ell_4} + \varepsilon_1^2) < 1, \qquad Ne^{-2\alpha\ell_4} < 2G, \qquad c_1\varepsilon_1 + N(\varepsilon_1^2 + e^{-2\alpha\ell_4}) < \varepsilon_0, \quad (3.7)$$

$$\bar{c}N\varepsilon_1 < b/2, \qquad \bar{c}Ne^{-\alpha\ell_4/2} < b/Q, \qquad e^{-3\alpha\ell_4/2} < \varepsilon_1/2. \quad (3.8)$$

Let now $m = n_\xi + \varphi \in \mathcal{S}_{\ell_4,\kappa,\varepsilon_1}$. By the first and the second inequality in (3.7) we have:

$$T_{m,N} > T_N \qquad \forall\, m \in \mathcal{S}_{\ell_4,\kappa,\varepsilon_1},$$

with $T_{m,N}$ as in (3.2). By (3.1) and the third inequality in (3.7) we thus get $\|S_t(m) - n_\xi\|_\infty < \varepsilon_0$ for all $t \in [0, T_N]$. Then, by Theorem 2.5, the function $\xi(t) = \Xi(S_t(m))$ is well defined for all $t \in [0, T_N]$ and

$$
\begin{aligned}
|\xi(t) - \xi| &\leq \bar{c}\|S_t(m) - n_\xi\|_\infty \leq \bar{c}\varepsilon_0, && (3.9)\\
\|\varphi(t)\|_\infty &\leq \bar{c}\|S_t(m) - n_\xi\|_\infty \\
&\leq \bar{c}(c_1 e^{-\omega_1 t} + N\varepsilon_1)\|\varphi\|_\infty + \bar{c}Ne^{-\alpha\ell_4/2}e^{-3\alpha\xi/2}, && (3.10)
\end{aligned}
$$

where $\varphi(t) \doteq S_t(m) - n_{\xi(t)}$ and we again used (3.1) for the last bound in (3.10). By (3.5), (3.8), and (3.10) it easily follows that:

$$
\|\varphi(t)\|_\infty \leq \frac{b}{2}\left(1 + e^{\omega_1(T_N - t)}\right)\|\varphi\|_\infty + \frac{b}{Q}e^{-3\alpha\xi/2}.
$$

Moreover, by (3.9),

$$
2be^{-3\alpha\xi/2} \leq e^{-3\alpha\xi(t)/2} \qquad \forall\, t \in [0, T_N].
$$

Then, recalling the definition (3.6) of Q and that $Q \geq 1$, we conclude that:

$$
\begin{aligned}
\|\varphi(t)\|_\infty &\leq b\left(Q\|\varphi\|_\infty - e^{-3\alpha\xi/2}\right) + e^{-3\alpha\xi(t)/2} \qquad \forall\, t \in [0, T_N], && (3.11)\\
\|\varphi(T_N)\|_\infty &\leq \frac{b}{Q}\left(Q\|\varphi\|_\infty - e^{-3\alpha\xi/2}\right) + \frac{1}{Q}e^{-3\alpha\xi(T_N)/2}. && (3.12)
\end{aligned}
$$

We have thus proved that if $m \in \mathcal{S}_{\ell_4,\kappa,\varepsilon_1}$ then $S_t(m) \in \mathcal{B}_{\ell_4,\kappa,\varepsilon_0}$ for all $t \in [0, T_N]$ and $\varphi(t) = S_t(m) - n_{\xi(t)}$ satisfies the bounds (3.11) and (3.12). It follows that $S_t(m)$ may leave $\mathcal{S}_{\ell_4,\kappa,\varepsilon_0}$ at a given time $t^* \in [0, T_N]$ only because $\xi(t^*)$ belongs to the boundary of $\Gamma_{\ell_4,\kappa}$. If this is not the case, since the third inequality in (3.8) implies $\|\varphi(T_N)\|_\infty < \varepsilon_1$, then $S_{T_N}(m) \in \mathcal{S}_{\ell_4,\kappa,\varepsilon_1}$ and we can repeat the same analysis for the solution in the interval $[T_N, 2T_N]$. Let n be the largest integer such that $S_t(m)$ leaves $\mathcal{S}_{\ell_4,\kappa,\varepsilon_0}$ at $t^* \geq t_n \doteq nT_N$, setting $n = +\infty$ if $S_t(m) \in \mathcal{S}_{\ell_4,\kappa,\varepsilon_0}$ forever. Then, for any integer $k \leq n$ we iterate (3.12) getting

$$
\|\varphi(t_k)\|_\infty \leq \frac{b^k}{Q}\left(Q\|\varphi(0)\|_\infty - e^{-3\alpha\xi(0)/2}\right) + \frac{1}{Q}e^{-3\alpha\xi(t_k)/2}, \qquad t_k \doteq kT_N
$$

from which, using (3.11) with $\varphi = \varphi(t_k)$, we finally obtain:

$$
\|\varphi(t)\|_\infty \leq b^{k+1}\left(Q\|\varphi(0)\|_\infty - e^{-3\alpha\xi(0)/2}\right) + e^{-3\alpha\xi(t)/2} \qquad \forall\, t \in [t_k, t_{k+1}],
$$

which implies (2.51) with $\nu = -(\log b)/T_N$. The theorem is thus proved. □

Proof of Theorem 2.8. Recalling (2.6),

$$
\begin{aligned}
&f(n_\xi + \varphi) = f(n_\xi) + L_\xi\varphi + R_\xi[\varphi], && (3.13)\\
&R_\xi[\varphi] = \beta^2(J * \varphi)^2 \int_0^1 ds\,(1 - s)\tanh''\{\beta[J * (n_\xi + s\varphi) + h]\}, && (3.14)
\end{aligned}
$$

so that, since $\pi_\xi(L_\xi\varphi) = \lambda_\xi\pi_\xi(\varphi) = 0$, Eq. (2.48) becomes:

$$
\dot{\xi} = \frac{\pi_\xi(f(n_\xi)) + \pi_\xi(R_\xi[\varphi])}{\pi_\xi(\partial_\xi n_\xi) - \partial_\xi\pi_\xi(\varphi)}.
$$

By (2.7), (2.20), (2.31), (2.38), and the definition of $\mathcal{Z}_{\ell,\kappa,1}$, (2.52) follows for a suitable $\delta^* > 0$. □

4. The critical droplet

In this section we prove Theorem 2.9. Recalling (2.48) and (2.49) we have that $m = n_\xi + \varphi \in \mathcal{S}_{\ell_3,\kappa,\varepsilon_0}$ is a solution of $f(m) = 0$ iff (ξ, φ) solve

$$f(n_\xi + \varphi) = 0, \qquad \pi_\xi(f(n_\xi + \varphi)) = 0, \qquad \pi_\xi(\varphi) = 0.$$

We rewrite the above equations as follows. For $\xi \in \Gamma_{\ell_3,\kappa}$ we define the projection T_ξ acting on $L_\infty^{\mathrm{sym}}(\mathbb{R})$ by:

$$T_\xi \psi = \psi - \pi_\xi(\psi) v_\xi. \tag{4.1}$$

Observe that by (2.26) and (2.31) there is C so that

$$\|T_\xi \psi\|_\infty \le C \|\psi\|_\infty \qquad \forall \psi \in L_\infty^{\mathrm{sym}}(\mathbb{R}). \tag{4.2}$$

Then, using the expansion (3.13) and the fact that $\pi_\xi(L_\xi \varphi) = 0$ we get that $f(n_\xi + \varphi) = 0$ iff (ξ, φ) solve

$$L_\xi \varphi + T_\xi \left(f(n_\xi) + R_\xi[\varphi] \right) = 0, \tag{4.3}$$

$$\pi_\xi \left(f(n_\xi) + R_\xi[\varphi] \right) = 0. \tag{4.4}$$

We now proceed in the following way: we first show in Lemma 4.1 below, that there exists a unique graph $\xi \mapsto \varphi_\xi$ in a tubular neighborhood $\mathcal{S}_{\ell,\kappa,\varepsilon}$ such that φ_ξ is solution of equation (4.3). Replacing φ by φ_ξ in (4.4) the problem reduces to study an equation in the real variable ξ.

Lemma 4.1. *For any $\kappa > 0$ there are $\varepsilon_2 \in (0, \varepsilon_0]$ and $\ell_6 \ge \ell_3$ such that, for any $h < e^{-2\alpha \ell_6}$ and $\xi \in \Gamma_{\ell_6,\kappa}$, the map $A_\xi : L_\infty^{\mathrm{sym}}(\mathbb{R}) \to L_\infty^{\mathrm{sym}}(\mathbb{R})$ defined by*

$$A_\xi(\varphi) \doteq -L_\xi^{-1} T_\xi \left(f(n_\xi) + R_\xi[\varphi] \right) \tag{4.5}$$

is a contraction in $Y_{\varepsilon_2} \doteq \{\varphi \in L_\infty^{\mathrm{sym}}(\mathbb{R}) : \|\varphi\|_\infty \le \varepsilon_2\}$.

Proof. From (2.7) and (2.34) we have:

$$\|A_\xi \varphi\|_\infty \le \frac{c_1}{\omega_1} \left[\|T_\xi f(n_\xi)\|_\infty + k_2 \|\varphi\|_\infty^2 \right].$$

We write

$$T_\xi f(n_\xi) = T_\xi (f(n_\xi) - V(\xi) \partial_\xi n_\xi) + V(\xi) T_\xi \partial_\xi n_\xi.$$

From (2.20) and (4.2) it follows that

$$\|T_\xi (f(n_\xi) - V(\xi) \partial_\xi n_\xi)\|_\infty \le C e^{-(2\alpha + \delta_0)\xi}.$$

Since for $\xi \in \Gamma_{\ell_3,\kappa}$ and, if $h < e^{-2\alpha \ell_3}$, then $h e^{-2\alpha\xi} \le e^{2\alpha\kappa}$, from (2.40) we get:

$$|V(\xi)| \|T_\xi \partial_\xi n_\xi\|_\infty \le C e^{-(2\alpha + \delta_1)\xi}.$$

By the above estimates we conclude that there is $\delta > 0$ so that

$$\|T_\xi f(n_\xi)\|_\infty \le C e^{-(2\alpha + \delta)\xi}. \tag{4.6}$$

Then, for any $\ell \ge \ell_3$, $h < e^{-2\alpha\ell}$, $\xi \in \Gamma_{\ell,\kappa}$, and $\varphi \in Y_\varepsilon$,

$$\|A_\xi(\varphi)\|_\infty \le \frac{c_1}{\omega_1} \left[C e^{-(2\alpha + \delta)\ell} + k_2 \varepsilon^2 \right]. \tag{4.7}$$

Moreover, recalling (3.14), there is $k_3 > 0$ for which, if $\varphi_1, \varphi_2 \in Y_\varepsilon$, then

$$\|R_\xi[\varphi_1] - R_\xi[\varphi_2]\|_\infty \le k_3 \varepsilon (1 + \varepsilon) \|\varphi_1 - \varphi_2\|_\infty$$

so that

$$\|A_\xi(\varphi_1) - A_\xi(\varphi_2)\|_\infty \le \frac{c_1 k_3}{\omega_1} \varepsilon (1 + \varepsilon) \|\varphi_1 - \varphi_2\|_\infty. \tag{4.8}$$

From (4.7) and (4.8) we conclude that, by choosing $\varepsilon_2 = 2Cc_1\omega_1^{-1}e^{-(2\alpha+\delta)\ell_6}$ and $\ell_6 \geq \ell_3$ large enough, A_ξ is a contraction in Y_{ε_2}. $\square$

Proposition 4.2. *For any $\kappa > 0$ there is $\ell_7 > \ell_6$ such that, for any $h < e^{-2\alpha\ell_7}$, the graph $\xi \to \varphi_\xi$ is differentiable and*

$$\|\varphi_\xi\|_\infty \leq Ce^{-(2\alpha+\delta_2)\xi}, \qquad \|\partial_\xi\varphi_\xi\|_\infty \leq Ce^{-\delta_2\xi} \tag{4.9}$$

with $\delta_2 = \min\{\delta_0, \delta_1, 2\alpha\}$ (δ_0 and δ_1 as in Theorem 2.2 and Lemma 2.4 respectively).

Proof. Since φ_ξ is the fixed point of the map (4.5), the bound for $\|\varphi_\xi\|_\infty$ follows immediately from (4.6) (with $\delta_2 = \delta$). Moreover, since the r.h.s. of (4.5) is a continuous function of ξ and φ, the continuity of $\xi \mapsto \varphi_\xi$ is also straightforward.

We next study the equation for $\partial_\xi\varphi_\xi$ which is obtained by differentiating (4.3) (for $\varphi = \varphi_\xi$) w.r.t. ξ. We shall prove that for any $\kappa > 0$ there is $\ell_7 \geq \ell_6$ such that it has a unique solution for any $\xi \in \Gamma_{\ell,\kappa}$. By standard arguments this solution defines the derivative $\partial_\xi\varphi_\xi$.

Let $G_\xi \doteq -(f(n_\xi) + R_\xi[\varphi_\xi])$, so that φ_ξ solves $L_\xi\varphi_\xi = T_\xi G_\xi$. By differentiating (recall (2.39)) we get:

$$\begin{aligned} L_\xi\partial_\xi\varphi_\xi + \partial_\xi p_\xi J * \varphi_\xi &= T_\xi\partial_\xi G_\xi - \partial_\xi\pi_\xi(G_\xi)v_\xi - \pi_\xi(G_\xi)\partial_\xi v_\xi \\ &= T_\xi\partial_\xi G_\xi - \pi_\xi(G_\xi)T_\xi\partial_\xi v_\xi \\ &\quad - [\pi_\xi(G_\xi)\pi_\xi(\partial_\xi v_\xi) + \partial_\xi\pi_\xi(G_\xi)]v_\xi. \end{aligned}$$

By substituting $G_\xi = L_\xi\varphi_\xi + \pi_\xi(G_\xi)v_\xi$ in the term $\partial_\xi\pi_\xi(G_\xi)$ we have:

$$\begin{aligned} L_\xi\partial_\xi\varphi_\xi &= -\partial_\xi p_\xi J * \varphi_\xi + T_\xi\partial_\xi G_\xi - \pi_\xi(G_\xi)T_\xi\partial_\xi v_\xi \\ &\quad - \big\{\pi_\xi(G_\xi)\big[\pi_\xi(\partial_\xi v_\xi) + \partial_\xi\pi_\xi(v_\xi)\big] + \partial_\xi\pi_\xi(L_\xi\varphi_\xi)\big\}v_\xi. \end{aligned}$$

Observing now that, by (2.29), $\pi_\xi(\partial_\xi v_\xi) + \partial_\xi\pi_\xi(v_\xi) = 0$ we get

$$L_\xi\partial_\xi\varphi_\xi = T_\xi\left[\partial_\xi G_\xi - \partial_\xi p_\xi J * \varphi_\xi - \pi_\xi(G_\xi)\partial_\xi v_\xi\right] + \lambda_\xi H(\xi)v_\xi, \tag{4.10}$$

with

$$\begin{aligned} H(\xi) &= -\frac{1}{\lambda_\xi}\left[\pi_\xi(\partial_\xi p_\xi J * \varphi_\xi) + \partial_\xi\pi_\xi(L_\xi\varphi_\xi)\right] \\ &= -\frac{1}{\lambda_\xi}\int_0^\infty dx\,\left[\frac{\partial_\xi p_\xi}{p_\xi}v_\xi J * \varphi_\xi + \left(\frac{\partial_\xi v_\xi}{p_\xi} - \frac{\partial_\xi p_\xi}{p_\xi^2}v_\xi\right)L_\xi\varphi_\xi\right](x) \\ &= -\frac{1}{\lambda_\xi}\int_0^\infty dx\,\left(\partial_\xi v_\xi J * \varphi_\xi + \frac{\partial_\xi p_\xi}{p_\xi^2}v_\xi\varphi_\xi - \frac{\partial_\xi v_\xi}{p_\xi}\varphi_\xi\right)(x) \\ &= -\frac{1}{\lambda_\xi}\int_0^\infty dx\,\varphi_\xi(x)\left(J * \partial_\xi v_\xi + \frac{\partial_\xi p_\xi}{p_\xi^2}v_\xi - \frac{\partial_\xi v_\xi}{p_\xi}\right)(x), \end{aligned} \tag{4.11}$$

where, in the last line, we used that, since $\partial_\xi v_\xi$, J, and φ_ξ are symmetric functions, then:

$$\int_0^\infty dx\,\partial_\xi v_\xi(x)J * \varphi_\xi(x) = \int_0^\infty dx\,\varphi_\xi(x)J * \partial_\xi v_\xi(x).$$

Now, by differentiating $L_\xi v_\xi = \lambda_\xi v_\xi$,

$$J * \partial_\xi v_\xi = \frac{d\lambda_\xi}{d\xi}\frac{v_\xi}{p_\xi} + \lambda_\xi\frac{\partial_\xi v_\xi}{p_\xi} - \frac{\partial_\xi p_\xi}{p_\xi}J * v_\xi + \frac{\partial_\xi v_\xi}{p_\xi},$$

so that

$$H(\xi) = -\frac{1}{\lambda_\xi}\int_0^\infty dx\, \varphi_\xi(x) \left(\frac{d\lambda_\xi}{d\xi}\frac{v_\xi}{p_\xi} + \lambda_\xi \frac{\partial_\xi v_\xi}{p_\xi} - \frac{\partial_\xi p_\xi}{p_\xi^2} L_\xi v_\xi \right)(x).$$

Using again $L_\xi v_\xi = \lambda_\xi v_\xi$ and observing the first term on the r.h.s. is proportional to $\pi_\xi(\varphi_\xi) = 0$, we finally obtain:

$$H(\xi) = -\int_0^\infty dx\, \varphi_\xi(x) \left(\frac{\partial_\xi v_\xi}{p_\xi} - \frac{\partial_\xi p_\xi}{p_\xi^2} v_\xi \right)(x). \tag{4.12}$$

Recalling now $L_\xi = Df|_{n_\xi}$ and (3.14), we have:

$$\partial_\xi G_\xi = -L_\xi \partial_\xi n_\xi + K_\xi + Z_\xi J * \partial_\xi \varphi_\xi, \tag{4.13}$$

where

$$K_\xi = -\beta^3 (J*\varphi_\xi)^2 \int_0^1 ds\, s(1-s) \tanh'''\{\beta[J*(n_\xi + s\varphi_\xi) + h]\} J * \partial_\xi n_\xi, \tag{4.14}$$

$$\begin{aligned} Z_\xi = {} & -2\beta^2 J*\varphi_\xi \int_0^1 ds\,(1-s)\tanh''\{\beta[J*(n_\xi + s\varphi_\xi)+h]\} \\ & - \beta^3 (J*\varphi_\xi)^2 \int_0^1 ds\, s(1-s)\tanh'''\{\beta[J*(n_\xi+s\varphi_\xi)+h]\}. \end{aligned} \tag{4.15}$$

Observing that $T_\xi L_\xi \partial_\xi n_\xi = L_\xi T_\xi \partial_\xi n_\xi$, from (4.10) and (4.13) we obtain $\partial_\xi \varphi_\xi$ by solving the linear equation:

$$\psi = U_\xi \psi + g_\xi,$$

where

$$\begin{aligned} U_\xi \psi & \doteq L_\xi^{-1} T_\xi Z_\xi J * \psi, \\ g_\xi & \doteq L_\xi^{-1} T_\xi \left[K_\xi - \partial_\xi p_\xi J * \varphi_\xi - \pi_\xi(G_\xi) \partial_\xi v_\xi \right] + H(\xi) v_\xi - T_\xi \partial_\xi n_\xi. \end{aligned}$$

From (2.34) and (4.15) $\|U_\xi\|_\infty \le C \|\varphi_\xi\|_\infty$, hence, by using the first bound in (4.9), there is $\ell_7 \ge \ell_6$ such that $\|U_\xi\|_\infty \le 1/2$ for $\xi \in \Gamma_{\ell,\kappa}$. Then for such values of ξ the equation has a unique solution $\partial_\xi \varphi_\xi = (1-U_\xi)^{-1} g_\xi$. The continuity of $\xi \mapsto \partial_\xi\varphi_\xi$ follows from that of $\xi \mapsto U_\xi$ and $\xi \mapsto g_\xi$, which can be easily proved.

We are left with the second bound in (4.9). Since $\|\partial_\xi \varphi_\xi\|_\infty \le 2\|g_\xi\|_\infty$ we have only to prove an analogous estimate for the known term g_ξ. From (2.34), (4.14), and (2.33),

$$\|g_\xi\|_\infty \le C\left(\|\varphi_\xi\|_\infty^2 + \|\varphi_\xi\|_\infty + |\pi_\xi(G_\xi)| + |H(\xi)| + \|\partial_\xi n_\xi - \pi_\xi(\partial_\xi n_\xi) v_\xi)\|_\infty \right),$$

where we have used $\|\partial_\xi p_\xi\|_\infty \le \text{const}\|\partial_\xi n_\xi\|_\infty$ and (2.19), and $\|\partial_\xi v_\xi\|_\infty \le \text{const}$, which follows by (2.26) and (2.36). Recalling $G_\xi = -(f(n_\xi) + R_\xi[\varphi_\xi])$, (2.7), (2.20), (2.21), and (2.31) we have

$$|\pi_\xi(G_\xi)| \le C\left(e^{-2\alpha\xi} + \|\varphi_\xi\|_\infty\right).$$

From (2.35), (2.36), and $\|\partial_\xi p_\xi\|_\infty \le \text{const}$, we get that also $|H(\xi)| \le C\|\varphi_\xi\|_\infty$. Then, from the first bound in (4.9), (2.40), and the previous estimates, we conclude that $\|g_\xi\|_\infty \le Ce^{-\min\{2\alpha;\delta_1\}\xi}$. The proposition is thus proved. □

To conclude the proof of Theorem 2.9 we are left with the equation $P(\xi) = 0$, where $P(\xi) = \pi_\xi\left(f(n_\xi) + R_\xi[\varphi_\xi]\right)$. We first observe that from (2.20), (2.21), Lemma

2.4, and Proposition 4.2, for any $\kappa > 0$ there are $\ell_8 \geq \ell_7$ such that, for any $h < e^{-2\alpha\ell_8}$ and $\xi \in \Gamma_{\ell_8,\kappa}$,

$$|P(\xi) - V(\xi)| \leq Ce^{-(2\alpha+\delta_2)\xi} \tag{4.16}$$

(recall $\delta_2 = \min\{\delta_0; \delta_1; 2\alpha\}$). We then have:

$$\begin{aligned} P(\ell) &\leq V(\ell) + Ce^{-2(\alpha+\delta_2)\ell} \\ &= -\left(\mu K - Ce^{-\delta_2\ell}\right) e^{-2\alpha\ell} + 2m_\beta\mu h, \\ P\left((2\alpha)^{-1}|\log h| + \kappa\right) &\geq V\left((2\alpha)^{-1}|\log h| + \kappa\right) - Ce^{-(2\alpha+\delta_2)\kappa} h^{1+(2\alpha)^{-1}\delta_2} \\ &= \left[\mu(2m_\beta - Ke^{-2\alpha\kappa}) - Ce^{-(2\alpha+\delta_2)\kappa} h^{(2\alpha)^{-1}\delta_2}\right] h. \end{aligned}$$

Let $\kappa_0 > 0$ be such that $2m_\beta - Ke^{-2\alpha\kappa_0} \geq m_\beta$ and, for any $\kappa \geq \kappa_0$, let ℓ_9 so large that $\mu K - Ce^{-\delta_2\ell_9} \geq \mu K/2$. Then, for any $\kappa \geq \kappa_0$ and $\ell \geq \ell_9$ there is $h_{\kappa,\ell}$ such that, for any $h < h_{\kappa,\ell}$,

$$P(\ell) < 0, \qquad P\left((2\alpha)^{-1}|\log h| + \kappa\right) > 0.$$

Since the function $\xi \mapsto P(\xi)$ is easily seen to be continuous (actually differentiable) it follows it has at least one zero in $\Gamma_{\ell,\kappa}$ for any h small enough. In order to prove uniqueness it is sufficient to show the function is strictly monotone. Recalling $G_\xi \doteq -(f(n_\xi) + R_\xi[\varphi_\xi])$, using (4.13), and $v_\xi^* L_\xi = \lambda_\xi v_\xi^*$ we have:

$$P'(\xi) = \lambda_\xi \pi_\xi(\partial_\xi n_\xi) - \pi_\xi(K_\xi + Z_\xi J * \partial_\xi\varphi_\xi) + \partial_\xi\pi_\xi(f(n_\xi)) + \partial_\xi\pi_\xi(R_\xi[\varphi_\xi]).$$

From (2.31), (2.38), (4.14), (4.15), and Proposition 4.2, it follows that, for all $h < e^{-2\alpha\ell_9}$ and $\xi \in \Gamma_{\ell_9,\kappa}$,

$$|\pi_\xi(K_\xi + Z_\xi J * \partial_\xi\varphi_\xi)| + |\partial_\xi\pi_\xi(R_\xi[\varphi_\xi])| \leq Ce^{-2(\alpha+\delta_2)\xi}.$$

On the other hand, from (2.20), (2.21), and using (2.42), we have $|\partial_\xi\pi_\xi(f(n_\xi))| \leq Ce^{-(2\alpha+\delta_2)\xi}$. From (2.25) and (2.41) we conclude that

$$P'(\xi) \geq \frac{e^{-2\alpha\xi}}{c_1\sqrt{\mu}} \left(1 - Ce^{-\delta_2\xi}\right).$$

Then, for any $\kappa \geq \kappa_0$, there is $\ell_5 \geq \ell_9$ such that, for each $h \leq h_{\kappa,\ell_5}$, the equation $P(\xi) = 0$ has a unique solution on $\Gamma_{\ell_5,\kappa}$. Theorem 2.9 thus follows with $\bar{\xi}$ equal to such a solution, $\bar{\varphi} = \varphi_{\bar{\xi}}$, and $h_0 = h_{\kappa,\ell_5}$. We have only to check the limits (2.53). By our choice of ℓ_9 and since $\ell_5 \geq \ell_9$,

$$P(\xi) \leq -\frac{\mu K}{2} e^{-2\alpha\xi} + 2m_\beta\mu h \qquad \forall\, \xi \in \Gamma_{\ell_5,\kappa} \quad \forall\, h \leq h_{\kappa,\ell_5},$$

which implies $\bar{\xi} \to +\infty$ as $h \downarrow 0$. On the other hand, by (4.16),

$$\left|e^{2\alpha\bar{\xi}} V(\bar{\xi})\right| = \left|2m_\beta\mu h e^{2\alpha\bar{\xi}} - \mu K\right| \leq Ce^{-\delta_2\bar{\xi}}.$$

Letting $h \downarrow 0$ in the above inequality we thus obtain the first limit in (2.53); the second one then follows by the first bound in (4.9). $\square$

5. Spatial properties of the bump

In this section we prove Proposition 2.10. The critical droplet is uniquely determined by its restriction to the semi-space $\mathbb{R}_+$, which solves

$$q(x) = \tanh\left\{\beta\left[(J_+q)(x)+h\right]\right\}, \qquad x\in\mathbb{R}_+, \tag{5.1}$$

where, for any $u\in C(\mathbb{R}_+)$,

$$(J_\pm u)(x) \doteq \int_0^{+\infty} dy\, J_\pm(x,y)u(y), \qquad J_\pm(x,y)\doteq J(x-y)\pm J(x+y). \tag{5.2}$$

Observe that, since J is non increasing on $\mathbb{R}_+$, $J_\pm(x,y)\ge 0$ for all $x,y\ge 0$; moreover, since $\sup\{x\in\mathbb{R}: J(x)>0\}=1$, if $x>1$ then $J_\pm(x,y)=J(x-y)$ for all $y\ge 0$.

To prove the proposition we will exploit the fact that the critical droplet is a solution of (5.1) which is close, for h small, to a suitable reflected instanton, see (2.12).

Let $\bar m_{\xi^*}$ be as in Theorem 2.1. Since $\beta(1-m_\beta^2)<1$, from (1.10) and (2.13) there are $\theta\in(0,1)$, $h_1\in(0,h_0]$, and a positive integer $\ell^*\in(1,\xi^*-1)$ such that

$$\beta\left(1-\bar m_{\xi^*}(x)^2\right)\le\theta \qquad \forall\,|x-\xi^*|\ge\ell^*-1, \quad \forall\,h\in(0,h_1]. \tag{5.3}$$

On the other hand, recalling the definition (2.57), (2.12) implies

$$\lim_{h\downarrow 0}\left\|p-\beta(1-\bar m_{\xi^*}^2)\right\|_\infty = 0. \tag{5.4}$$

From (5.3) and (5.4) it follows there are $\delta\in(\theta,1)$ and $h_2\in(0,h_1]$ such that, for $\ell^*\in(1,\xi^*-1)$ as before,

$$p(x)\le\delta \qquad \forall\,|x-\xi^*|\ge\ell^*-1, \quad \forall\,h\in(0,h_2]. \tag{5.5}$$

Lemma 5.1. *Let $h\in(0,h_2]$, h_2 and ℓ^* be as in (5.5). Then, for each $k\in[0,\xi^*-\ell^*]$ and $s\ge\xi^*+\ell^*$, we have:*

$$q'(x) = \int_k^{k+1} dy\, H_k(x,y)q'(y) \qquad \forall\,x\in[0,k), \tag{5.6}$$

$$q'(x) = \int_{s-1}^{s} dy\, K_s(x,y)q'(y) \qquad \forall\,x\in(s,+\infty), \tag{5.7}$$

where $H_k(x,y)$, $x\in(0,k)$, and $K_s(x,y)$, $x>s$, are non negative continuous functions of y, strictly positive for some $y\in[k,k+1]$, $y\in[s-1,s]$ respectively.

Proof. We start with the proof of (5.6). We differentiate (5.1) at $x\in[0,k)$, obtaining (recall (5.2))

$$q'(x) = p(x)\int_0^k dy\, J_-(x,y)q'(y) + p(x)\int_k^{k+1} dy\, J_-(x,y)q'(y).$$

After N iteration we get

$$q'(x) = \int_k^{k+1} dy\, H_k^{(N)}(x,y)q'(y) + \int_0^k dy\, D_k^{(N)}(x,y)q'(y), \tag{5.8}$$

where

$$H_k^{(N)}(x,y)\doteq\sum_{n=1}^N D_k^{(n)}(x,y), \qquad D_k^{(1)}(x,y)\doteq p(x)J_-(x,y),$$

and, for $n > 1$, setting $x = y_0$ and $y = y_n$,

$$D_k^{(n)}(y_0, y_n) = \int_0^k dy_1 \cdots \int_0^k dy_{n-1} \prod_{i=1}^n p(y_{i-1}) J_-(y_{i-1}, y_i).$$

The assumptions on J imply

$$0 \le J_-(x,y) \le J(x-y), \quad J_-(0,y) \equiv 0 \qquad \forall\, x,y \in \mathbb{R}_+ \tag{5.9}$$

and

$$\sup\{|y-x| \in \mathbb{R}_+ \ :\ J_-(x,y) > 0\} > 0 \qquad \forall\, x > 0. \tag{5.10}$$

From (2.57), (5.5), and (5.9) we get

$$0 \le D_k^{(n)}(y_0, y_n) \le \delta^{n-1} J^n(y_0, y_n). \tag{5.11}$$

Since $J^n(y_0, y_n)$ is a probability density and $\|q'\|_\infty < \infty$, the second integral in the r.h.s. of (5.8) vanishes as $N \to +\infty$ and we obtain (5.6) with

$$H_k(x,y) = \sum_{n=1}^{\infty} D_k^{(n)}(x,y), \tag{5.12}$$

and the series converges exponentially fast. Clearly $H_k(x,\cdot)$ is non negative and continuous. Moreover, from (5.10), it is strictly positive for some $y \in [k, k+1]$.

The case $x > s$ can be treated in the same manner, getting

$$K_s(x,y) = \sum_{n=1}^{\infty} R_s^{(n)}(x,y), \qquad R_s^{(1)}(x,y) \doteq p(x) J(x-y), \tag{5.13}$$

where, for $n > 1$, setting $x = y_0$ and $y = y_n$,

$$R_s^{(n)}(y_0, y_n) = \int_s^{+\infty} dy_1 \cdots \int_s^{+\infty} dy_{n-1} \prod_{i=1}^n p(y_{i-1}) J(y_{i-1} - y_i). \tag{5.14}$$

In (5.13) we used that $J_-(u,v) = J(u-v)$ for $u > s > 1$ and $v \ge 0$. $\square$

Proof of the monotonicity property. We first prove that there is $h_3 \in (0, h_2]$ such that

$$q'(x) < 0 \qquad \forall\, |x - \xi^*| \le \ell^*, \quad \forall\, h \in (0, h_3]. \tag{5.15}$$

To prove (5.15) we differentiate (5.1) for $|x - \xi^*| \le \ell^*$. Recalling $J_+(x,y) = J(x-y)$ when $x + y > 1$, we get,

$$q'(x) = p(x)(J * q)'(x) = p(x)(J' * (q - \bar m_{\xi^*}))(x) + p(x)(J * \bar m'_{\xi^*})(x). \tag{5.16}$$

Since $\bar m_{\xi^*}$ is strictly decreasing on $\mathbb{R}_+$, from (2.12) we get (5.15).

From (5.15) and Lemma 5.1 it follows $q'(x) < 0$ for all $x > 0$ and $h \in (0, h_3]$, thus getting the monotonicity property of the bump. $\square$

We will prove Proposition 2.10 with $h^* = h_3$. We are thus left with the proof of (2.55). We follow the same strategy used in [6, Sect. 3]. In fact large part of that strategy can be adapted to our context without modification. We first need a weaker result.

Lemma 5.2. *There are $\eta > 0$ and $c > 0$ such that*

$$|q'(x)| \le c e^{-\eta(x - \xi_q)} \qquad \forall\, x \in \mathbb{R}_+, \quad \forall\, h \in (0, h^*], \tag{5.17}$$

where $\xi_q = \xi_q(h)$ is the (unique) positive zero of the bump.

Proof. By the definition (5.14), $R_s^{(n)}(x,y) = 0$ if $x > n+s$ and $y \in [s-1,s]$, and it satisfies a bound analogous to (5.11). Then, from (5.7), for any $x > s \geq \xi^* + \ell^*$, we have

$$|q'(x)| \leq \beta \|q'\|_\infty \sum_{n \geq x-s} \delta^{n-1} \leq \delta^{-1}\beta\|q'\|_\infty e^{-(x-s)|\log\delta|}. \tag{5.18}$$

Let ξ_q be the (unique) zero of $q(x)$ in $\mathbb{R}_+$. By (2.12) $\bar{m}_{\xi^*}(\xi_q) = \bar{m}(\xi^* - \xi_q)$ vanishes as $h \downarrow 0$, hence

$$\lim_{h\downarrow 0} [\xi_q(h) - \xi^*(h)] = 0. \tag{5.19}$$

In particular (5.19) implies there is $\ell < \infty$ such that

$$\xi_q + \ell \geq \xi^* + \ell^* \qquad \forall\, h \in [0,h^*]. \tag{5.20}$$

and (5.17) follows from (5.18) with $s = \xi_q + \ell$. □

From (5.7), we have, for each $s \geq \ell$,

$$q'(x+\xi_q) = \int_{s-1}^{s} dy\, G_s(x,y) q'(y+\xi_q), \qquad \forall\, h \in (0,h^*], \tag{5.21}$$

where, setting $p_{\xi_q}(x) \doteq p(x+\xi_q)$,

$$G_s(x,y) \doteq \sum_{n=1}^{\infty} \int_s^{+\infty} dy_1 \cdots \int_s^{+\infty} dy_{n-1} \prod_{i=1}^{n} p_{\xi_q}(y_{i-1}) J(y_{i-1}-y_i). \tag{5.22}$$

We observe that $p_{\xi_q}(x)$ is a strictly decreasing function of x for $x > 0$,

$$p_{\xi_q}(x) > \inf_{x>0} p_{\xi_q}(x) = p_\infty = \beta\left(1 - (m_{\beta,h}^-)^2\right) < 1 \tag{5.23}$$

and, by Lemma 5.2, there is $c' > 0$ such that

$$p_{\xi_q}(x) \leq p_\infty + c' e^{-\eta x}, \qquad \forall\, h \in [0,h^*]. \tag{5.24}$$

In [6, Thm. 3.1] the asymptotics of $\bar{m}'(x)$ follows from an analogous (to (5.21)) expression for $\bar{m}'(x)$, where $p_{\xi_q}(x)$ is replaced by $p_{\bar{m}}(x) \doteq \beta(1-\bar{m}(x)^2)$ in the definition of $G_s(x,y)$. The proof does not depend on the specific form of the function $p_{\bar{m}}(x)$, but only on the monotonicity property and the analogous of (5.23) and (5.24). Then a result as [6, Thm. 3.1] holds in our case. We conclude that there exist $M > 0$ and $\delta \in (0,\gamma)$, γ as in (2.54), such that

$$\lim_{x\to+\infty} e^{\gamma x} q'(x+\xi_q) = -M, \qquad \lim_{x\to+\infty} e^{\delta x}\left(e^{\gamma x} q'(x+\xi_q) + M\right) = 0. \tag{5.25}$$

As in [6] the constant M is non zero because of the monotonicity property of $q'(x)$. Moreover, since $0 < p_\infty < 1$ and (5.24) holds uniformly in h, the constant $M = M(h)$ appearing in (5.25) remains bounded away from 0 as $h \downarrow 0$.

Analogously we obtain (2.55) (with $A = M\gamma^{-1}$) from (5.25) by arguing exactly as in the proofs of [6, Thms. 3.2, 3.3] where (2.9) follows as a corollary of [6, Thm. 3.1]. We omit the details. □

6. The invariant manifold $\mathcal{W}$

In this section we prove Theorem 2.12, i.e. the existence of a one dimensional, invariant, expanding manifold $\mathcal{W}$ in $C^{\mathrm{sym}}(\mathbb{R};[-1,1])$ consisting of two branches that originate from the bump q.

For $h \in (0,h^*]$ (h^* as in Proposition 2.10) let L, λ, v be as in Proposition 2.11. We next derive some properties of the evolution $S_t(q+u_0)$ starting from an initial datum $q+u_0$ with u_0 small. We set $u_t \doteq S_t(q+u_0)-q$. Since $S_t(q)=q$ and $f(q)=0$ we have

$$\frac{du_t}{dt} = Lu_t + [f(q+u_t)-f(q)-Lu_t]\,, \tag{6.1}$$

which implies

$$u_t = e^{Lt}u_0 + \int_0^t ds\, e^{L(t-s)}\,[f(q+u_s)-f(q)-Lu_s]\,. \tag{6.2}$$

Then by (2.7) and (2.64) (with $\zeta=0$)

$$\left\|u_t - e^{Lt}u_0\right\|_\infty \le C_2\int_0^t ds\, e^{\lambda(t-s)}\|u_s\|_\infty^2, \tag{6.3}$$

where

$$C_2 \doteq C_1 k_2. \tag{6.4}$$

Lemma 6.1. *There is $N>0$ such that if $u_0\in L_\infty(\mathbb{R})$ satisfies*

$$\sigma(u_0) \doteq \frac{1}{\lambda}\log\frac{1}{N\|u_0\|_\infty} > 0, \tag{6.5}$$

then, for all $t<\sigma(u_0)$,

$$\left\|u_t - e^{Lt}u_0\right\|_\infty \le N\left(e^{\lambda t}\|u_0\|_\infty\right)^2 \tag{6.6}$$

and

$$\|u_t\|_\infty \le (1+C_1)e^{\lambda t}\|u_0\|_\infty. \tag{6.7}$$

with C_1 as in (2.64).

Proof. The lemma will follow with $N \doteq 4C_2\lambda^{-1}$. We prove (6.6) by contradiction. Fix $\tau<\sigma(u_0)$ and define $\rho_\tau \doteq e^{\lambda\tau}\|u_0\|_\infty$. Let $T\le\tau$ be the first time when the inequality (6.6) becomes an equality. Then, by (6.3) with $t=T$ (and supposing without loss of generality that $\|u_0\|_\infty>0$),

$$\begin{aligned}
N\left(e^{\lambda T}\|u_0\|_\infty\right)^2 &\le C_2\int_0^T ds\, e^{\lambda(T-s)}\left[e^{\lambda s}\|u_0\|_\infty + N\left(e^{\lambda s}\|u_0\|_\infty\right)^2\right]^2 \\
&\le C_2\, e^{\lambda T}\|u_0\|_\infty\int_0^T ds\, e^{\lambda s}\|u_0\|_\infty\,(1+N\rho_\tau)^2 \\
&\le C_2\,(1+N\rho_\tau)^2\,\lambda^{-1}\left(e^{\lambda T}\|u_0\|_\infty\right)^2 < N\left(e^{\lambda T}\|u_0\|_\infty\right)^2,
\end{aligned} \tag{6.8}$$

where in the last inequality we used $N\rho_\tau<1$. We have thus reached a contradiction and (6.6) is proved for all $t\le\tau$. Hence, by (2.64),

$$\begin{aligned}
\|u_t\|_\infty &\le C_1 e^{\lambda t}\|u_0\|_\infty + N\left(e^{\lambda t}\|u_0\|_\infty\right)^2 \\
&\le (C_1+N\rho_\tau)\,e^{\lambda t}\|u_0\|_\infty \le (1+C_1)e^{\lambda t}\|u_0\|_\infty
\end{aligned} \tag{6.9}$$

for all $t\le\tau$, and Lemma 6.1 is proved. □

We use in the sequel the following notation. For v, N as in Proposition 2.11 and Lemma 6.1 we denote by ρ any positive number such that

$$N\rho\|v\|_\infty < 1 \tag{6.10}$$

and define

$$\psi_{\pm\varepsilon} \doteq q \pm \varepsilon v, \qquad \varepsilon \in [0,\rho], \tag{6.11}$$

$$\tau(\rho,\varepsilon) \doteq \frac{1}{\lambda}\log\frac{\rho}{\varepsilon}, \qquad \text{i.e. } e^{\lambda\tau(\rho,\varepsilon)} = \frac{\rho}{\varepsilon}. \tag{6.12}$$

We observe that $\pm\varepsilon v$, $\varepsilon \in [0,\rho]$ satisfy the hypothesis of Lemma 6.1 and that $\tau(\rho,\varepsilon) < \sigma(\pm\varepsilon v)$, $\sigma(\cdot)$ as in (6.5). Hence, for any $t \le \tau(\rho,\varepsilon)$,

$$\left\|S_t\left(\psi_{\pm\varepsilon}\right) - \left(q \pm e^{\lambda t}\varepsilon v\right)\right\|_\infty \le N\left(e^{\lambda t}\varepsilon\|v\|_\infty\right)^2 \tag{6.13}$$

and

$$\left\|S_t\left(\psi_{\pm\varepsilon}\right) - q\right\|_\infty \le (1+C_1)e^{\lambda t}\varepsilon\|v\|_\infty. \tag{6.14}$$

Theorem 6.2. *For any $h \in (0,h^*]$ (h^* as in Proposition 2.10), there are $\rho > 0$ and $w_s^\pm \in C_0^{\mathrm{sym}}(\mathbb{R})$, $s \le 0$, such that, for any $s \le 0$,*

$$\lim_{\varepsilon\downarrow 0}\|S_{\tau(\rho,\varepsilon)+s}(\psi_{\pm\varepsilon}) - w_s^\pm\|_\infty = 0. \tag{6.15}$$

Moreover

$$\lim_{s\to-\infty}\|w_s^\pm - q\|_\infty = 0; \qquad S_t(w_s^\pm) = w_{s+t}^\pm \quad \text{if } s+t \le 0. \tag{6.16}$$

A uniformity in $s \le 0$ of the limit (6.15) is proved in Proposition 6.6 below to which we refer for a precise statement.

Proof of Theorem 6.2. We will next prove that if ρ is small enough then $\{S_{\tau(\rho,\varepsilon)}(\psi_{\pm\varepsilon}) : \varepsilon \in (0,\rho]\}$ is a Cauchy sequence as $\varepsilon \downarrow 0$. Without loss of generality we restrict to the case with the plus sign. Then we need to estimate

$$S_{\tau(\rho,\varepsilon')}\left(\psi_{\varepsilon'}\right) - S_{\tau(\rho,\varepsilon)}\left(\psi_\varepsilon\right), \qquad 0 < \varepsilon' < \varepsilon. \tag{6.17}$$

Observing that

$$\psi_\varepsilon = q + e^{\lambda\tau(\varepsilon,\varepsilon')}\varepsilon' v,$$

by (6.13),

$$\left\|S_{\tau(\varepsilon,\varepsilon')}\left(\psi_{\varepsilon'}\right) - \psi_\varepsilon\right\|_\infty \le N\|v\|_\infty^2\varepsilon^2. \tag{6.18}$$

We thus need to compare $S_t\left(\psi_\varepsilon\right)$ and $S_t(\tilde m)$, $t \le \tau(\rho,\varepsilon)$, for all functions $\tilde m$ such that

$$\|\tilde m - \psi_\varepsilon\|_\infty \le N\|v\|_\infty^2\varepsilon^2. \tag{6.19}$$

Let

$$\Delta_t \doteq S_t\left(\psi_\varepsilon\right) - S_t(\tilde m). \tag{6.20}$$

By (2.7) we have

$$\frac{d\Delta_t}{dt} = L_{S_t(\psi_\varepsilon)}\Delta_t + R_t^{(1)}, \qquad \|R_t^{(1)}\|_\infty \le k_2\|\Delta_t\|_\infty^2. \tag{6.21}$$

Since $\|L_{m+u}\Delta - L_m\Delta\|_\infty \le c'\|u\|_\infty\|\Delta\|_\infty$ with c' a suitable constant, by (6.14) there is C_3 so that

$$R_t^{(2)} \doteq L_{S_t(\psi_\varepsilon)}\Delta_t - L\Delta_t, \qquad \|R_t^{(2)}\|_\infty \le C_3\rho\|\Delta_t\|_\infty \qquad \forall\, t \le \tau(\rho,\varepsilon). \tag{6.22}$$

Thus

$$\frac{d\Delta_t}{dt} = L\Delta_t + R_t^{(2)} + R_t^{(1)} \tag{6.23}$$

and

$$\Delta_t = e^{Lt}\Delta_0 + \int_0^t ds\, e^{L(t-s)}[R_s^{(2)} + R_s^{(1)}]. \tag{6.24}$$

Then by (2.64) and the bounds in (6.21)–(6.22), for any $t \le \tau(\rho,\varepsilon)$,

$$\|\Delta_t\|_\infty \le C_1 e^{\lambda t}\Delta_0 + C_1 \int_0^t ds\, e^{\lambda(t-s)}[C_3\rho\|\Delta_s\|_\infty + k_2\|\Delta_s\|_\infty^2].$$

Calling

$$\lambda^* \doteq \lambda + C_2C_3\rho, \tag{6.25}$$

we have, by iteration and recalling (6.4),

$$\|\Delta_t\|_\infty \le C_1 e^{\lambda^* t}\Delta_0 + C_2 \int_0^t ds\, e^{\lambda^*(t-s)}\|\Delta_s\|_\infty^2. \tag{6.26}$$

Setting $W_t \doteq e^{-\lambda^* t}\|\Delta_t\|_\infty$ and using (6.19), from (6.26) we get, for all $t \le \tau(\rho,\varepsilon)$,

$$W_t \le C_1 N\|v\|_\infty^2\varepsilon^2 + C_2\int_0^t ds\, W_s^2, \tag{6.27}$$

which implies:

$$W_t \le c\varepsilon^2 \sum_{n=0}^\infty \left(c\varepsilon^2 t\right)^n, \qquad c \doteq C_1(1\vee C_2)N\|v\|_\infty^2. \tag{6.28}$$

Since $\varepsilon\tau(\rho,\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, we can choose $\varepsilon_1 \in (0,\rho]$ so that the series converges and $W_t \le 2c\varepsilon^2$ for all $\varepsilon \in (0,\varepsilon_1]$ and $t \le \tau(\rho,\varepsilon)$. Choosing ρ small enough so that

$$C_2C_3\rho \le \frac{\lambda}{2} \qquad \text{i.e.} \quad e^{\lambda^*\tau(\rho,\varepsilon)} \le \left(\frac{\rho}{\varepsilon}\right)^{3/2} \tag{6.29}$$

and recalling (6.12), (6.25), and the definition of W_t, we get

$$\|\Delta_t\|_\infty \le C_4\sqrt{\varepsilon}, \qquad \forall\, \varepsilon \in (0,\varepsilon_1] \quad \forall\, t \le \tau(\rho,\varepsilon), \tag{6.30}$$

with $C_4 \doteq 2c\rho^{3/2}$.

By (6.18) and (6.30) we conclude that

$$\left\|S_{\tau(\rho,\varepsilon')}(\psi_{\varepsilon'}) - S_{\tau(\rho,\varepsilon)}(\psi_\varepsilon)\right\|_\infty \le C_4\sqrt{\varepsilon} \qquad \text{if } 0 < \varepsilon' < \varepsilon \le \varepsilon_1, \tag{6.31}$$

which shows $\{S_{\tau(\rho,\varepsilon)}(\psi_\varepsilon)\}$ is a Cauchy sequence as $\varepsilon \downarrow 0$ for all ρ small enough. Analogously we argue for the case with the minus sign.

The same argument shows that also $S_{\tau(\rho,\varepsilon)+s}(\psi_{\pm\varepsilon})$ is, for each $s \le 0$, a Cauchy sequence. Then $S_{\tau(\rho,\varepsilon)+s}(\psi_{\pm\varepsilon})$ converges in sup norm as $\varepsilon \downarrow 0$ to a function $w_s^\pm$, hence (6.15). Moreover if $t+s \le 0$, $t \ge 0$, then $S_t\left(S_{\tau(\rho,\varepsilon)+s}(\psi_\varepsilon)\right) = S_{\tau(\rho,\varepsilon)+s+t}(\psi_\varepsilon)$. By (2.3) for each $t \ge 0$, $S_t(m)$ depends continuously on m, thus $S_t\left(S_{\tau(\rho,\varepsilon)+s}(\psi_\varepsilon)\right) \to S_t(w_s^\pm)$ as $\varepsilon \downarrow 0$. On the other hand $S_{\tau(\rho,\varepsilon)+s+t}(\psi_\varepsilon) \to w_{s+t}^\pm$ as $\varepsilon \downarrow 0$, hence $S_t(w_s^\pm) = w_{s+t}^\pm$, proving the second relation in (6.16). Finally, from (6.14),

$$\left\|S_{\tau(\rho,\varepsilon)+s}(\psi_{\pm\varepsilon}) - q\right\|_\infty \le C_5\, e^{\lambda s}, \qquad C_5 \doteq (1+C_1)\rho\|v\|_\infty, \tag{6.32}$$

from which, letting $\varepsilon \downarrow 0$,

$$\left\|w_s^\pm - q\right\|_\infty \le C_5\, e^{\lambda s}, \tag{6.33}$$

proving the first statement in (6.16), Theorem 6.2 is proved. □

Proof of Theorem 2.12. The manifold

$$\mathcal{W} \doteq \mathcal{W}^+ \cup \mathcal{W}^-, \quad \mathcal{W}^\pm \doteq \left\{S_t(w_s^\pm)\,:\, s \le 0 \le t\right\} \tag{6.34}$$

and both its branches $\mathcal{W}^\pm$ are invariant under S_t which is invertible on $\mathcal{W}^\pm$. By (6.16) $\mathcal{W}^\pm$ originate at $s=-\infty$ from q.

Recalling (6.12), from (6.13)

$$\left\|S_{\tau(\rho,\varepsilon)+s}(\psi_{\pm\varepsilon}) - \left(q \pm e^{\lambda s}\rho v\right)\right\|_\infty \le C_6\, e^{2\lambda s}, \qquad C_6 \doteq N\left(\rho\|v\|_\infty\right)^2, \tag{6.35}$$

from which, letting $\varepsilon \downarrow 0$,

$$\left\|w_s^\pm - \left(q \pm e^{\lambda s}\rho v\right)\right\|_\infty \le C_6\, e^{2\lambda s}. \tag{6.36}$$

Next, by (6.1) and recalling that $f(q)=0$,

$$\begin{aligned}\frac{dw_s^\pm}{ds} &= L(w_s^\pm - q) + \left[f(w_s^\pm) - f(q) - L\left(w_s^\pm - q\right)\right] = L\left[w_s^\pm - \left(q \pm e^{\lambda s}\rho v\right)\right] \\ &\quad \pm \lambda e^{\lambda s}\rho v + \left[f(w_s^\pm) - f(q) - L\left(w_s^\pm - q\right)\right]. \end{aligned} \tag{6.37}$$

Denoting by $\|L\|_\infty$ the norm of the operator L (which is finite), by (2.7), (6.33), and (6.36) we have:

$$\left\|\frac{dw_s^\pm}{ds} \mp \lambda e^{\lambda s}\rho v\right\|_\infty \le C_7\, e^{2\lambda s}, \qquad C_7 \doteq \|L\|_\infty C_6 + k_2 C_5^2. \tag{6.38}$$

Recalling that $v(x) \approx \tilde m'(\xi^* - x)$ in the sense of (2.69), we set

$$s_0 :\ \rho e^{\lambda s_0} = \frac{1}{\sqrt{\mu}}, \qquad m_s^\pm \doteq w_{s+s_0}^\pm. \tag{6.39}$$

Then (6.38) implies

$$\left\|\frac{dm_s^\pm}{ds} \mp \lambda e^{\lambda s}\frac{1}{\sqrt{\mu}} v\right\|_\infty \le C_7\, e^{2\lambda s},$$

which gives (2.73).

The proofs of the monotonicity property of $m_s^\pm$ and of the bound (2.74) will be given in Proposition 6.3 below, Theorem 2.12 is then proved. □

Proposition 6.3. *For any $s \in \mathbb{R}$, the symmetric functions $m_s^\pm$ are non increasing on $\mathbb{R}_+$ and* (2.74) *holds.*

In order to prove Proposition 6.3 we need the following properties of the flow S_t.

Theorem 6.4. (The Comparison Theorem, [8]**)** *Let $m, \tilde m \in L_\infty(\mathbb{R})$ be such that $m(x) \le \tilde m(x)$ for all $x \in \mathbb{R}$. Then $S_t(m)(x) \le S_t(\tilde m)(x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.*

Lemma 6.5. *Let $m \in L_\infty^{\rm sym}(\mathbb{R})$ be a non increasing function on $\mathbb{R}_+$. Then $S_t(m)$ has the same monotonicity property for all $t \in \mathbb{R}_+$.*

Proof. The flow solution $S_t(m)$ solves the integral equation

$$S_t(m) = e^{-t}m + \int_0^t ds\, e^{-(t-s)} \tanh\left\{\beta\left[J * S_s(m) + h\right]\right\}.$$

Since J is smooth, the function $g_t(x) \doteq S_t(m)(x) - e^{-t}m(x)$ is differentiable. Further its spatial derivative $g_t'(x)$ is an antisymmetric function which satisfies, for any $x \in \mathbb{R}_+$,

$$g_t'(x) = \int_0^t ds\, p_s(x)\left(J_- g_s'\right)(x) + z_t(x), \tag{6.40}$$

$$p_s(x) \doteq \frac{\beta}{\cosh^2\left\{\beta\left[\left(J * S_s(m)\right)(x) + h\right]\right\}}, \quad z_t(x) \doteq \int_0^t ds\, e^{-t} p_s(x)(J_- * m')(x).$$

To get (6.40) we used that, since g_s is differentiable, and since both g'_s and m' are odd functions,

$$\frac{d}{dx}\left(J*S_s(m)\right)(x)=\left(J*g'_s\right)(x)+e^{-s}(J*m')(x)=\left(J_-g'_s\right)(x)+e^{-s}(J_-m')(x),$$

where for any function u on $\mathbb{R}_+$, J_-u, is defined in (5.2). By iteration of (6.40), calling $(t,x)=(s_0,x_0)$, we get

$$\begin{aligned} g'_t(x) &= \sum_{n=1}^{\infty}\int_0^{s_0}ds_1\dots\int_0^{s_{n-1}}ds_n \\ &\quad\times\int_0^{+\infty}dx_1\dots\int_0^{+\infty}dx_n\prod_{k=1}^{n-1}p_s(x_k)J_-(x_k,x_{k+1})z_{s_n}(x_n). \end{aligned} \tag{6.41}$$

From the fact that $J_-\geq 0$ and $m'(x)\leq 0$ for $x\geq 0$ it follows that z_s is a non positive function on $\mathbb{R}_+$. Since $J_-(x,y)$ and $p_s(x)$ are non negative for $x,y\geq 0$, we conclude from (6.41) that also g'_t is a non positive function on $\mathbb{R}_+$. Then $S_t(m)$ is non increasing on $\mathbb{R}_+$ because sum of two functions with this property. The lemma is proved. □

Proof of Proposition 6.3. Since the difference between $m_s^{\pm}$ and $w_s^{\pm}$ is only a time shift, see (6.39), it is enough to prove the proposition for $w_s^{\pm}$.

We start with the monotonicity property. We use Theorem 6.2 and Lemma 6.5. Thus the first step is to show that for ε small the functions $\psi_{\pm\varepsilon}=q\pm\varepsilon v$ are non increasing on $\mathbb{R}_+$. To this purpose we first notice that, by definition (see (2.59)),

$$v'(x)=-\frac{2\beta}{1+\lambda}q(x)q'(x)(J*v)(x)+\frac{p}{1+\lambda}(J'*v)(x). \tag{6.42}$$

By (2.63), for a suitable constant C_8,

$$\sup_{x>1}\left|e^{\gamma_v x}v'(x)\right|\leq C_8\sup_{x>1}e^{\gamma_v x}v(x)<\infty. \tag{6.43}$$

Then from (2.55), since $\gamma_v>\gamma$, we get

$$\sup_{x>1}\left|\frac{v'(x)}{q'(x)}\right|<\infty. \tag{6.44}$$

For $x\in[0,1]$, since $q'(0)=0$, we need to show that $q''(0)\neq 0$. This is easily seen by noticing that since $q'(0)=0$ and both J' and q' are antisymmetric functions,

$$q''(0)=-2\beta(1-q(0)^2)\int_0^1 dy\,J'(y)q'(y)<0. \tag{6.45}$$

In the last inequality we used that, by our assumptions on the function J and Proposition 2.10, $J'(x)q'(x)>0$ for $x\in(0,1)$. From (6.44) and (6.45) we then get

$$\sup_{x\in\mathbb{R}_+}\left|\frac{v'(x)}{q'(x)}\right|<\infty. \tag{6.46}$$

Lemma 6.5 and (6.46) imply that for any $s\leq 0$ there is $\varepsilon_s\in(0,\rho]$ such that $\{S_{\tau(\rho,\varepsilon)+s}(\psi_{\pm\varepsilon})\,:\,\varepsilon\in(0,\varepsilon_s]\}$ is a sequence of non increasing functions on $\mathbb{R}_+$. Hence from (6.15) the same property holds for $w_s^{\pm}$, $s\leq 0$. Then the monotonicity property of $w_s^{\pm}$ for all $s\in\mathbb{R}$ follows from Lemma 6.5.

We are left with the bound (2.74). Since q solves (1.7) and it is strictly decreasing on $\mathbb{R}^+$, it follows that $m^-_{\beta,h}<q(x)<m^+_{\beta,h}$ for all $x\in\mathbb{R}$. We also recall that q

satisfies (2.55). Since v is a positive function which satisfies (2.63) with $\gamma_v > \gamma$, we conclude that, for all ε small enough,

$$m^-_{\beta,h} \le \psi_{-\varepsilon}(x) < q(x) < \psi_\varepsilon(x) < m^+_{\beta,h}. \tag{6.47}$$

Then (2.74) follows from Theorem 6.2 and the Comparison Theorem. □

We conclude this section by proving Proposition 6.6 below, which is a stronger version of Theorem 6.2, since we show that the curves $\{w^\pm_s\}$ are the limits, in sup norm, of the curves $S_{\tau(\rho,\delta)}\mathcal{C}_\delta$ where, for any $\delta > 0$,

$$\mathcal{C}_\delta \doteq \{\psi_\varepsilon \,:\, 0 < \varepsilon < \delta\}\,.$$

Proposition 6.6. *Let $\delta > 0$, $s \le 0$, and*

$$\delta(s) \doteq e^{\lambda s}\delta. \tag{6.48}$$

Then

$$\lim_{\delta\downarrow 0}\sup_{s\le 0}\left\|S_{\tau(\rho,\delta)}\left(\psi_{\pm\delta(s)}\right) - w^\pm_s\right\|_\infty = 0. \tag{6.49}$$

Proof. Without loss of generality we restrict to the case with the plus sign in (6.48) and (6.49). We need to show that for any $\eta > 0$ there is $\delta_\eta > 0$ so that for any $\delta < \delta_\eta$ and $s \le 0$

$$\left\|S_{\tau(\rho,\delta)}\left(\psi_{\delta(s)}\right) - w^+_s\right\|_\infty \le \eta. \tag{6.50}$$

We approximate w^+_s by $S_t(\psi_\varepsilon)$ for suitable values of ε and t: given $s \le 0$ let ε_0 be such that for $\varepsilon \in (0, \varepsilon_0]$

$$\left\|S_{\tau(\rho,\varepsilon)+s}\left(\psi_\varepsilon\right) - w^+_s\right\|_\infty \le \frac{\eta}{2}. \tag{6.51}$$

For $\delta < \rho$ we have

$$S_{\tau(\rho,\varepsilon)+s}\left(\psi_\varepsilon\right) = S_{\tau(\rho,\delta)}\left(S_{\tau(\delta(s),\varepsilon)}\left(\psi_\varepsilon\right)\right). \tag{6.52}$$

By (6.13),

$$\left\|S_{\tau(\delta(s),\varepsilon)}\left(\psi_\varepsilon\right) - \psi_{\delta(s)}\right\|_\infty = \left\|S_{\tau(\delta(s),\varepsilon)}\left(\psi_\varepsilon\right) - q - e^{\lambda\tau(\delta(s),\varepsilon)}\varepsilon v\right\|_\infty = N\|v\|^2_\infty\delta(s)^2. \tag{6.53}$$

We define

$$D_t \doteq S_t\left(\psi_{\delta(s)}\right) - S_t\left(S_{\tau(\delta(s),\varepsilon)}\left(\psi_\varepsilon\right)\right), \tag{6.54}$$

so that

$$\left\|D_{\tau(\rho,\delta)}\right\|_\infty = \left\|S_{\tau(\rho,\varepsilon)+s}\left(\psi_\varepsilon\right) - S_{\tau(\rho,\delta)}\left(\psi_{\delta(s)}\right)\right\|_\infty. \tag{6.55}$$

The analysis of D_t is identical to that of Δ_t in the proof of Theorem 6.2. In fact, by comparing (6.19)–(6.20) with (6.53)–(6.54), we see that D_t satisfies the conditions defining the function Δ_t when the parameter ε appearing in (6.19)–(6.20) is replaced by $\delta(s)$. Then the bound (6.30) applied to D_t becomes

$$\|D_t\|_\infty \le C_4\sqrt{\delta(s)} \qquad \forall\,\delta \in (0,\varepsilon_1] \quad \forall\, t \le \tau(\rho,\delta(s)),$$

which implies

$$\left\|D_{\tau(\rho,\delta)}\right\|_\infty \le C_4\sqrt{\delta} \qquad \forall\,\delta \in (0,\varepsilon_1]. \tag{6.56}$$

We then choose $\delta_\eta \in (0,\varepsilon_1]$ so small that

$$C_4\sqrt{\delta_\eta} \le \frac{\eta}{2}, \tag{6.57}$$

and we have by (6.51) and (6.55)–(6.57) that, for all $\delta < \delta_\eta$,

$$\begin{aligned}\left\| S_{\tau(\rho,\delta)}\left(\psi_{\delta(s)}\right) - w_s^+\right\|_\infty &\le \left\| S_{\tau(\rho,\varepsilon)+s}\left(\psi_\varepsilon\right) - w_s^+\right\|_\infty \\ &+ \left\| S_{\tau(\rho,\varepsilon)+s}\left(\psi_\varepsilon\right) - S_{\tau(\rho,\delta)}\left(\psi_{\delta(s)}\right)\right\|_\infty \le \eta. \end{aligned} \tag{6.58}$$

Proposition 6.6 is thus proved. □

7. Global structure of $\mathcal{W}$

In this section we prove Theorem 2.13. To this purpose we will define suitable functions $Q_a^+ \le q \le Q_a^-$, a a small parameter, which are close to q, see (7.10) below. We shall prove that the functions m_s^+ (resp. m_s^-) at a certain time s are above Q_a^- (resp. below Q_a^+). Then, by the Comparison Theorem it is enough to study the evolution of $Q_a^\pm$. Using the spectral properties of the linear operator L, we show that, for a time interval $T_a \sim |\log a|$, the evolution $S_{T_a}(Q_a^+)$ (resp. $S_{T_a}(Q_a^-)$) can be bounded from above (resp. below) by the same functions Q_a^+ (resp. Q_a^-) suitably translated in space, see Theorem 7.2 below. By the Comparison Theorem we can iterate the argument, thus getting bounds at longer times which, combined with general properties of the flow S_t, lead to the desired result, Corollary 7.3 below, from which Theorem 2.13 will follow.

In the sequel we shall need a more refined *apriori* bound on the evolution around the critical droplet, which is the content of the following lemma.

Lemma 7.1. *There is $K>0$ such that if $u_t \doteq S_t(q+u_0) - q$, $u_0 \in L_\infty(\mathbb{R})$, then, for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,*

$$\left|u_t(x) - e^{Lt}u_0(x)\right| \le K \int_0^t ds\, e^{L(t-s)} \left(J * e^{Ls}u_0\right)^2(x) + K\mathcal{R}_t[u_\cdot], \tag{7.1}$$

where

$$\mathcal{R}_t[u_\cdot] = e^{\lambda t} \sup_{s\in[0,t]} \left\{ \|u_s\|_\infty^3 + \|u_s\|_\infty \left\|u_s - e^{Ls}u_0\right\|_\infty + \left\|u_s - e^{Ls}u_0\right\|_\infty^2 \right\}. \tag{7.2}$$

Proof. We recall u_t solves (6.2). Expanding the tanh appearing in (2.2) around $\beta(J*q+h)$ and using that q solves (1.7) we have

$$f\left(q+u_s\right) - f(q) - Lu_s = \Phi\left(J*u_s\right)^2 + \frac{\beta^3}{3!}\tanh'''(\theta_s)\left(J*u_s\right)^3, \tag{7.3}$$

where

$$\Phi(x) \doteq -\beta^2 q(x)\left(1-q(x)^2\right), \tag{7.4}$$

while θ_s is a number in the interval with end-points $\beta[J*q+h]$ and $\beta[J*(q+u_s)+h]$. Then we rewrite (6.2) as

$$u_t = e^{Lt}u_0 + \int_0^t ds\, e^{L(t-s)}\left[\Phi\left(J*e^{Ls}u_0\right)^2 + R[u_s]\right], \tag{7.5}$$

where, using (7.3),

$$R[u_s] = \Phi\left[\left(J*u_s\right)^2 - \left(J*e^{Ls}u_0\right)^2\right] + \frac{\beta^3}{3!}\tanh'''(\theta_s)\left(J*u_s\right)^3. \tag{7.6}$$

Since Φ is a bounded function on $\mathbb{R}$, the first integral on the r.h.s. of (7.5) is bounded by the first term on the r.h.s. of (7.1) with $K = \|\Phi\|_\infty$.

We next rewrite the square bracket on the r.h.s. of (7.6) as

$$(J * u_s)^2 - \left(J * e^{Ls} u_0\right)^2 = \left[J * (u_s - e^{Ls} u_0)\right]^2 + 2\left(J * e^{Ls} u_0\right)\left[J * (u_s - e^{Ls} u_0)\right].$$

Then using $\tanh'''$ is bounded and J has compact support, from (2.64) we have, for any K large enough,

$$\left\| \int_0^t ds\, e^{L(t-s)} R[u_s] \right\|_\infty \le K \mathcal{R}_t[u_\cdot].$$

The lemma is proved. □

Warning: For the rest of the section we shall denote by C a generic constant whose numerical value may change from line to line. From the statements it will appear clear on which parameters it depends on.

Let γ and λ be as in (2.54) and (2.61) respectively. We fix δ and R_0 such that

$$0 < \delta < \frac{1}{8}, \qquad \frac{3}{2} < \gamma R_0 < 2 - 4\delta, \tag{7.7}$$

and we set, for any $a \in (0,1]$,

$$T_a \doteq \frac{\delta}{\lambda} |\log a|, \qquad R_a \doteq R_0 |\log a|, \qquad \Delta_a \doteq a^{1-\delta/2}. \tag{7.8}$$

Recalling (2.61), (2.62), and (2.68), there exists an $\bar{h} \in (0, h^*]$, h^* as in Proposition (2.10), such that

$$(\gamma_v - \gamma) R_0 \le \frac{\delta}{4} \quad \text{and} \quad \delta \lambda^{-1} \omega > 3 \qquad \forall\, h \in [0, \bar{h}]. \tag{7.9}$$

We define the symmetric functions

$$Q_a^\pm(x) \doteq q_a^\pm(x) \mathbf{1}_{|x| \le R_a} + \left[m_{\beta,h}^- \pm a^{3/2}\right] \mathbf{1}_{|x| > R_a}, \tag{7.10}$$

where

$$q_a^+(x) \doteq q(|x| + a), \qquad q_a^-(x) \doteq q(0) \mathbf{1}_{|x| \le a} + q(|x| - a) \mathbf{1}_{|x| > a}. \tag{7.11}$$

The main result of this section is the content of the following theorem.

Theorem 7.2. *Let $h \in [0, \bar{h}]$ with $\bar{h}$ as in (7.9). Then there is $a_0 \in (0,1]$ such that, for any $a \in (0, a_0]$,*

$$S_{T_a}\left(Q_a^+\right)(x) \le Q_a^+(x + \Delta_a) \tag{7.12}$$

and

$$S_{T_a}\left(Q_a^-\right)(x) \ge Q_a^-(x - \Delta_a), \tag{7.13}$$

with T_a and Δ_a as in (7.8).

Proof. From Proposition 2.10 there is a constant c such that $|q''(x)| \le c|q'(x)|$ for any $|x| \ge 1$. Then, by expanding to second order $q_a^\pm(x)$ around $q(x)$ for $|x| \ge 1$, and using $q_a^+(x) \le q(x) \le q_a^-(x)$ for all $x \in \mathbb{R}$ (see (7.11)) we have, for any a small enough,

$$q_a^+(x) \le q(x) + \frac{a}{2} q'(|x|) \mathbf{1}_{|x| \ge 1}, \qquad q_a^-(x) \ge q(x) - \frac{a}{2} q'(|x|) \mathbf{1}_{|x| \ge 1}. \tag{7.14}$$

Observing $q'(|x|) = -|q'(x)|$ for any $x \in \mathbb{R}$, if we define

$$\varphi(x) \doteq \frac{1}{2} |q'(x)| \mathbf{1}_{|x| \ge 1}, \tag{7.15}$$

from (7.10) and (7.14) we obtain:

$$Q_a^+(x) \le q(x) - a\varphi(x) + \left[m_{\beta,h}^- - q(x) + a^{3/2} + a\varphi(x)\right] \mathbf{1}_{|x|>R_a}, \tag{7.16}$$

$$Q_a^-(x) \ge q(x) + a\varphi(x) + \left[m_{\beta,h}^- - q(x) - a^{3/2} - a\varphi(x)\right] \mathbf{1}_{|x|>R_a}. \tag{7.17}$$

Moreover, from (2.55) and (7.7), for any a small enough,

$$|m_{\beta,h}^- - q(x)| + a|\varphi(x)| \le \frac{1}{2}a^{3/2} \qquad \forall\, |x| > R_a,$$

so that, if we define

$$U_0^\pm(x) \doteq \mp a\varphi(x) \pm \frac{3}{2}a^{3/2}\mathbf{1}_{|x|>R_a}, \tag{7.18}$$

from (7.16) and (7.17) we get, for any a small enough,

$$Q_a^+(x) \le q(x) + U_0^+(x), \qquad Q_a^-(x) \ge q(x) + U_0^-(x). \tag{7.19}$$

We shall now obtain good bounds on $S_{T_a}\left(q + U_0^\pm\right)$. We can apply Lemma 7.1 to $U_t^\pm \doteq S_t\left(q + U_0^\pm\right) - q$, getting:

$$\left|U_t^\pm - e^{Lt}U_0^\pm\right| \le K\int_0^t ds\, e^{L(t-s)}\left(J * e^{Ls}U_0^\pm\right)^2 + K\mathcal{R}_t\left[U_\cdot^\pm\right]. \tag{7.20}$$

We will use (7.20) to estimate $U_{T_a}^\pm$, by analyzing separately the various terms.

Estimate on $e^{LT_a}U_0^\pm$. Since $e^{\lambda T_a} = a^{-\delta}$, see (7.8), we have

$$e^{LT_a}U_0^\pm = \mp a^{1-\delta}\pi(\varphi)v \mp a e^{LT_a}\left[\varphi - \pi(\varphi)v\right] \pm \frac{1}{2}a^{3/2}e^{LT_a}\mathbf{1}_{|x|>R_a}, \tag{7.21}$$

where, recalling (2.66) and (7.15),

$$\pi(\varphi) = \int_1^\infty dx\, \frac{v(x)}{p(x)}\, |q'(x)| > 0. \tag{7.22}$$

From the spectral gap property (2.67) and (7.9),

$$\left\|e^{LT_a}\left[\varphi - \pi(\varphi)v\right]\right\|_\infty \le e^{-\omega T_a}\|\varphi - \pi(\varphi)v\|_\infty \le Ca^3. \tag{7.23}$$

Analogously we estimate:

$$e^{LT_a}\mathbf{1}_{|x|>R_a} = a^{-\delta}\pi\left(\mathbf{1}_{|x|>R_a}\right)v + e^{LT_a}\left[\mathbf{1}_{|x|>R_a} - \pi\left(\mathbf{1}_{|x|>R_a}\right)v\right] \le Ca^{3/2-\delta}, \tag{7.24}$$

where we used $\pi\left(\mathbf{1}_{|x|>R_a}\right) \le Ca^{\gamma_v R_0}$ with $\gamma_v R_0 > \gamma R_0 > 3/2$. From (7.21), (7.23) and (7.24) we obtain:

$$\left|e^{LT_a}U_0^\pm \pm a^{1-\delta}\pi(\varphi)v\right| \le Ca^{3-\delta}. \tag{7.25}$$

Estimate on $\int_0^{T_a} ds\, e^{L(T_a-s)}\left(J * e^{Ls}U_0^\pm\right)^2$. Using (2.64) with $\zeta = 0$ and (7.24), we get, for any a small enough,

$$\begin{aligned}\int_0^{T_a} ds\, e^{L(T_a-s)}\left(J * e^{Ls}U_0^\pm\right)^2 &\le Ca^2\int_0^{T_a} ds\, e^{L(T_a-s)}\Big[\left(J * e^{Ls}\varphi\right)^2 \\ &\quad + a\left(J * e^{Ls}\mathbf{1}_{|x|\ge R_a}\right)^2 + \sqrt{a}\left(J * e^{Ls}\varphi\right)\left(J * e^{Ls}\mathbf{1}_{|x|\ge R_a}\right)\Big] \\ &\le Ca^{3-2\delta} + Ca^2\int_0^{T_a} ds\, e^{L(T_a-s)}(J * e^{Ls}\varphi)\left[(J * e^{Ls}\varphi) + a^{2-\delta}\right]. \end{aligned}\tag{7.26}$$

Now, recalling the definitions (7.15) and (2.58), from the asymptotics (2.55) it follows that $\varphi \in X_\gamma$. Since $\gamma < \gamma_v$ we can use (2.64) with $\zeta = \gamma$. Hence, since J has compact support,

$$\left| \left(J * e^{Ls}\varphi \right)(x) \right| \le C e^{\lambda s - \gamma |x|}.$$

Therefore, by applying again (2.64) with $\zeta = \gamma$,

$$\int_0^{T_a} ds\, e^{L(T_a - s)} \left(J * e^{Ls}\varphi \right)^2 \le C a^{-2\delta} e^{-\gamma|x|},$$

so that from (7.26), for all a small enough,

$$\int_0^{T_a} ds\, e^{L(t-s)} \left(J * e^{Ls} U_0^{\pm} \right)^2 \le C \left(a^{2-2\delta} e^{-\gamma|x|} + a^{3-2\delta} \right). \tag{7.27}$$

Estimate on $\mathcal{R}_t\left[U_\cdot^{\pm}\right]$. We use Lemma 6.1 to obtain *apriori* bounds. Since $\|U_0^\pm\|_\infty \le Ca$, comparing the definitions (6.5) and (7.8) and using $\delta < 1$ we conclude that for all a small enough $\sigma(U_0^\pm) > T_a$. Therefore from (6.6) and (6.7)

$$\|U_t^\pm\|_\infty \le (1 + C_1) a^{1-\delta} \qquad \forall\, t \le T_a \tag{7.28}$$

and

$$\left\| U_t^\pm - e^{Lt} U_0^\pm \right\|_\infty \le N a^{2-2\delta} \qquad \forall\, t \le T_a. \tag{7.29}$$

Recalling (7.2), from (7.28) and (7.29) we get

$$\mathcal{R}_{T_a}\left[U_\cdot^\pm\right] \le C a^{3-4\delta}. \tag{7.30}$$

Collecting (7.20), (7.25), (7.27), and (7.30), we conclude that, for any a small enough,

$$\left| U_{T_a}^\pm(x) \pm a^{1-\delta}\pi(\varphi) v(x) \right| \le C \left(a^{2-2\delta} e^{-\gamma|x|} + a^{3-4\delta} \right). \tag{7.31}$$

Therefore, from the Comparison Theorem and (7.19), recalling $U_t^\pm = S_t\left(q + U_0^\pm\right) - q$, we finally get

$$S_{T_a}\left(Q_a^+\right)(x) \le q(x) - C\left[a^{1-\delta} v(x) - a^{2-2\delta} e^{-\gamma|x|} - a^{3-4\delta} \right], \tag{7.32}$$

$$S_{T_a}\left(Q_a^-\right)(x) \ge q(x) + C\left[a^{1-\delta} v(x) - a^{2-2\delta} e^{-\gamma|x|} - a^{3-4\delta} \right]. \tag{7.33}$$

Next we shall find good bounds on $Q_a^\pm(x \pm \Delta_a)$. We first observe that, since $\Delta_a \le 1$, $|x| \le R_a - 1$ implies $|x + \Delta_a| \le R_a$ while $|x| > R_a + 1$ implies $|x + \Delta_a| > R_a$. Moreover, from (2.55), (7.7), and (7.11), $|q_a^\pm(x \pm \Delta_a) - m_{\beta,h}^-| \le a^{3/2}$ if $R_a - 1 < |x| \le R_a + 1$ and a is small enough. Hence:

$$Q_a^+(x + \Delta_a) \ge q_a^+(x + \Delta_a)\mathbf{1}_{|x| \le R_a + 1} + \left[m_{\beta,h}^- + a^{3/2} \right] \mathbf{1}_{|x| > R_a + 1},$$

$$Q_a^-(x - \Delta_a) \le q_a^-(x - \Delta_a)\mathbf{1}_{|x| \le R_a + 1} + \left[m_{\beta,h}^- - a^{3/2} \right] \mathbf{1}_{|x| > R_a + 1}.$$

Now we notice $q_a^+(x + \Delta_a) \ge q_{a+\Delta_a}^+(x)$ and $q_a^-(x - \Delta_a) \le q_{a+\Delta_a}^-(x)$ for all $x \in \mathbb{R}$. Moreover, since $|q''(x)| \le c|q'(x)|$ for $|x| \ge 1$, by expanding to the second order for $|x| > 1$ and to the first one for $|x| \le 1$, we get, if a is small enough,

$$q_{a+\Delta_a}^+(x) \ge q(x) - (a + \Delta_a)\psi(x), \qquad q_{a+\Delta_a}^-(x) \le q(x) + (a + \Delta_a)\psi(x), \tag{7.34}$$

where

$$\psi(x) \doteq 2\left[|q'(x)| + \|q'\|_\infty \mathbf{1}_{|x| \le 1} \right], \tag{7.35}$$

hence

$$Q_a^+(x+\Delta_a) \geq [q(x)-(a+\Delta_a)\psi(x)]\,\mathbf{1}_{|x|\leq R_a+1} + \left[m_{\beta,h}^- + a^{3/2}\right]\mathbf{1}_{|x|>R_a+1}, \quad (7.36)$$

$$Q_a^-(x-\Delta_a) \leq [q(x)+(a+\Delta_a)\psi(x)]\,\mathbf{1}_{|x|\leq R_a+1} + \left[m_{\beta,h}^- - a^{3/2}\right]\mathbf{1}_{|x|>R_a+1}. \quad (7.37)$$

We can now conclude the proof of the theorem. We consider first the case $|x| \leq R_a + 1$. Since v is strictly positive and obeys the asymptotics (2.63), and q satisfies (2.55), from (7.9) and (7.35) we have

$$v(x) \geq Ca^{\delta/4}\psi(x) \qquad \forall\,|x| \leq R_a + 1. \quad (7.38)$$

On the other hand, using (7.7),

$$a^{2-2\delta}e^{-\gamma|x|} + a^{3-4\delta} \leq Ca\psi(x) \qquad \forall\,|x| \leq R_a + 1.$$

Therefore, for any a small enough,

$$a^{1-\delta}v(x) - a^{2-2\delta}e^{-\gamma|x|} - a^{3-4\delta} \geq Ca^{1-3\delta/4}\psi(x) \qquad \forall\,|x| \leq R_a + 1. \quad (7.39)$$

Since $(a+\Delta_a)a^{-1+3\delta/4}$ vanishes as $a \downarrow 0$, (7.12) and (7.13) for $|x| \leq R_a + 1$ follow from (7.32), (7.33), (7.36), (7.37), and (7.39).

Finally we consider the case $|x| > R_a + 1$. Using $\gamma_v R_0 > \gamma R_0$ and (7.7), from (7.32) and (7.33) we get

$$\lim_{a\downarrow 0} a^{-3/2} \sup_{|x|>R_a+1} \left| S_{T_a}\left(Q_a^{\pm}\right)(x) - m_{\beta,h}^-\right| = 0. \quad (7.40)$$

Then (7.12) and (7.13) for $|x| > R_a + 1$ and a small enough follow from (7.36), (7.37), and (7.40). □

Corollary 7.3. *In the same hypothesis of Theorem 7.2, there is $a_1 \in (0, a_0]$ such that, for any $a \in (0, a_1]$,*

$$\lim_{t\to+\infty} \left\| S_t\left(Q_a^+\right) - m_{\beta,h}^-\right\|_\infty = 0, \quad (7.41)$$

$$\lim_{t\to+\infty} S_t\left(Q_a^-\right)(x) = m_{\beta,h}^+ \qquad \forall\, x \in \mathbb{R}. \quad (7.42)$$

To prove the above Corollary we need the following Barrier Lemma.

Lemma 7.4. (The Barrier Lemma, [8]**)** *There are V and C^* positive so that if $m, \tilde{m} \in L_\infty(\mathbb{R}; [-1,1])$ and, for some $x_0 \in \mathbb{R}$ and $T > 0$, $m(x) = \tilde{m}(x)$ for all $|x - x_0| \leq VT$, then:*

$$|S_t(m)(x_0) - S_t(\tilde{m})(x_0)| \leq C^* e^{-T}.$$

Proof of Corollary 7.3. We first prove (7.41). By (7.12) and the Comparison Theorem, for any integer n,

$$S_{nT_a}\left(Q_a^+\right)(x) \leq Q_a^+(x + n\Delta_a) \qquad \forall\, x \in \mathbb{R}.$$

From (7.10) the function on the r.h.s. of the above inequality is identically equal to $m_{\beta,h}^- + a^{3/2}$ for all $x > R_a - n\Delta_a$. On the other hand $S_{nT_a}\left(Q_a^+\right)$ is a symmetric function for all integer n, then

$$S_{nT_a}\left(Q_a^+\right)(x) \leq m_{\beta,h}^- + a^{3/2} \qquad \forall\, x \in \mathbb{R} \quad \forall\, n > \frac{R_a}{\Delta_a}.$$

Using again the Comparison Theorem and recalling $m^-_{\beta,h} \le Q^+_a$, we conclude that:

$$m^-_{\beta,h} \le S_t\left(Q^+_a\right) \le S_t\big(m^-_{\beta,h} + a^{3/2}\big) \qquad \forall\, t > \left(1 + \frac{R_a}{\Delta_a}\right) T_a. \tag{7.43}$$

We now observe that $S_t\big(m^-_{\beta,h} + a^{3/2}\big)$ solves the homogeneous equation:

$$\frac{d\rho(t)}{dt} = -\rho(t) + \tanh\{\beta[\rho(t) + h]\}, \tag{7.44}$$

with initial datum $m^-_{\beta,h} + a^{3/2}$. Since the free energy density (1.1) is a Lyapunov functional for the flow evolution (7.44), it is easy to verify that the intervals $(-1, m^0_{\beta,h})$ and $(m^0_{\beta,h}, 1)$ are basins of attraction of the stationary solutions $m^-_{\beta,h}$ and $m^+_{\beta,h}$. Then for any a small enough,

$$\lim_{t\to+\infty} S_t\big(m^-_{\beta,h} + a^{3/2}\big) = m^-_{\beta,h}. \tag{7.45}$$

From (7.43) and (7.45) we get (7.41).

We shall next prove (7.42). We need to show that for any $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ there is T_{ε,x_0} so that

$$\left| S_t\left(Q^-_a\right)(x_0) - m^+_{\beta,h} \right| < \varepsilon \qquad \forall\, t > T_{\varepsilon,x_0}. \tag{7.46}$$

By (7.13) and the Comparison Theorem, for any integer n,

$$S_{nT_a}\left(Q^-_a\right)(x) \ge Q^-_a(x - n\Delta_a) \qquad \forall\, x \in \mathbb{R}.$$

Recalling the definition (7.10) we have

$$S_{nT_a}\left(Q^-_a\right)(x) \ge q(x - n\Delta_a - a) \qquad \forall\, x > n\Delta_a.$$

Since Q^-_a is symmetric and non increasing for $x > 0$, from Lemma 6.5,

$$S_{nT_a}\left(Q^-_a\right)(x) \ge q(0) \qquad \forall\, x \in [0, n\Delta_a].$$

Hence:

$$S_{nT_a}\left(Q^-_a\right)(x) \ge q(0)\mathbf{1}_{|x|\le n\Delta_a} + q(|x| - n\Delta_a)\mathbf{1}_{|x|\ge n\Delta_a}.$$

We conclude that for any $R > 0$ we can find a time T_R so that

$$S_t\left(Q^-_a\right)(x) \ge q(0) \qquad \forall\, |x| \le R \quad \forall\, t > T_R. \tag{7.47}$$

We can now prove (7.46). Given any $\varepsilon > 0$ we choose T_ε so large that $C^* e^{-T_\varepsilon} < \varepsilon/2$ and $R \ge |x_0| + VT_\varepsilon$, C^*, V as in the Barrier Lemma 7.4. Hence from the Comparison Theorem, (7.47), and the Barrier Lemma it follows that

$$S_t\left(Q^-_a\right)(x_0) > S_t\left(q(0)\right) - \frac{\varepsilon}{2} \qquad \forall\, t > T_R + T_\varepsilon. \tag{7.48}$$

On the other hand since $q(0)$ belongs to the basin of attraction of $m^+_{\beta,h}$ w.r.t. the dynamics (7.44) (in fact $m^0_{\beta,h} < 0 < q(0) < m^+_{\beta,h}$), there is $\bar T$ such that

$$\left| S_t\left(q(0)\right) - m^+_{\beta,h} \right| < \frac{\varepsilon}{2} \qquad \forall\, t > \bar T. \tag{7.49}$$

Recalling $Q^-_a \le m^+_{\beta,h}$, from the Comparison Theorem, (7.48), and (7.49) we finally get

$$m^+_{\beta,h} - \varepsilon < S_t\left(Q^-_a\right) \le m^+_{\beta,h} \qquad \forall\, t > \bar T \vee (T_\varepsilon + T_R),$$

which implies (7.46) with $T_{\varepsilon,x_0} = \bar T \vee (T_\varepsilon + T_R)$. □

Proof of Theorem 2.13. Let $w_s^\pm$, $s \le 0$, be as in Theorem 6.2. We will next prove that for any $a > 0$ small enough there is $s_a < 0$ such that

$$w_{s_a}^-(x) \le Q_a^+(x) \quad \text{and} \quad Q_a^-(x) \le w_{s_a}^+(x) \qquad \forall\, x \in \mathbb{R}. \tag{7.50}$$

Theorem 2.13 will then follows from the Comparison Theorem, Corollary 7.3, (2.74), and (7.50) (recall that the relation between $w_s^\pm$ and $m_s^\pm$ is only a time shift, see (6.39)).

To prove (7.50) we need a more accurate estimate on the difference $w_s^\pm - \big(q \pm e^{\lambda s}\rho v\big)$. This is the content of Proposition 7.5 below.

Proposition 7.5. *Let $w_s^\pm$ be as in Theorem 6.2 and γ as in* (2.54)*. Then there is a constant $\bar{C}$ so that, for all $x \in \mathbb{R}$ and $s \le 0$,*

$$\left|w_s^\pm(x) - q \mp e^{\lambda s}\rho v(x)\right| \le \bar{C}\left(e^{2\lambda s - \gamma|x|} + e^{3\lambda s}\right). \tag{7.51}$$

Proof. We apply Lemma 7.1 with $u_0 = \pm\varepsilon v$, getting

$$\left|S_t\left(\psi_{\pm\varepsilon}\right) - q \mp e^{\lambda s}\varepsilon v(x)\right| \le K\varepsilon^2 \int_0^t dt'\, e^{2\lambda t'} e^{L(t-t')}(J * v)^2 + K\mathcal{R}_t\left[S_\cdot\left(\psi_{\pm\varepsilon}\right) - q\right]. \tag{7.52}$$

We bound $(J*v)^2$ by $\|v\|_\infty J*v$; we next observe that since J has compact support and v satisfies (2.63) with $\gamma_v > \gamma$, hence $J*v \in X_\gamma$, see (2.58). Then, by applying (2.64) with $\zeta = \gamma$,

$$K\varepsilon^2 \int_0^t dt'\, e^{2\lambda t'} e^{L(t-t')}(J*v)^2 \le KC_1\|J*v\|_\gamma \|v\|_\infty \lambda^{-1}\varepsilon^2 e^{2\lambda t - \gamma|x|}. \tag{7.53}$$

We now observe that for $t = \tau(\rho,\varepsilon) + s$, $s \le 0$, the r.h.s. of (7.53) is bounded, uniformly as $\varepsilon \downarrow 0$, by const $e^{2\lambda s - \gamma|x|}$. Analogously, from (6.32), (6.35), and (7.2), we get that $K\mathcal{R}_{\tau(\rho,\varepsilon)+s}\left[S_\cdot\left(\psi_{\pm\varepsilon}\right) - q\right]$ is bounded by const $e^{3\lambda s}$. The Proposition is proved. □

Proof of (7.50). We first observe that, arguing as in getting (7.36) and (7.37), for all a small enough we have:

$$Q_a^+(x) \ge \left[q(x) - a\psi(x)\right]\mathbf{1}_{|x|\le R_a} + \left[m_{\beta,h}^- + a^{3/2}\right]\mathbf{1}_{|x|>R_a}, \tag{7.54}$$

$$Q_a^-(x) \le \left[q(x) + a\psi(x)\right]\mathbf{1}_{|x|\le R_a} + \left[m_{\beta,h}^- - a^{3/2}\right]\mathbf{1}_{|x|>R_a}, \tag{7.55}$$

where ψ is defined in (7.35). We then set

$$s_a \doteq \frac{r}{\lambda}\log a \qquad \text{with } \frac{1}{2} < r < 1 - \frac{\delta}{4} \text{ and } \delta \text{ as in (7.7)}. \tag{7.56}$$

Recalling (2.63) and that $\gamma_v R_0 > \gamma R > 3/2$, from (7.51) it follows

$$\lim_{a\downarrow 0} a^{-3/2} \sup_{|x|>R_a}\left|w_{s_a}^\pm(x) - m_{\beta,h}^-\right| = 0. \tag{7.57}$$

On the other hand, by using (7.38), we also have, if a is small enough,

$$e^{\lambda s_a}\rho v(x) > a\psi(x) \qquad \forall\, x \in \mathbb{R}. \tag{7.58}$$

Then (7.50) follows from (7.51), (7.54)–(7.58). Theorem 2.13 is proved. □

8. The quasi-invariant manifold and the spectral estimates

In this section we prove Theorem 2.2, Proposition 2.3, Lemma 2.4, and Theorem 2.5. Most of the statements follow from the results proved in [6] and [7] as we are going to explain.

The first step is the definition of the function n_ξ which characterizes the quasi-invariant manifold, see (2.22). As we already mentioned in the Introduction, the existence of the critical droplet has been proved in [7] by using the Newton method: n_ξ will be a slight modification of the starting point of the Newton map.

Recalling the definition (2.11) and the properties of the instanton stated in (2.9), we consider the following symmetric function:

$$m_\xi^0(x) \doteq \bar{m}_\xi(x) - a e^{-\alpha(\xi+|x|)}. \tag{8.1}$$

The operator L_m (see (2.5)) for m in a neighbor of m_ξ^0 has been studied in details in [6, Thms. 2.1, 2.3, 2.4], see also [7, Thm. 3.2], and for the reader convenience we now recall these results.

We first define the neighbor of m_ξ^0 as the set $G(c^*,\xi,\delta^*)$, $\xi > 1$, $c^*, \delta^* > 0$ of all functions $m \in C^{\mathrm{sym}}(\mathbb{R},[-1,1])$ such that

$$\left|m(x) - m_\xi^0(x)\right| \le c^* \begin{cases} e^{-2\alpha\xi} e^{\alpha(\xi - x)} & \text{for } 0 \le x \le \xi \\ e^{-2\alpha\xi} & \text{for } \xi < x \end{cases} \tag{8.2}$$

and

$$-\int_{|x-\xi|\le\sqrt{\xi}} dx\, [m(x) - m_\xi^0(x)]\bar{m}'(\xi - x)^2 \bar{m}(\xi - x) \ge -c^* e^{-(2\alpha+\delta^*)\xi}. \tag{8.3}$$

Spectral analysis in $G(c^*,\xi,\delta^*)$. Some of the results that we need are true for ξ sufficiently large, so we assume this condition even if it is not always necessary. In [6] the results are stated in the interval $[0,\ell]$ with Neumann boundary conditions, but since the estimates are uniform in ℓ they include the case $\ell = \infty$ treated here.

For all $m \in G(c^*,\xi,\delta^*)$ the operator L_m has a strictly positive eigenvalue λ_m with strictly positive right and left eigenfunctions denoted, respectively, by v_m^* and $v_m = p_m v_m^*$ (recall the definition of p_m in (2.5)), which we assume normalized in such a way that:

$$\int_0^\infty dx\, \frac{v_m(x)^2}{p_m(x)} \equiv \int_0^\infty dx\, v_m^*(x) v_m(x) = 1. \tag{8.4}$$

In [6, Thm. 2.3] it has been proved that there are $c_\pm$ and c' all positive so that:

$$c_- e^{-2\alpha\xi} \le \lambda_m \le c_+ e^{-2\alpha\xi}, \tag{8.5}$$

$$v_m(x) \le c_+ e^{-\alpha'|\xi - x|}, \qquad \alpha' = \alpha - c' e^{-2\alpha\xi}, \tag{8.6}$$

$$|v_m(x) - \tilde{m}_\xi'(x)| \le c_+ e^{-2\alpha\xi} e^{\alpha|\xi-x|}\xi^4 \qquad \text{for } |\xi - x| \le \xi/2. \tag{8.7}$$

where μ is defined in (2.10) and $\tilde{m}_\xi(x)$ in (2.27). Let us remark that by choosing ξ large enough the parameter α' can be done as close to α as needed.

For any $m \in G(c^*,\xi,\delta^*)$ we define the linear functional π_m on $L_\infty^{\mathrm{sym}}(\mathbb{R})$ as:

$$\pi_m(\psi) \doteq \int_0^\infty dx\, v_m^*(x)\psi(x) \tag{8.8}$$

(observe that $\pi_m(v_m)=1$ by (8.4)). An easy consequence of (8.6) and (8.7) is the existence of a constant $c_3>0$ such that:

$$|\pi_m(\psi)| \le c_3\|\psi\|_\infty \qquad \forall\,\psi\in L^{\mathrm{sym}}_\infty(\mathbb{R}). \tag{8.9}$$

In [6, Thm. 2.4] it is proved that there exists $\xi_0>1$ such that for any $\xi>\xi_0$ and $m\in G(c^*,\xi,\delta^*)$ there exist constants $d_\pm>0$ so that:

$$\|e^{L_m t}\psi\|_\infty \le d_+ e^{\lambda_m t}\|\psi\|_\infty \qquad \forall\,\psi\in L^{\mathrm{sym}}_\infty(\mathbb{R}), \tag{8.10}$$

and, if ψ is such that $\pi_m(\psi)=0$,

$$\|e^{L_m t}\psi\|_\infty \le d_+ e^{-d_- t}\|\psi\|_\infty. \tag{8.11}$$

Moreover there is $c_4>0$ such that the inverse L_m^{-1} exists and

$$\|L_m^{-1}\psi\|_\infty \le c_4\lambda_m^{-1}\|\psi\|_\infty \qquad \forall\,\psi\in L^{\mathrm{sym}}_\infty(\mathbb{R}). \tag{8.12}$$

For ψ such that $\pi_m(\psi)=0$ we also have:

$$L_m^{-1}\psi = -\int_0^{+\infty} dt\, e^{L_m t}\psi, \qquad \|L_m^{-1}\psi\|_\infty \le c_4\|\psi\|_\infty. \tag{8.13}$$

In the case $m=m^0_\xi$ stronger results hold. In order to avoid heavy notation we abbreviate:

$$\bar L_\xi = L_{m^0_\xi},\quad \bar p_\xi = p_{m^0_\xi},\quad \bar\lambda_\xi=\lambda_{m^0_\xi},\quad \bar v^*_\xi = v^*_{m^0_\xi},\quad \bar v_\xi = v_{m^0_\xi},\quad \bar\pi_\xi=\pi_{m^0_\xi}.$$

In [6, Thm. 11.1] it is proved that there are $\eta\in(\alpha,3\alpha/2)$, $\xi_1>\xi_0$, and $c_5>0$ such that, for all $\xi>\xi_1$,

$$\begin{aligned}
\left|\partial_\xi \bar v_\xi(x) - \tilde m''_\xi(x)\right| &\le c_5 e^{-\alpha\xi}\xi^4 &&\text{for}\quad |\xi-x|\le \xi/2, &(8.14)\\
\left|\partial_\xi \bar v_\xi(x)\right| &\le c_5\xi^2\bar v_\xi(x) &&\text{for}\quad |\xi-x|\ge\xi/2, &(8.15)\\
\left|\frac{d\bar\lambda_\xi}{d\xi}\right| &\le c_5 e^{-\eta\xi}. && &(8.16)
\end{aligned}$$

Setting

$$\partial_\xi\bar\pi_\xi(\psi) = \int_0^\infty dx\,\partial_\xi\bar v^*_\xi(x)\psi(x), \qquad \psi\in L^{\mathrm{sym}}_\infty(\mathbb{R}), \tag{8.17}$$

from (8.14) and (8.15) it follows that, for some $c_6>0$,

$$|\partial_\xi\bar\pi_\xi(\psi)| \le c_6\|\psi\|_\infty. \tag{8.18}$$

Furthermore from [6, Prop. 6.3] (see also [7, Eq. (6.38)]), for any $\zeta\in(0,\alpha)$ there exists $c_7>0$ so that, for any $t,x,y>0$,

$$e^{\bar L_\xi t}(x,y) \le c_7\bar\lambda_\xi t\, e^{\bar\lambda_\xi t}e^{-\zeta|x-y|}. \tag{8.19}$$

We now come back to the construction of n_ξ. The definition (8.1) is motivated by the fact that $f(m^0_\xi)$ (see (2.2)) is small for h small and ξ large. With this in mind and following [7] we define:

$$b_\xi(x) \doteq -m^0_\xi(x) + \tanh\left\{\beta(J*m^0_\xi)(x)\right\}, \tag{8.20}$$

so that

$$f(m^0_\xi) = b_\xi + h\bar p_\xi + O(h^2).$$

In [7, Lemma 5.1] it is shown that given $\alpha_0>\alpha$ as in (2.9) there is $c_8>0$ so that, for all $\xi>1$ and $x\ge 0$,

$$\left|b_\xi(x) + e^{-2\alpha\xi}k^0_\xi(x)\right| \le c_8\left(e^{-\alpha_0\xi}\mathbf{1}_{0\le x\le 1} + e^{-2\alpha(\xi+x)}\right), \tag{8.21}$$

where $k^0_\xi(x) = k^0(\xi - x)$ and

$$k^0(y) \doteq \frac{a\, e^{\alpha y}}{1 - m_\beta^2} \left[m_\beta^2 - \bar m^2(y) \right]. \tag{8.22}$$

Observe that from (2.9) it follows that there is $c_9 > 0$ such that

$$\|k^0\|_\infty \le c_9. \tag{8.23}$$

From the estimate (8.21) the leading term in the projection of $f(m^0_\xi)$ is $V(\xi)/\sqrt{\mu}$ (with $V(\xi)$ as in (2.21)): this is the content of the next lemma.

Lemma 8.1. *There is $\ell_0 > 1$ such that for any $h < e^{-2\alpha\ell_0}$ the following holds. There exists $\delta_3 > 0$ such that, for any $\kappa > 0$ and any $\xi \in \Gamma_{\ell_0,\kappa}$,*

$$\left| \sqrt{\mu}\, \bar\pi_\xi(b_\xi + h\bar p_\xi) - V(\xi) \right| \le C e^{-(2\alpha + \delta_3)\xi}. \tag{8.24}$$

Proof. We choose $\ell_0 > \xi_1$ so large that $\alpha_0 + \alpha' > 2\alpha$ (α' as in (8.6)). Then, from (8.6) and (8.21), it is easy to check we can find $\delta_4 > 0$ such that:

$$\left| \bar\pi_\xi(b_\xi) + e^{-2\alpha\xi} \bar\pi_\xi(k^0_\xi) \right| \le C e^{-(2\alpha+\delta_4)\xi}. \tag{8.25}$$

From (8.6) and (8.23) we have:

$$\bar\pi_\xi(k^0_\xi) = \int_{\xi/2}^{3\xi/2} dx\, \frac{\bar v_\xi(x)}{\bar p_\xi(x)} k^0_\xi(x) + O\left(e^{-\alpha'\xi/2} \right).$$

On the other hand, by the definitions (2.14), (8.22), recalling that $p_{\bar m} = \beta(1 - \bar m^2)$, and using the estimates (2.9) and (8.23), we also have:

$$\sqrt{\mu}\, K = \sqrt{\mu} \int dx\, \frac{\bar m'(x)}{p_{\bar m}(x)} k^0(x) = \int_{\xi/2}^{3\xi/2} dx\, \frac{\tilde m'_\xi(x)}{p_{\tilde m_\xi}(x)} k^0_\xi(x) + O\left(e^{-\alpha\xi/2} \right).$$

From (8.1) and (8.7) we conclude that

$$\left| \bar\pi_\xi(k^0_\xi) - \sqrt{\mu}\, K \right| \le C e^{-\alpha'\xi/2}. \tag{8.26}$$

We finally observe that

$$\begin{aligned} \bar\pi_\xi(\bar p_\xi) &= \int_0^\infty dx\, \bar v_\xi(x) = \int_{-\infty}^{\xi} dx\, \tilde m'(x) + \int_0^\infty dx\, (\bar v_\xi - \tilde m'_\xi)(x) = \\ &= 2m_\beta\sqrt{\mu} - \int_\xi^\infty dx\, \tilde m'(x) + \int_0^\infty dx\, (\bar v_\xi - \tilde m'_\xi)(x) \\ &= 2m_\beta\sqrt{\mu} + O\left(e^{-\alpha'\xi/2} \right), \end{aligned} \tag{8.27}$$

where in the last equality we used again (2.9), (8.6), and (8.7). From (8.25), (8.26), and (8.27) we get (8.24) for a suitable $\delta_3 > 0$. □

We now construct the function n_ξ in such a way that the main contribution to $f(n_\xi)$ is given by $\bar\pi_\xi(b_\xi + h\bar p_\xi)\bar v_\xi$.

Definition 8.2. *Given $\xi > \ell_0$ we define the symmetric function ψ_ξ as the solution of:*

$$\bar L_\xi \psi_\xi + b_\xi + h\bar p_\xi - \bar\pi_\xi(b_\xi + h\bar p_\xi)\bar v_\xi = 0. \tag{8.28}$$

The quasi-invariant manifold (2.22) is then defined via the symmetric functions:

$$n_\xi(x) \doteq m^0_\xi(x) + \psi_\xi(x), \qquad \xi \in \Gamma_{\ell_0,+\infty}. \tag{8.29}$$

We note that the function ψ_ξ is well defined, indeed from (8.13),

$$\psi_\xi = -\int_0^{+\infty} dt\, e^{\bar L_\xi t} \bar T_\xi \left(b_\xi + h\bar p_\xi\right),$$

where $\bar T_\xi$ is the projection acting on $L_\infty^{\rm sym}(\mathbb{R})$ defined by:

$$\bar T_\xi \psi = \psi - \bar\pi_\xi(\psi)\bar v_\xi.$$

Observe that, by (8.6) and (8.9),

$$\|\bar T_\xi \psi\|_\infty \le C\|\psi\|_\infty. \tag{8.30}$$

In the next lemma we estimate the sup-norm of both ψ_ξ and its derivative with respect to ξ.

Lemma 8.3. *In the same hypothesis of Lemma 8.1, there exists $\delta_5 > 0$ such that, for any $\kappa > 0$ and any $\xi \in \Gamma_{\ell_0,\kappa}$,*

$$\|\psi_\xi\|_\infty \le Ce^{-(\alpha+\delta_5)\xi}, \qquad \|\partial_\xi\psi_\xi\|_\infty \le Ce^{-(\alpha+\delta_5)\xi}. \tag{8.31}$$

Proof. From (8.21) and (8.23),

$$\|b_\xi\|_\infty \le C\left(e^{-2\alpha\xi} + e^{-\alpha_0\xi}\right).$$

By (2.16), since $\bar p_\xi \le \beta$,

$$\|b_\xi + h\bar p_\xi\|_\infty \le C\left(e^{-2\alpha\xi} + e^{-\alpha_0\xi}\right), \tag{8.32}$$

so that, by (8.13) and (8.30),

$$\|\psi_\xi\|_\infty \le C\left(e^{-2\alpha\xi} + e^{-\alpha_0\xi}\right). \tag{8.33}$$

To compute the derivative of ψ_ξ we differentiate (8.28) and, recalling (8.17), we get:

$$\begin{aligned} \bar L_\xi \partial_\xi\psi_\xi + \partial_\xi \bar p_\xi J * \psi_\xi \;&=\; -\partial_\xi(b_\xi + h\bar p_\xi) + \bar\pi_\xi\big(\partial_\xi(b_\xi + h\bar p_\xi)\big)\bar v_\xi \\ &\quad + \partial_\xi\bar\pi_\xi(b_\xi + h\bar p_\xi)\bar v_\xi + \bar\pi_\xi(b_\xi + h\bar p_\xi)\partial_\xi\bar v_\xi. \end{aligned} \tag{8.34}$$

We now proceed as in the proof of Proposition 4.2. Recalling the definition of the operator $\bar T_\xi$, we rewrite (8.34) as follows:

$$\begin{aligned} \bar L_\xi\partial_\xi\psi_\xi \;&=\; -\bar T_\xi(\partial_\xi\bar p_\xi J * \psi_\xi) - \bar T_\xi\big(\partial_\xi(b_\xi + h\bar p_\xi)\big) + \bar\pi_\xi(b_\xi + h\bar p_\xi)\bar T_\xi\partial_\xi\bar v_\xi \\ &\quad - \big[\bar\pi_\xi(\partial_\xi\bar p_\xi J * \psi_\xi) - \partial_\xi\bar\pi_\xi(b_\xi + h\bar p_\xi) - \bar\pi_\xi(b_\xi + h\bar p_\xi)\bar\pi_\xi(\partial_\xi\bar v_\xi)\big]\bar v_\xi. \end{aligned}$$

From the definition of ψ_ξ,

$$b_\xi + h\bar p_\xi = -\bar L_\xi\psi_\xi + \bar\pi_\xi(b_\xi + h\bar p_\xi)\bar v_\xi, \tag{8.35}$$

then

$$\partial_\xi\bar\pi_\xi(b_\xi + h\bar p_\xi) = -\partial_\xi\bar\pi_\xi(\bar L_\xi\psi_\xi) + \bar\pi_\xi(b_\xi + h\bar p_\xi)\partial_\xi\bar\pi_\xi(\bar v_\xi). \tag{8.36}$$

We next observe that, by (8.4), $\bar\pi_\xi(\partial_\xi\bar v_\xi) + \partial_\xi\bar\pi_\xi(\bar v_\xi) = 0$. From (8.35) and (8.36) we then get:

$$\begin{aligned} \bar L_\xi\partial_\xi\psi_\xi \;&=\; -\bar T_\xi\big[\partial_\xi(b_\xi + h\bar p_\xi) + \partial_\xi\bar p_\xi J * \psi_\xi - \bar\pi_\xi(b_\xi + h\bar p_\xi)\partial_\xi\bar v_\xi\big] \\ &\quad - \big[\bar\pi_\xi(\partial_\xi\bar p_\xi J * \psi_\xi) + \partial_\xi\bar\pi_\xi(\bar L_\xi\psi_\xi)\big]\bar v_\xi. \end{aligned} \tag{8.37}$$

By standard arguments the derivative $\partial_\xi\psi_\xi$ is defined by the solution of (8.37), whose existence is guaranteed from the existence of $\bar L_\xi^{-1}$, see (8.12) and (8.13), In order to prove the second inequality in (8.31) we need to bound the various terms on the right hand side of (8.37). We start with the term $\partial_\xi b_\xi$. Let R_ξ be the function defined as:

$$\begin{aligned} R_\xi(x) &= \left\{ m_\xi^0(x) - [\bar{m}(\xi - x) - ae^{-\alpha(\xi+x)}] \right\} \mathbf{1}_{x\in[-1,0]} \\ &= \left\{ [\bar{m}(\xi + x) + ae^{-\alpha(\xi+x)}] - [\bar{m}(\xi - x) + ae^{-\alpha(\xi-x)}] \right\} \mathbf{1}_{x\in[-1,0]}. \end{aligned} \tag{8.38}$$

Furthermore:

$$\partial_\xi R_\xi(x) = \left\{ \bar{m}'(\xi + x) - a\alpha e^{-\alpha(\xi+x)} - \bar{m}'(\xi - x) + a\alpha e^{-\alpha(\xi-x)} \right\} \mathbf{1}_{x\in[-1,0]}. \tag{8.39}$$

Then, by (2.9),

$$\|R_\xi\|_\infty + \|\partial_\xi R_\xi\|_\infty \le Ce^{-\alpha_0\xi}. \tag{8.40}$$

Since

$$m_\xi^0(x) = \bar{m}(\xi - x) - ae^{-\alpha(x+\xi)} + R_\xi(x) \qquad \text{if } \ x > -1,$$

by Taylor expansion and using that $L_{\bar{m}}\bar{m}' = 0$, after some simple computations we easily get:

$$\begin{aligned} \partial_\xi b_\xi(x) &= -a\alpha L_{\bar{m}} e^{-\alpha(\cdot+\xi)}(x) + L_{\bar{m}}\partial_\xi R_\xi(x) \\ &\quad + \beta^2 \tanh''(\zeta)[-ae^{-\alpha(x+\xi)} + R_\xi(x)](J * \partial_\xi m_\xi^0)(x) \end{aligned} \tag{8.41}$$

where $\zeta = \zeta(x)$ is a number in the interval with end-points $\beta(J * \bar{m})(\xi - x)$ and $\beta(J * \bar{m})(\xi - x) - a(J * e^{-\alpha(\cdot+\xi)})(x) + (J * R_\xi)(x)$. By using (2.8) and (2.9), we have, for all $x \ge 0$,

$$|L_{\bar{m}} e^{-\alpha(\cdot+\xi)}(x)| = e^{-\alpha(\xi+x)} \left| 1 - \frac{(p_{\bar{m}} J * \bar{m})(\xi - x)}{1 - m_\beta^2} \right| \tag{8.42}$$

$$\le Ce^{-\alpha(\xi+x)} e^{-\alpha|\xi-x|} \le Ce^{-3\alpha\xi/2}. \tag{8.43}$$

By (8.40) $\|L_{\bar{m}}\partial_\xi R_\xi\|_\infty \le Ce^{-\alpha_0\xi}$. Then, since $\|\tanh''\|_\infty \le 2$, we have, for all $x \ge 0$,

$$\begin{aligned} &\left|\beta^2 \tanh''(\zeta)[-ae^{-\alpha(x+\xi)} + R_\xi(x)](J * \partial_\xi m_\xi^0)(x)\right| \\ &\qquad \le C\left(e^{-\alpha(\xi+x)} + e^{-\alpha_0\xi}\right) e^{-\alpha|\xi-x|} \le C\left(e^{-3\alpha\xi/2} + e^{-\alpha_0\xi}\right). \end{aligned}$$

By (8.42) and the above estimates it follows that:

$$\|\partial_\xi b_\xi(x)\|_\infty \le C\left(e^{-3\alpha\xi/2} + e^{-\alpha_0\xi}\right).$$

From (2.16) and the fact that $\partial_\xi \bar{p}_\xi$ is uniformly bounded, we conclude that:

$$\|\partial_\xi(b_\xi + h\bar{p}_\xi)\|_\infty \le C\left(e^{-3\alpha\xi/2} + e^{-\alpha_0\xi}\right). \tag{8.44}$$

To bound the second term on the right hand side of (8.37) we use again that $\partial_\xi \bar{p}_\xi$ is uniformly bounded and the bound (8.33), getting:

$$\|\partial_\xi \bar{p}_\xi J * \psi_\xi\|_\infty \le C\left(e^{-2\alpha\xi} + e^{-\alpha_0\xi}\right). \tag{8.45}$$

The third term on the right hand side of (8.37) is bounded by using (8.24), (2.16) and (8.14), (8.15), obtaining

$$\|\bar{\pi}_\xi(b_\xi + h\bar{p}_\xi)\bar{T}_\xi(\partial_\xi \bar{v}_\xi)\|_\infty \le Ce^{-2\alpha\xi}. \tag{8.46}$$

Finally, for the last terms on the right hand side of (8.37) we apply the same argument as in equations (4.11)–(4.12) to which we refer for details. Then:

$$\bar{\pi}_\xi(\partial_\xi \bar{p}_\xi J * \psi_\xi) + \partial_\xi \bar{\pi}_\xi(\bar{L}_\xi \psi_\xi) = \bar{\lambda}_\xi \int_0^\infty dx\, \psi_\xi(x) \left(\frac{\partial_\xi \bar{v}_\xi}{\bar{p}_\xi} - \frac{\partial_\xi \bar{p}_\xi}{\bar{p}_\xi^2} \bar{v}_\xi \right)(x). \tag{8.47}$$

Therefore, by (8.33),

$$\|\bar{\pi}_\xi(\partial_\xi \bar{p}_\xi J * \psi_\xi) + \partial_\xi \pi_\xi(\bar{L}_\xi \psi_\xi)\|_\infty \le C\bar{\lambda}_\xi \left(e^{-2\alpha\xi} + e^{-\alpha_0 \xi}\right). \tag{8.48}$$

From (8.12), (8.13), (8.37), and the estimates (8.44), (8.45), (8.46), and (8.48) we finally obtain:

$$\|\partial_\xi \psi_\xi\|_\infty \le C \left(e^{-3\alpha\xi/2} + e^{-\alpha_0 \xi}\right). \tag{8.49}$$

Setting $\delta_5 = \min\{\alpha/2; \alpha_0 - \alpha\}$, the inequalities (8.31) follow from (8.33) and (8.49). □

Proof of Theorem 2.2. Equations (2.18) and (2.19) follow from the definition (8.29) of n_ξ and the estimates (8.31). Recalling the definition (8.20), we next write:

$$f(n_\xi) = \tanh\{\beta \left[J * (m_\xi^0 + \psi_\xi) + h\right]\} - \tanh\{\beta J * m_\xi^0\} + b_\xi - \psi_\xi,$$

from which, by Taylor expansion to second order and using (8.28),

$$f(n_\xi) = \bar{L}_\xi \psi_\xi + h\bar{p}_\xi + b_\xi + R(\psi_\xi, h) = \bar{\pi}_\xi(b_\xi + h\bar{p}_\xi)\bar{v}_\xi + R(\psi_\xi, h),$$

where

$$|R(\psi_\xi, h)| \le C \left(\|\psi_\xi\|_\infty^2 + h^2\right).$$

From (8.24), (8.31), and (2.16) we get (for $\xi \in \Gamma_{\ell_0,\kappa}$):

$$\begin{aligned} \|f(n_\xi) - V(\xi)\partial_\xi n_\xi\|_\infty \quad &\le \quad Ce^{-2\alpha\xi}\|\bar{v}_\xi - \sqrt{\mu}\,\partial_\xi n_\xi\|_\infty \\ &\qquad + C\left(e^{-(2\alpha+\delta_3)\xi} + e^{-2(\alpha+\delta_5)\xi} + e^{-4\alpha\xi}\right). \end{aligned}$$

On the other hand, by (8.6), (8.7), and the second inequality in (8.31),

$$\|\bar{v}_\xi - \sqrt{\mu}\,\partial_\xi n_\xi\|_\infty \le C\left(e^{3\alpha\xi/2}\xi^4 + e^{-\alpha'\xi} + e^{-(\alpha+\delta_5)\xi}\right).$$

From the above bounds the inequality (2.20) easily follows with a suitable $\delta_0 > 0$. □

For the proof of Proposition 2.3 we have to show that $n_\xi + h \in G(c^*, \xi, \delta^*)$. To this end we need more refined estimates on ψ_ξ, and this is done in the next lemma. We first introduce some notation. We define

$$\Gamma_\xi \doteq \int_0^{\xi^2} dt\, e^{\bar{L}_\xi t}\bar{T}_\xi(b_\xi + h\bar{p}_\xi).$$

Then, from (8.11), (8.30), and (8.32),

$$\|\psi_\xi - \Gamma_\xi\|_\infty \le Ce^{-d_- \xi^2}\left(e^{-2\alpha\xi} + e^{-\alpha_0\xi}\right). \tag{8.50}$$

Recalling (8.21) and (8.22), we write:

$$\begin{aligned} \Gamma_\xi \quad &= \quad P_\xi + Q_\xi, \\ P_\xi \quad &\doteq \quad \int_0^{\xi^2} dt\, e^{\bar{L}_\xi t}\bar{T}_\xi(b_\xi + k_\xi), \qquad k_\xi(x) \doteq e^{-2\alpha\xi}k_\xi^0(x), \\ Q_\xi \quad &\doteq \quad -\int_0^{\xi^2} dt\, e^{\bar{L}_\xi t}\bar{T}_\xi(k_\xi - h\bar{p}_\xi). \end{aligned} \tag{8.51}$$

Lemma 8.4. *For each $\alpha^* < \alpha$ there exists $C_* > 0$ so that*

$$|P_\xi(x)| \le C_* \left[e^{-\alpha^*(\xi+x)} + e^{-2\alpha\xi} \left(e^{-\alpha^*\xi/4} \mathbf{1}_{|x-\xi|\le\sqrt{\xi}} + \mathbf{1}_{|x-\xi|>\sqrt{\xi}} \right) \right]. \tag{8.52}$$

Furthermore:

$$|Q_\xi(x)| \le C e^{-2\alpha\xi}. \tag{8.53}$$

Proof. The proof of this lemma is similar to the proof of the second inequality in [7, Eq. (6.1)], therefore we only give an outline of the argument, by referring to [7] for more details. We first prove (8.53). By (8.11), (8.23), (8.30), and (2.16) we have:

$$\|Q_\xi\|_\infty \le C \int_0^{\xi^2} dt\, e^{-d_- t} (\|h\bar{p}_\xi\|_\infty + \|k_\xi\|_\infty) \le C e^{-2\alpha\xi}. \tag{8.54}$$

Since $\bar{T}_\xi \psi = \psi - \bar{\pi}_\xi(\psi)\bar{v}_\xi$ and $\|\bar{v}_\xi\|_\infty \le C$, by (8.21) and (8.25),

$$|P_\xi(x)| \le C\xi^2 e^{-(2\alpha+\delta_4)\xi} + \int_0^{\xi^2} dt \int_0^{+\infty} dy\, e^{\bar{L}_\xi t}(x,y) \left(e^{-\alpha_0\xi} \mathbf{1}_{y\in[0,1]} + e^{-2\alpha(\xi+y)} \right). \tag{8.55}$$

An estimate of the last integral in (8.55) can be found in [7] but, since it is not written in the form we need, we give the argument below. From (8.5) and (8.19) we get, for any $x \ge 0$,

$$\int_0^{\xi^2} dt \int_0^{+\infty} dy\, e^{\bar{L}_\xi t}(x,y) e^{-\alpha_0\xi} \mathbf{1}_{y\in[0,1]} \le C\xi^2 e^{-(\alpha_0\xi + \zeta x)}. \tag{8.56}$$

Again by (8.19),

$$\int_0^{\xi^2} dt \int_0^{+\infty} dy\, e^{\bar{L}_\xi t}(x,y) e^{-2\alpha(\xi+y)} \le C\xi^2 e^{-2\alpha\xi} \int_0^{+\infty} dy\, e^{-\zeta|x-y|} e^{-2\alpha y}.$$

The integral on the r.h.s. is uniformly bounded; moreover, if $|\xi - x| \le \sqrt{\xi}$ then, since $|x-y| \ge |y-\xi| - |x-\xi|$,

$$\begin{aligned} \int_0^{+\infty} dy\, e^{-\zeta|x-y|} e^{-2\alpha y} &= \int_0^{+\infty} dy\, e^{-\zeta|x-y|} e^{-2\alpha y} \left(\mathbf{1}_{|y-\xi|\le\xi/2} + \mathbf{1}_{|y-\xi|>\xi/2} \right) \\ &\le C \left(e^{-3\alpha/2} + e^{\zeta(\sqrt{\xi}-\xi/2)} \right). \end{aligned}$$

Collecting all the above estimates we obtain (8.52) by choosing $\alpha^* < \zeta$ (recall that in (8.19) the parameter $\zeta \in (0,\alpha)$ can be fixed arbitrarily close to α). □

Proof of Proposition 2.3. From (2.16), (8.50), (8.52), and (8.53) we easily get that $\psi_\xi + h$ satisfies the bounds on the right hand side of (8.2) for a suitable constant c^* if $\xi > \ell_0$ is chosen large enough. Instead the estimate (8.3) is not immediate. By (8.50), (8.51), and (8.52) there is $\delta_6 > 0$ so that

$$\left| \int_{|x-\xi|\le\sqrt{\xi}} dx\, [\psi_\xi(x) + h - Q_\xi(x)]\, \bar{m}'(\xi-x)^2 \bar{m}(\xi-x) \right| \le C e^{-(2\alpha+\delta_6)\xi}. \tag{8.57}$$

(note that since $(\bar{m}')^2\bar{m}$ is antisymmetric the constant $+h$ does not contribute to the integral). The contribution of Q_ξ to the above integral has been estimated in [7, Sect. 8]. More precisely, setting

$$M_2 \doteq \int_{|x-\xi|\le\sqrt{\xi}} dx\, Q_\xi(x) \bar{m}'(\xi-x)^2 \bar{m}(\xi-x),$$

it is proved that there is δ_7 such that $|M_2 - M_3| \le Ce^{-(2\alpha+\delta_7)\xi}$, where

$$M_3 \doteq -\int_{|x-\xi|\le\sqrt{\xi}} dx\, \bar m'(\xi-x)^2 \bar m(\xi-x) \int_0^{\xi^2} dt \int dy\, G_{\xi,t}(x,y)\,(h\bar p_\xi - k_\xi)\,(y),$$

with $G_{\xi,t}(x,y)$ explicitly given in [7, Eq. (8.10)] and satisfying $G_{\xi,t}(\xi-x, y-\xi) = G_{\xi,t}(\xi+x, y+\xi)$. It is finally shown that M_3 is bounded by the r.h.s. of (8.3): here we only remark that, since $(\bar m')^2\bar m$ is antisymmetric, only the odd part (w.r.t. ξ) of $h\bar p_\xi - k_\xi$ contributes to the integral, and since $\bar p_\xi$ is even, only the odd part of k_ξ survives. The estimate of this last term uses the fact that J is non increasing, we refer to [7] for the details.

We have thus proved that for any $\kappa > 0$ there exists $\ell_1 > \ell_0$ and suitable positive constants c^* and δ^* such that $n_\xi + h \in G(c^*, \xi, \delta^*)$ for all $\xi \in \Gamma_{\ell_1,\kappa}$. Therefore Proposition 2.3 as well as the estimates (2.33) and (2.34) follow from the spectral analysis in $G(c^*, \xi, \delta^*)$. □

Proof of Lemma 2.4. The proof of of inequalities (2.35) (2.36) and (2.37) can be done following line by line that one of (8.14), (8.15), and (8.16) given in [6, Sect. 11]. The arguments given there apply also in our case (where m^0_ξ is replaced by n_ξ): the key tool is Lemma 8.3, which gives sufficiently strict bounds on $n_\xi - m^0_\xi$. In order to make shorter the paper we omit the details. Finally, recalling that $v_\xi = p_\xi v^*_\xi$ and that $\|\partial_\xi p_\xi\|_\infty \le C$, (2.38) follows from (2.26) and (2.36).

We next prove (2.40)–(2.42). By the second inequality in (8.31) and recalling the definitions (2.27) and (8.1) we have:

$$\|\sqrt{\mu}\,\partial_\xi n_\xi - \tilde m'_\xi\|_\infty \le Ce^{-\alpha\xi}. \tag{8.58}$$

On the other hand, since $n_\xi \in G(c^*, \xi, \delta^*)$, the inequalities (8.6) and (8.7) apply to $v_{n_\xi} = v_\xi$, from which it follows that

$$\|v_\xi - \tilde m'_\xi\|_\infty \le Ce^{-\alpha'\xi/2}.$$

Then, since $\pi_\xi(v_\xi) = 1$, from the previous estimates and (2.31) we get

$$\left|\sqrt{\mu}\,\pi_\xi(\partial_\xi n_\xi) - 1\right| \le Ce^{-\alpha'\xi/2}.$$

Collecting together the above estimates, (2.40) and (2.41) follow. We are left with the proof of (2.42). By (2.26), (2.35), (2.36), (8.31), and (8.58) we have:

$$\left|\partial_\xi \pi_\xi(\partial_\xi n_\xi) - \sqrt{\mu}\int_0^{+\infty} dx\, \frac{\bar m''(\xi-x)\bar m'(\xi-x)}{p_{\bar m}(\xi-x)}\right| \le Ce^{-\alpha'\xi/2}.$$

But, since $\bar m$ is odd and $p_{\bar m}$ even,

$$\int dx\, \frac{\bar m''(\xi-x)\bar m'(\xi-x)}{p_{\bar m}(\xi-x)} = 0 \qquad \forall\, \xi > 0.$$

Then by (2.9) and the previous estimate (2.42) follows. □

Proof of Proposition 2.11. The critical droplet $q = n_{\bar\xi} + \bar\varphi$ belongs to the set $G(c^*, \bar\xi, \delta^*)$ for h small. Therefore items 1), 3), and 4) of Proposition 2.11 follow from the spectral analysis in the set $G(c^*, \bar\xi, \delta^*)$. Let us prove item 2). The existence and uniqueness of $\gamma_v > \gamma$ solving (2.62) follows from [6, Lemma 3.1]. The proof of (2.63) is not given in [6], but it can be done in the same way as the proof of (2.55) in Sect. 5. In fact, recalling the definition (2.56) of the linear operator L, we can rewrite the equation for v in (2.59) as $v = (1+\lambda)^{-1} pJ * v$; then, by arguing as in Sect. 5, we have:

$$v(x+\xi_q) = \int_{s-1}^{s} dy\, \tilde G_s(x,y) v(y+\xi_q), \qquad \forall\, x > s \ge \ell,$$

with ξ_q, ℓ as in (5.20) and

$$\tilde G_s(x,y) \doteq \sum_{n=1}^{\infty} \frac{1}{(1+\lambda)^n} \int_s^{+\infty} dy_1 \cdots \int_s^{+\infty} dy_{n-1} \prod_{i=1}^{n} p_{\xi_q}(y_{i-1}) J(y_{i-1}-y_i).$$

Finally (2.64) is exactly [6, Eq. (2.23)], the only difference here is that this estimate can be proved also for $\zeta = \gamma_v$, as it can be checked by exploiting the proof in [6, Sect. 10], we omit the details. $\square$

Proof of Theorem 2.5. In Theorem 2.2, Proposition 2.3, and Lemma 2.4 we have introduced functions $\ell_i = \ell_i(\kappa)$ and $c_i = c_i(\kappa)$ for $i = 0,1,2$ (which we can assume non decreasing). In the sequel, given $\bar\kappa > 0$, we shorthand $\bar\ell_i = \ell_i(\bar\kappa)$ and $\bar c_i = c_i(\bar\kappa)$, $i = 0,1,2$. For $\varepsilon > 0$ and $\bar\ell > \bar\ell_2$ to be fixed, we define:

$$F : \Gamma_{\bar\ell,\bar\kappa} \times \mathcal{B}_{\bar\ell,\bar\kappa,\varepsilon} \to \mathbb{R} \quad : \quad F(\xi,m) = \pi_\xi(n_\xi - m).$$

Clearly $F(\xi,n_\xi) = 0$ for all $\xi \in \Gamma_{\bar\ell,\bar\kappa}$. Given $\ell_3 > \bar\ell$ and $\kappa < \bar\kappa$, we next apply the contraction mapping principle in the first argument of the function

$$G : \Gamma_{\bar\ell,\bar\kappa} \times \mathcal{B}_{\bar\ell,\bar\kappa,\varepsilon} \times \Gamma_{\ell_3,\kappa} \to \mathbb{R} \quad : \quad G(\xi,m,\xi^*) = \xi - \left[\partial_\xi F(\xi^*,n_{\xi^*})\right]^{-1} F(\xi,m).$$

Since $\partial_\xi F(\xi^*,n_{\xi^*}) = \pi_{\xi^*}(\partial_{\xi^*} n_{\xi^*})$ then

$$\partial_\xi G = \pi_{\xi^*}(\partial_{\xi^*} n_{\xi^*})^{-1} \left[\pi_{\xi^*}(\partial_{\xi^*} n_{\xi^*}) - \pi_\xi(\partial_\xi n_\xi) + \partial_\xi \pi_\xi(n_\xi - m)\right],$$

from which, by using (2.38) and (2.41),

$$|\partial_\xi G| \le (\sqrt{\mu} + \bar c_2)\big(|\pi_{\xi^*}(\partial_{\xi^*} n_{\xi^*}) - \pi_\xi(\partial_\xi n_\xi)| + \bar c_2 \|m - n_\xi\|_\infty\big).$$

But (2.19) implies $\|n_\xi - n_{\xi^*}\|_\infty \le (\bar c_0 + \|\bar m'\|_\infty)|\xi - \xi^*|$, hence there is a constant $C = C(\bar\kappa)$ so that:

$$|\partial_\xi G| \le C\big(|\pi_{\xi^*}(\partial_{\xi^*} n_{\xi^*}) - \pi_\xi(\partial_\xi n_\xi)| + |\xi - \xi^*| + \|m - n_{\xi^*}\|_\infty\big).$$

Using again (2.41), we can now fix $\bar\ell$ so large that (for all $h < e^{-2\alpha\ell_3}$)

$$C\big|\pi_{\xi^*}(\partial_{\xi^*} n_{\xi^*}) - \pi_\xi(\partial_\xi n_\xi)\big| < \frac{1}{8}.$$

Then, if $|\xi - \xi^*| < r$ and $\|m - n_{\xi^*}\|_\infty < \bar\varepsilon$ with r, $\bar\varepsilon > 0$ small enough, we find $|\partial_\xi G| < 1/4$ (for all $h < e^{-2\alpha\ell_3}$). Moreover, if we assume also $\ell_3 > \bar\ell + r$ and $\kappa < \bar\kappa - r$, the function $G(\cdot,m,\xi^*)$ is defined for all $\xi \in (\xi^* - r, \xi^* + r)$ and

$$\begin{aligned} |G(\xi,m,\xi^*) - \xi^*| &\le |G(\xi,m,\xi^*) - G(\xi,n_{\xi^*},\xi^*)| + |G(\xi,n_{\xi^*},\xi^*) - G(\xi^*,n_{\xi^*},\xi^*)| \\ &\le |\pi_{\xi^*}(\partial_{\xi^*} n_{\xi^*})|^{-1} |\pi_\xi(n_{\xi^*} - m)| + \frac{|\xi - \xi^*|}{4} \\ &\le \bar c_2(\sqrt{\mu} + \bar c_2)\|n_{\xi^*} - m\|_\infty + \frac{|\xi - \xi^*|}{4} \\ &\le \bar c_2(\sqrt{\mu} + \bar c_2)\varepsilon + \frac{r}{4} < \frac{r}{2}, \end{aligned}$$

where we used (2.31), (2.41), and we further assumed $\bar\varepsilon$ so small that $4\bar c_2(\sqrt{\mu}+\bar c_2)\bar\varepsilon < r$. Then we can apply the contraction mapping principle, by obtaining there is a unique solution $\xi = \Xi(m,\xi^*)$ of the equation $F(\xi,m) = 0$ in $(\xi^* - r, \xi^* + r)$ for any

$\xi^* \in \Gamma_{\ell_3,\kappa}$ and m such that $\|m - n_{\xi^*}\|_\infty < \bar{\varepsilon}$. Moreover $m \mapsto \Xi(m, \xi^*)$ is C^1 and $|\Xi(m, \xi^*) - \xi^*| < r/2$.

We finally observe that if ℓ_3 is large enough there is $\varepsilon_0 \in (0, \bar{\varepsilon}]$ such that the following holds. If ξ^*, $\xi^{**} \in \Gamma_{\ell_3,\kappa}$ and $\|n_{\xi^*} - n_{\xi^{**}}\|_\infty < 2\varepsilon_0$ then $|\xi^* - \xi^{**}| < r/2$. In fact, by using (2.9) and (2.19), from the condition $|\xi^* - \xi^{**}| \geq r/2$ it follows there is $C = C(\bar{\kappa})$ so that

$$|n_{\xi^*}(0) - n_{\xi^{**}}(0)| \geq \left(\alpha a e^{-\alpha \ell_3} - C e^{-\min\{\alpha_0;\alpha+\delta_0\}\ell_3}\right)\frac{r}{2},$$

and, for ℓ_3 large enough, the r.h.s. is not smaller than $2\varepsilon_0$ for any sufficiently small ε_0.

We now prove Theorem 2.5. Given $\kappa > 0$, we fix $\bar{\kappa} > \kappa$ and then we introduce the parameters ε_0, $\bar{\ell}$, and ℓ_3 so that all the previous statements hold. Then we define the map $\Xi : \mathcal{B}_{\ell_3,\kappa,\varepsilon_0} \to \Gamma_{\bar{\ell},\bar{\kappa}}$ by setting $\Xi(m) = \Xi(m, \xi^*)$ if $\|m - n_{\xi^*}\|_\infty < \varepsilon_0$ (the existence of such $\xi^* \in \Gamma_{\ell_3,\kappa}$ follows from (2.43)). We see that the definition is well posed: assume $\|m - n_{\xi^{**}}\|_\infty < \varepsilon_0$ for some other $\xi^{**} \in \Gamma_{\ell_3,\kappa}$, hence $\|n_{\xi^*} - n_{\xi^{**}}\|_\infty < 2\varepsilon_0$ and then $|\xi^* - \xi^{**}| \leq r/2$ for what stated before. Then $|\Xi(m, \xi^{**}) - \xi^*| \leq |\Xi(m, \xi^{**}) - \xi^{**}| + |\xi^{**} - \xi^*| < r$ and, by local uniqueness, $\Xi(m, \xi^{**}) = \Xi(m, \xi^*)$. Moreover, to prove (2.45), we observe that $\|m - n_\xi\|_\infty \leq \|m - n_{\xi^*}\|_\infty + (\sqrt{\mu} + \bar{c}_2)|\xi - \xi^*|$ and, on the other hand, if $\xi = \Xi(m, \xi^*)$ then $|\xi - \xi^*| \leq \mathrm{const}|\pi_{\xi^*}(\partial_{\xi^*} n_{\xi^*})|^{-1}|\pi_{\xi^*}(n_{\xi^*} - m)| \leq C\|m - n_{\xi^*}\|_\infty$, for some $C = C(\bar{\kappa})$. This last bound also prove (2.46), since $\|m - n_{\xi_0}\|_\infty < \varepsilon_0$ implies $\xi = \Xi(m, \xi_0)$. Finally Corollary 2.6 is an immediate consequence of the above reasoning.

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