THERMODYNAMIC LIMITS FOR HAMILTONIANS DEFINED AS PSEUDODIFFERENTIAL OPERATORS.

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Abstract.

If $P(h)$ is a $h-$pseudodifferential operator in $\mathbf{R}^n$ associated to an holomorphic semi-bounded symbol in some neighborhood of the real phase space, with bounded derivatives, we describe the symbol of $e^{-tP(h)}$, by inequalities where the constants depend on the bounds for the derivatives of the symbol of $P(h)$, but not on the dimension n. Some applications to thermodynamic limits (free energy) are given.

1. Introduction.

In [16], J. Sjöstrand describes the exponential of an $n-$dimensional semiclassical Schrödinger operator

$$P_n(h) \ = \ -h^2\Delta \ + \ V_n(x) \ . \tag{1.1}$$

Since the dimension n is variable, the given object is rather a sequence $(V_n)_{n\geq 1}$, where $V_n \in C^\infty(\mathbf{R}^n)$, belonging to a suitable class of such sequences, called $0-$*standard.* This class is defined by inequalities where the constants are independent on the dimension n. Instead of recalling the definition of $0-$standard sequences, let us say that, if there are some constants $a > 0$ and $M > 0$, independent on n, such that V_n extends to an holomorphic function V_n on

$$\Omega_n(a) \ := \ \{x \in \mathbf{C}^n, \ |Im \ x|_\infty < a \ \} \tag{1.2}$$

and such that

$$\left| \frac{\partial V_n}{\partial x_j}(x) \right| \ \leq \ M \qquad \forall x \in \Omega_n(a), \qquad \forall n \geq 1, \tag{1.3}$$

then $(V_n)_{(n\geq 1)}$ is $0-$standard, by Proposition 1.1 of [15]. Then, assuming also that V_n is real valued and lower bounded for real x, Sjöstrand gives (in [16], sect. 7), a description of the kernel of the operator $e^{-\frac{t}{h}P_n(h)}$, $(t > 0)$, near the diagonal, where the constants in the inequalities are independent on n, and he gives some applications to thermodynamic limits when $n \to +\infty$ (see below).

The aim of this paper is to replace the Schrödinger operator by pseudodifferential operators $P_n(h)$ associated, by the semiclassical Weyl calculus (see sect. 2), to symbols $p_n(x, \xi)$ which extend to

$$\Omega_{2n}(a) \ := \ \{(x, \xi) \in \mathbf{C}^{2n}, \ |Im \ (x, \xi)|_\infty < a \ \} \tag{1.4}$$

and satisfy

$$
(1.5) \qquad \left| \frac{\partial p_n}{\partial x_j}(x,\xi) \right| \ + \ \left| \frac{\partial p_n}{\partial \xi_j}(x,\xi) \right| \ \leq \ M \qquad \forall (x,\xi) \in \Omega_{2n}(a), \qquad \forall n \geq 1,
$$

for some constants $a > 0$ and $M > 0$, independent on n. Moreover, we shall be interested in $e^{-tP_n(h)}$ instead of $e^{-\frac{t}{h}P_n(h)}$. Our aim is to give, assuming also that $Re\ p_n(x,\xi)$ is lower bounded, a description of the *symbol* of the operator $e^{-tP_n(h)}$, (and not of its *kernel*, like in [16]), where the constants will be independent on the dimension and, more precisely, to show that the symbol of this operator can be written $e^{-q_n(x,\xi,t,h)}$, where q_n satisfies inequalities similar to (1.5). This result will rely on the theorem of composition of pseudodifferential operators in large dimension, (where good estimations are not satisfied by the composed symbols, but by their logarithms), proved in a previous paper [1] with L. Amour and Ph. Kerdelhué. In [1], the results proved here were conjectured, and proved in the formal level. In [10], the results were proved under the assumption that the symbols were bounded, and not only semi-bounded. To weaken the hypothesis, we had to make important change to the technique of proof.

Let us explain (in a particular case to simplify the notations), the physical model given by J. Sjöstrand ([16], sect.8) as an application of his results. At each point j of the one-dimensional lattice $\mathbf{Z}$, we consider a particle A_j described by a Schrödinger operator in $\mathbf{R}^k$ (where $k \geq 1$ is fixed)

$$
P_1(h) \ = \ -h^2\Delta + V(x) \ .
$$

The assumptions of [16] are satisfied if V extends to an holomorphic function in $\Omega_k(a)$ $(a > 0)$, if its derivatives are bounded in $\Omega_k(a)$, if V is real for real x, and if $V(x)$ is greater than a positive power of $|x|$ for large $|x|$. The interaction between each particle and its neighbours in the lattice is given by a potential $W(x,y)$, and the hypotheses of [16] are satisfied if W extends to a bounded holomorphic function in $\Omega_{2k}(a)$, real for real (x,y), and such that $W(y,x) = W(x,y)$. For each $n \geq 1$, the Hamiltonian $P_n(h)$ describing the system of particles A_j $(-n \leq j \leq n)$ in interaction is defined by (1.1), (with n replaced by $(2n+1)k$), where

$$
(1.6) \qquad V_n(x) \ = \ \sum_{j=-n}^{n} V(x^{(j)}) \ + \ \sum_{j=-n}^{n-1} W(x^{(j)}, x^{(j+1)}) \ .
$$

The variable of $\mathbf{R}^{(2n+1)k}$ is denoted by $x = (x^{(-n)}, \dots, x^{(n)})$, with $x^{(j)} \in \mathbf{R}^k$. This sequence (V_n) satisfies (1.3). Such type of Hamiltonians, given at each point of a lattice, each of them interacting with its neighbors, is common in the literature (see Toda [17] in the classical mechanics). In [16], the lattice is not only $\mathbf{Z}$, but be multidimensional, and one motivation can be the theory of anharmonic crystals. In E. Lieb [9] and B. Simon [14], a similar model (the *quantum Heisenberg model*), is studied, but with another Hamiltonian at each point of the lattice, and another type of interaction.

Then, Sjöstrand [16] proves that, for each $h > 0$ and $t > 0$, the sequence

$$
(1.7) \qquad \Lambda_n(t,h) \ := \ \frac{1}{(2n+1)}\ ln\ \left[(2\pi h)^{(2n+1)k} Tr\ \left(e^{-\frac{t}{h}P_n(h)} \right) \right]
$$

has a limit $\Lambda_\infty(t,h)$ when $n \to +\infty$, that

$$
(1.8) \qquad |\Lambda_n(t,h) \ - \ \Lambda_\infty(t,h)| \ \leq \ \frac{C}{n} \ ,
$$

and, with some more hypotheses, that $\Lambda_\infty(t,h)$, (the *thermodynamic limit*), has an asymptotic expansion in powers of h when $h \to 0$. (See Ruelle [13] for the notion of thermodynamic limit, and also Helffer-Sjöstrand [6] for the Lemma used to prove its existence and (1.8)). For an anharmonic crystal, the derivative of $\Lambda_\infty(t,h)$ with respect to t (if it exists), is the mean energy at the temperature $1/t$. In [16], the potentials are typically $\mathcal{O}(|x|)$ at infinity. For exactly quadratic potentials, see also Royer [12], which gives a rigorous proof and a generalization of some usual physical formulas on harmonic crystals.

As an application of our theorem 2.2, we shall prove similar results when the particles A_j ($j \in \mathbf{Z}$) are no more described by a Schrödinger operator, and with $e^{-\frac{t}{h}P_n(h)}$ replaced by $e^{-tP_n(h)}$. Our result can be applied, be applied, for example, to

$$P_n(h) \ := \ \sum_{j=-n}^{n} \sqrt{I - h^2\Delta_j} \ + \ V_n(x)$$

where Δ_j is the Laplacian for the variable $x^{(j)} \in \mathbf{R}^k$, and $V_n(x)$ is defined by (1.6). For such operators, we shall prove the analogous of the result of Sjöstrand [16]. The similar result for the quantum Heisenberg model seems to be still an open problem. (In this model, h is the inverse of the spin, and Lieb [9] and Simon [14] proved only the continuity of $\Lambda_\infty(t,h)$ at $h=0$). In [16], the asymptotic expansion of the thermodynamic limit in powers of h relies on a study of Laplace integrals in large dimension (cf also [2]). In our sect. 9, it relies on the study of the greatest eigenvalue of some integral operator on $\mathbf{R}^p$ (for a suitable p, depending on the order of the asymptotic expansion in powers of h). The study of this integral operator has some common points with the works of B. Helffer (cf [3] and the references given in [3]) and Helffer-Ramond (cf [4]) for the Kac operator.

The results of Lascar [8] on pseudodifferential calculus in infinite dimension have another motivation, related to the quantum field theory.

The main result is theorem 2.2, and its application to thermodynamic limit is given in theorems 9.1 and 9.3. In sect.3, a result on the composition of pseudodifferential operators in large dimension may have its own interest.

In a preprint [10], a similar result was proved under the additional hypothesis that p_n was bounded, and an application to thermodynamic limits for periodic symbols (with a suitable notion of trace) was given. In this preprint some results on inversion and exponential of pseudodifferential operators in large dimension, (which may have also perhaps their own interest), were used.

2. Statement of the result on the exponential.

For each $(x,\xi) \in \mathbf{R}^{2n}$, we set

$$\|(x,\xi)\|_\infty \ = \ \sup_{j \le n} sup(|x_j|, |\xi_j|). \tag{2.1}$$

For each $a > 0$, the set $\Omega_{2n}(a)$ is defined in (1.4).

Definition 2.1. *For each $a > 0$, we denote by $S(a)$ the set whose elements are sequences $(f_n)_{(n \ge 1)}$,where $f_n = f_n(h) = f_n(.,h)$ is an holomorphic function in $\Omega_{2n}(a)$, depending on parameter h in an interval $]0, h_n]$, where $h_n > 0$, and satisfying the following condition. There exists*

$M > 0$, independent on n, and a constant $C_n > 0$ (which may depend on n) such that, if $1 \leq j \leq n$ and if $0 < h \leq h_n$,

$$Re\ f_n(x,\xi,h) \ \geq \ C_n, \qquad \forall (x,\xi) \in \Omega_{2n}(a) \quad \forall h \in]0,h_n], \tag{2.2}$$

$$\left|\frac{\partial f_n}{\partial x_j}(x,\xi,h)\right| + \left|\frac{\partial f_n}{\partial \xi_j}(x,\xi,h)\right| \ \leq \ M, \qquad \forall (x,\xi) \in \Omega_{2n}(a) \quad \forall h \in]0,h_n]. \tag{2.3}$$

If f is a bounded function in $\Omega_{2n}(a)$ (defined in (1.4)), we set

$$\|f\|_a \ = \ \sup_{X \in \Omega_{2n}(a)} |f(X)|. \tag{2.4}$$

If the derivatives are also bounded, we set

$$\|\nabla f\|_a \ = \ \sup_{X \in \Omega_{2n}(a)} \ \sup_{j \leq n} \ sup \left|\frac{\partial f}{\partial x_j}(X)\right|, \ \left|\frac{\partial f}{\partial \xi_j}(X)\right|. \tag{2.5}$$

By the Weyl calculus, we associate to a suitable function $f \in C^\infty(\mathbf{R}^{2n})$ the $h-$pseudodifferential operator $Op_h(f)$ by

$$(Op_h(f)u)(x) \ = \ (2\pi h)^{-n} \int_{\mathbf{R}^{2n}} e^{\frac{i}{h}(x-y).\xi} f(\frac{x+y}{2},\xi) u(y) \ dy d\xi \ . \tag{2.6}$$

If (p_n) is in $S(a)$, we know, by the functional calculus of Helffer-Robert [6], that, for each fixed dimension n, the exponential $e^{-tOp_h(p_n)}$ is also a $h-$pseudodifferential operator, and we want to describe its symbol uniformly with respect to the dimension n. The main result of this paper is the following.

Theorem 2.2. *Let $b > 0$, and $p = (p_n)$ a family of symbols in $S(b)$, independent on h. Then, for each $a \in]0,b[$, and for each integer $m \geq 1$, there exist $\varepsilon_m > 0$, $C_m > 0$ (independent on the dimension n), and a family of functions $q = (q_n)(x,\xi,t,h)$ in $S(a)$ ($0 \leq t \leq 1$) such that*

$$e^{-tOp_h(p_n)} \ = \ Op_h(e^{-q_n(.,t,h)}) \qquad if \quad 0 < nh^m \leq \varepsilon_m, \qquad n \geq 1 \ . \tag{2.7}$$

Moreover, the symbol q_n has an asymptotic expansion in powers of h

$$q_n(x,\xi,t,h) \ = \ \sum_{j=0}^{m-1} E_n^{(j)}(x,\xi,t) h^j + h^m R_n^{(m)}(x,\xi,t,h) \ , \tag{2.8}$$

where the families of functions $(E_n^{(j)})$ are in $S(a)$ and satisfy the inequalities (2.3), with the same conditions as above. Moreover, $E_n^{(0)}(x,\xi,t) = tp_n(x,\xi)$ and, for $j \geq 1$, we have

$$\|E_n^{(j)}(.,t)\|_a \leq nC_j \qquad \|R_n^{(m)}(.,t,h)\|_a \leq nC_m \ .$$

The explicit construction of the functions $E_n^{(j)}$ will be made more precise in sections 6 (in general) and 9 (in some particular cases).

The paper is organized as follows. In section 3 we recall a result of [1] on the composition of pseudodifferential operators in large dimension. In section 4, we recall the result of [1] on the composition of operators associated to symbols which are the exponential of functions p_n in $S(a)$. In [1], the result was stated for the standard calculus and the function p_n was assumed to be bounded, but the proof is the same with our hypotheses. In section 5, following the thesis of the second author [11], we state and prove a result which is like a linearized of the previous one. In section 6, we give the formal construction of the $E_n^{(j)}$ of (2.8), and in sections 7 and 8, the end of the proof of Theorem 2.2. The precise statement for applications to thermodynamic limit is given and proved in section 9.

3. Oscillating integrals and composition of symbols in large dimension (result of [1]).

We denote by σ the symplectic form in $\mathbf{C}^{2n}$ defined by $\sigma\big((x,\xi),\ (y,\eta)\big) = y.\xi - x.\eta$. If f and g are C^∞ functions on $\mathbf{R}^{2n}$, bounded with all their derivatives, we denote by $f\natural_h g$ the function defined by the oscillatory integral

$$(f\natural_h g)(X) \;=\; (\pi h)^{-2n}\int_{\mathbf{R}^{4n}} e^{-\frac{2i}{h}\sigma(Y,Z)} f(X+Y)g(X+Z)dYdZ\ . \tag{3.1}$$

More generally, we shall have to work with oscillating integrals of the following form :

$$(I_h(F))\ (X) \;=\; (\pi h)^{-2n}\int_{\mathbf{R}^{4n}} e^{-\frac{2i}{h}\sigma(Y,Z)} F(X,Y,Z)\ dYdZ\ . \tag{3.2}$$

where X is in $\mathbf{R}^p$ and F is a suitable function in an open set of $\mathbf{C}^p\times\mathbf{C}^{2n}\times\mathbf{C}^{2n}$.

The following theorem gives an estimation of such integrals.

Theorem 3.1. *Let $a>0$ and $\rho>0$, and $\Omega_{p,2n,2n}(a,\rho)$ be the set of $(X,Y,Z)\in\mathbf{C}^p\times\mathbf{C}^{2n}\times\mathbf{C}^{2n}$ such that X is in $\Omega_p(a)$ and Y and Z in $\Omega_{2n}(\rho)$. Let F be a bounded holomorphic function in $\Omega_{p,2n,2n}(a,\rho)$. Let $I_h(F)$ be integral defined in (3.2). Then we have*

$$\|I_h(F)\|_a \;\leq\; \sup_{(X,Y,Z)\in\Omega_{p,2n,2n}(a,\rho)} |F(X,Y,Z)|\ \left(1+\sqrt{\frac{2}{\pi}}\frac{\sqrt{h}}{\rho}e^{-\frac{\rho^2}{h}}\right)^{4n}\ . \tag{3.3}$$

This result is very similar to the Lemma 5.1 of [1]. The only difference is that the integral was in $\mathbf{R}^{2n}$ and that, instead of the quadratic form $\sigma(Y,Z)$ in $\mathbf{R}^{4n}$, we had the form $y.\eta$ in $\mathbf{R}^{2n}$, but the idea of the proof is the same: we diagonalize the quadratic form in order to reduce the problem to a one dimensional integral, like in [1]. Therefore, we shall not give the details of the proof.

The following corollary may have its own interest. It was not explicitly written in [1].

Theorem 3.2. *Let a and b be such that $0 < a < b$, and f and g be holomorphic bounded functions in $\Omega_{2n}(b)$. Then $f \sharp_h g$ is holomorphic and bounded in $\Omega_{2n}(a)$ and we have*

$$\|f \sharp_h g\|_a \;\leq\; \|f\|_b \; \|g\|_b \; \left(1 + \sqrt{\frac{2}{\pi}} \, \frac{\sqrt{h}}{b-a} e^{-\frac{(b-a)^2}{h}} \right)^{4n} . \tag{3.4}$$

For later use, let us recall that, if the integral $I_h(F)$ is of the following form (with $p = 4n$)

$$(I_h(F))\,(X, X') \;=\; (\pi h)^{-2n} \int_{\mathbf{R}^{4n}} e^{-\frac{2i}{h}\sigma(Y,Z)} F(X+Y, X'+Z) \; dY dZ, \tag{3.5}$$

then it satisfies the differential equation

$$2i \frac{\partial I_h}{\partial h}(X, X') = \sigma \left(\left(\frac{\partial}{\partial x} \, , \, \frac{\partial}{\partial \xi} \right) \; , \; \left(\frac{\partial}{\partial x'} \, , \, \frac{\partial}{\partial \xi'} \right) \right) I_h. \tag{3.6}$$

4. Composition of exponentials of symbols (result of [1]).

We shall use the theorems 2.4 and 2.6 of the paper [1] with L. Amour and Ph. Kerdelhué. Unfortunately, these results are written in [1] with the standard calculus, and here we use the Weyl calculus. Moreover, the exponent in the symbol was supposed to be bounded, but the proof is the same if it is semi-bounded. It is useful to write here the statement for the Weyl calculus, and to recall the formal construction, but the proof, which is the same as in [1], will be omitted.

Theorem 4.1. *Let f and g be functions in $S(b)$ $(b > 0)$, and let h'_n and h''_n be the sequences associated to them as in Definition 2.1. Let $a \in]0, b[$ and $m \geq 1$. Then, there exist an element φ of $S(a)$ and constants ε_m and $C_m > 0$, depending only on m, a and b, and on the bounds of $\|\nabla f_n\|_b$ and $\|\nabla g_n\|_b$, but not of the dimension n, such that*

$$e^{-\varphi_n(h)} \;=\; e^{-f_n(h)} \sharp_h e^{-g_n(h)} \qquad if \quad 0 < h \leq h'_n, \quad h \leq h''_n \quad and \quad nh^m \leq \varepsilon_m \; . \tag{4.1}$$

Moreover, we can write, omitting the subscript n:

$$\varphi(x, \xi, h) \;=\; \sum_{k=0}^{m-1} c^{(k)}(f, g)(x, \xi) h^k \;+\; h^m \; R^{(m)}(x, \xi, h) \; , \tag{4.2}$$

where $c^{(k)}(f, g)$ and $R^{(m)}$ are in $S(a)$, $c^{(0)}(f, g) = f + g$, and, with the preceding conditions, if m and $k \geq 1$

$$\frac{1}{n}\|c^{(k)}(f, g)\|_a \;+\; \frac{1}{n}\|R^{(m)}(., h)\|_a \;+\; \|\nabla c^{(k)}(f, g)\|_a \;\leq\; C_m \; . \tag{4.3}$$

5. Linearization of the previous result.

Given four suitable functions p_1, p_2, f_1 and f_2 on $\mathbf{R}^{2n}$, we have to estimate quotients of the form

$$\psi_h(f_1, f_2) = \frac{(e^{-p_1} f_1)\sharp_h (e^{-p_2} f_2)}{e^{-p_1}\sharp_h e^{-p_2}} \ . \tag{5.1}$$

These estimations will be useful in sections 7 and 8 in order to prove Theorem 2.2 by a fixed point theorem. We hope that the bounds of ψ_h will be bounded by the product of the bounds of f_1 and f_2, and that the constant in the inequality will depend on ∇p_1 and ∇p_2, and of course not of the dimension n, and moreover that this constant will be as near as possible to 1. This will be made more precise in the following

Theorem 5.1. *Let p_1 and p_2 holomorphic functions in $\Omega_{2n}(R)$ ($R > 0$), with bounded derivatives. Let f_1 and f_2 are holomorphic functions, bounded respectively on $\Omega_{2n}(\mu_1)$ and $\Omega_{2n}(\mu_2)$, where $0 < \mu_j < R$ ($j = 1, 2$). Then the function $\psi_h(f_1, f_2)$ defined in (3.1) is holomorphic and bounded in $\Omega_{2n}(\lambda)$ for each λ in $]0, \inf(\mu_1, \mu_2)[$, and satisfies, for each integer m, if we set $\mu = \min(\mu_1, \mu_2)$ and $R' = \max(\mu_1, \mu_2)$*

$$||\psi_h(f_1, f_2)||_\lambda \leq \left(1 + \frac{K_m h^m}{(\mu - \lambda)^{2m}}\right)^n ||f_1||_{\mu_1} ||f_2||_{\mu_2} \ , \tag{5.2}$$

if h satisfies the following conditions:

$$\begin{cases} h||\nabla p_2||_R \leq \frac{\mu_1 - \lambda}{4} \\ h||\nabla p_1||_R \leq \frac{\mu_2 - \lambda}{4} \end{cases} \quad \text{and} \quad nh^m \leq \varepsilon_m \ , \tag{5.3}$$

where $K_m > 0$ and $\varepsilon_m > 0$ depend on R, R', m and on $\|\nabla p_j\|_R$, ($j = 1, 2$), but not on n.

Before to prove this result, we shall first introduce some notations, recall some results of [1], and examine a similar result in small dimension (proposition 5.7) which will next lead us by induction to the conclusion.

Let $\mu_1 > 0$ and $\mu_2 > 0$, we denote by $\Omega_4(\mu_1, \mu_2)$ the following open set of $\mathbf{C}^4$:

$$\Omega_4(\mu_1, \mu_2) = \{(x, \xi, x', \xi') \in \mathbf{C}^4 \mid |\mathrm{Im}(x, \xi)|_\infty < \mu_1, |\mathrm{Im}(x', \xi')|_\infty < \mu_2\} \ ,$$

If $\mu_1 = \mu_2 = \mu$, we have $\Omega_4(\mu_1, \mu_2) = \Omega_4(\mu)$.

If f is a bounded holomorphic function defined on $\Omega_4(\mu_1, \mu_2)$, we set:

$$||f||_{\mu_1, \mu_2} = \sup_{X \in \Omega_4(\mu_1, \mu_2)} |f(X)| \ .$$

If f is an holomorphic function defined on $\Omega_4(\mu)$, having all its derivatives bounded, we set:

$$||\nabla_{x,\xi} f||_\mu = \max(||\partial_x f||_\mu, ||\partial_\xi f||_\mu) \ ,$$

$$||\nabla_{x',\xi'} f||_\mu = \max(||\partial_{x'} f||_\mu, ||\partial_{\xi'} f||_\mu) \ ,$$

$$||\nabla f||_\mu = \max(||\nabla_{x,\xi} f||_\mu, ||\nabla_{x',\xi'} f||_\mu) \ .$$

If Ω is an open set in $\mathbf{C}^p$ with $p \in \mathbf{N}^*$, we denote by $K^1(\Omega)$ the space of holomorphic functions in Ω with bounded derivatives and lower bounded real part in Ω .

In this section, we shall use the results proved in [1] with the standard pseudodifferential calculus which is here replaced by the Weyl one. Hence, we can prove, exactly as in [1], the following propositions.

Proposition 5.2 (Proposition 3.1 of [1]). *Let a and b such that $0 < a < b$. Let $u \in K^1(\Omega_1(b))$, with $\Omega_1(b) = \{z \in \mathbf{C}/|\mathrm{Im} z| < b\}$. Then there exists $h_1 > 0$ depending only on $b - a$ and $||u'||_b$, independent on $||u||_b$, and two functions v_+ and v_-, C^∞ on $[0, h_1]$, valued in $K^1(\Omega_1(a))$, verifying*

$$e^{-v_\pm(x,h)} = e^{\mp i\frac{\pi}{4}}\sqrt{\frac{2}{\pi h}}\int_{\mathbf{R}} e^{\frac{\pm 2iy^2}{h}} e^{-u(x-y)}dy \qquad if 0 < h \le h_1 \qquad v_\pm(x,0) = u(x).$$

$$||e^{-v_\pm(.,h)+u(.)} - 1||_a \le \frac{1}{2} .$$

Proposition 5.3 (Proposition 6.1 of [1]). *Let R and R' such that $0 < R' < R$ and $u \in K^1(\Omega_4(R))$. Then for all $m \ge 1$ there exists some functions $\alpha_m(R - R', ||\nabla u||_R)$ (decreasing in its last variable) and $C_m(R - R', ||\nabla u||_R)$ (increasing in its last variable) and a function $h \mapsto v(., h)$, C^∞ on $[0, \alpha_m(R - R', ||\nabla u||_R)]$ valued in $K^1(\Omega_4(R'))$ such that, if $h \le \alpha_m$:*

$$e^{-v(x,\xi,x',\xi',h)} = I_h(e^{-u})(x, \xi, x', \xi') , \quad \forall (x, \xi, x', \xi') \in \Omega_4(R') , \tag{5.4}$$

where I_h is the integral of (3.5) (with $n = 1$). Moreover, we have:

$$||\nabla v(., h)||_{R'} \le (1 + C_m(R - R', ||\nabla u||_R)h^m)||\nabla u||_R , \quad \forall h \in [0, \alpha_m] . \tag{5.5}$$

In [1], I_h was the similar integral in $\mathbf{R}^2$, with the quadratic form $\sigma(Y, Z)$ replaced by $y.\eta$, and the function u was supposed to be bounded, and not only semi-bounded, but the ideas in the proof are the same, and we can omit them.

In lemma 5.7 we shall also use the following result proved in [1].

Lemma 5.4 (Lemma 4.4 of [1]). *Let p an integer ≥ 1, then there exists a constant $C_p > 0$ having the following properties. For all $\rho > 0$ and $\mu > 0$ such that $C_p\rho < \mu$ and for all holomorphic function f bounded in the closed ball $B(0, \mu)$ in $\mathbf{C}^4$ with the $|.|_\infty$ norm, we have:*

$$\sum_{0\le|j|\le p} |f^{(j)}(0)|\frac{\rho^{|j|}}{j!} \le \left[1 + \left(\frac{C_p\rho}{\mu - C_p\rho}\right)^{p+1}\right] \sup_{|z|_\infty\le\mu} |f(z)| .$$

After recalling these results of [1], we shall use them to estimate quotients of the type (5.1), or more generally of the form

$$\psi(., h)(X, X') = \frac{I_h(e^{-u}f)(X, X')}{I_h(e^{-u})(X, X')} , \tag{5.6}$$

where I_h is the integral defined in (3.5), with $n = 1$, f is bounded and holomorphic on $\Omega_4(\mu_1, \mu_2)$ and $u \in K^1(\Omega_4(R))$ with $0 < \mu_j < R$, $j = 1, 2$. We begin by the one dimensional case.

Lemma 5.5. *Let $u \in K^1(\Omega_4(R))$ $(R > 0)$, and f be a bounded holomorphic function on $\Omega_4(\mu_1, \mu_2)$, $(0 < \mu_j < R$, $j = 1, 2)$. Then the function $\psi(., h)$ defined by (5.6) and by $\psi(., 0) = f$, is well defined and holomorphic in $\Omega_4(\lambda)$ if $0 < \lambda < \inf(\mu_1, \mu_2)$, depends smoothly on h, and satisfies, if we set $R' = \max(\mu_1, \mu_2)$ and $\mu = \inf(\mu_1, \mu_2)$,*

$$||\psi(., h)||_\lambda \leq 64 \left(1 + \frac{2\sqrt{h}}{(\mu - \lambda)} e^{-(\mu-\lambda)^2/4h}\right)^4 ||f||_{\mu_1,\mu_2} \ , \tag{5.7}$$

if h satisfies the following conditions:

$$\begin{cases} h||\nabla_{x',\xi'} u||_R \leq \mu_1 - \lambda \\ h||\nabla_{x,\xi} u||_R \leq \mu_2 - \lambda \end{cases} \quad and \quad h \leq \varepsilon(R - R', ||\nabla u||_R) \ , \tag{5.8}$$

where $\varepsilon(R - R', ||\nabla u||_R)$ is some positive function of the two variables.

Proof. First step. Estimation of the quotient $e^{-u(.)}/I(e^{-u})(., h)$. We set, for all $a > 0$,

$$\tilde{\Omega}_4(a) = \{X = (X_i)_{1\leq i\leq 4} \in \mathbf{C}^4 / |\mathrm{Im} X_1| + |\mathrm{Im} X_4| < a, |\mathrm{Im} X_2| + |\mathrm{Im} X_3| < a\} \ ,$$

and for each function g bounded on $\tilde{\Omega}_4(a)$,

$$|||g|||_a = \sup_{X \in \tilde{\Omega}_4(a)} |g(X)| \ .$$

Since u is holomorphic in $\Omega_4(R)$, the function $\tilde{u}$ defined by

$$\tilde{u} = u(X_1 + X_4, X_3 + X_2, X_3 - X_2, X_1 - X_4) \ ,$$

is holomorphic on $\tilde{\Omega}_4(R)$ and it verifies the same inequalities as u. We introduce so the function

$$\tilde{I}_h(X) = I_h(e^{-u})(X_1 + X_4, X_3 + X_2, X_3 - X_2, X_1 - X_4) \ ,$$

where I_h is defined in (3.5). We obtain:

$$\tilde{I}_h(X) = \frac{4}{(\pi h)^2} \int_{\mathbf{R}^4} e^{\frac{2i}{h}(Y_1^2+Y_2^2-Y_3^2-Y_4^2)} e^{-\tilde{u}(X-Y)} dY \qquad X \in \Omega_4(R') \ . \tag{5.9}$$

By Proposition 5.2, if $0 < a < b$, for each function $f \in K^1(\Omega_4(b))$, there exists $h_1 > 0$ depending only on $b - a$ and $||\nabla u||_b$, and functions $v_+^{(j)}$ and $v_-^{(j)}$, $(1 \leq j \leq 4)$ in $K^1(\Omega_1(a))$, depending smoothly on $h \in [0, h_1]$, verifying if $0 \leq h \leq h_1$:

$$e^{-v_\pm^{(j)}(x,h)} = e^{\mp i\frac{\pi}{4}} \sqrt{\frac{2}{\pi h}} \int_{\mathbf{R}} e^{\frac{\pm 2iy^2}{h}} e^{-f(x_1,\ldots,x_j-y,\ldots,x_4)} dy \ ,$$

$$||e^{-v_\pm^{(j)}(.,h)+f(.)} - 1||_a \leq \frac{1}{2} \ .$$

Let us denote $v_\pm^{(j)}$ by $V_\pm^{(j)}(h)f$. Therefore, the function

$$\tilde{g}_h = V_-^{(4)}(h)V_-^{(3)}(h)V_+^{(2)}(h)V_+^{(1)}(h)\tilde{u}$$

is holomorphic in $\tilde{\Omega}_4(R')$ and satisfies

$$e^{-\tilde{g}_h} = \tilde{I}_h \ , \qquad |||\frac{e^{-\tilde{u}}}{e^{-\tilde{g}_h}}|||_{R'} \leq 16 \ .$$

This is valid if $h \leq \varepsilon_1$, where ε_1 is a function of $R - R'$ and $||\nabla u||_R$. Therefore:

$$\left\| \frac{e^{-u(.)}}{I(e^{-u})(.,h)} \right\|_{R'} \leq 16 \ , \quad if \quad h \leq \varepsilon_1(R - R', ||\nabla u||_R) \ . \tag{5.10}$$

Second step. Estimation of the product $e^{u(.)}I(e^{-u}f)(.,h)$. For each function $\Phi(X,X') \in C^\infty(\mathbf{R}^{4n})$ let us set, for each $X = (x,\xi)$ and $X' = (x',\xi')$ in $\mathbf{R}^{2n}$,

$$H_\Phi(X,X') \ = \ \left(-\frac{\partial \Phi}{\partial \xi}, \ \frac{\partial \Phi}{\partial x} \right) \ ,$$

and we define similarly $H'_{\Phi(X,X')}$ by the derivatives with respect to x' and ξ'. Then we set, for each X, X', Y and Z in $\mathbf{R}^{2n}$

$$Q_\Phi(X,X',Y,Z) \ = \ \int_0^1 H_\Phi(X+tY, X'+tZ) \ dt \ ,$$

and we define similarly Q'_Φ using H'_Φ. With these notations, we can write, using the symplectic form σ

$$u(X+Y,X'+Z) - u(X,X') = -\sigma\left(Y, Q_u(X,X',Y,Z)\right) - \sigma\left(Z, Q'_u(X,X',Y,Z)\right) \ .$$

If I_h is the integral defined in (3.5), it follows that, for $(X,X') \in \Omega_4(\lambda)$:

$$e^{u(X,X')}I_h(e^{-u}f)(X,X') = (\pi h)^{-2} \int_{\mathbf{R}^4} e^{-\frac{2i}{h}\varphi(X,X',Y,Z,h)} f(X+Y,X'+Z) dY dZ \ , \tag{5.11}$$

with,

$$\varphi(.,h) = \sigma\left(Y + \frac{h}{2i}Q'_u(X,X',Y,Z), Z - \frac{h}{2i}Q_u(X,X',Y,Z) \right) + ...$$

$$... + \ \frac{h^2}{4}\sigma\left(Q_u(X,X',Y,Z), Q'_u(X,X',Y,Z)\right) \ .$$

The *third step* of the proof will be stated as a lemma.

Lemma 5.6. *If we suppose, setting now* $M = ||\nabla u||_R$*:*

$$h||\nabla_{x',\xi'} u||_R \leq \mu_1 - \lambda \ , \quad h||\nabla_{x,\xi} u||_R \leq \mu_2 - \lambda \quad and \quad \frac{hM}{R - R'} \leq \frac{1}{4} \ , \tag{5.12}$$

then, for each (X,X') *in* $\Omega_4(\lambda)$ *and for each* (Y,Z) *in* $\Omega_4(\frac{\mu_1-\lambda}{2},\frac{\mu_2-\lambda}{2})$, *the map*

$$(\widetilde{Y},\widetilde{Z}) \longrightarrow \left(Y - \frac{h}{2i}Q'_u(X,X',\widetilde{Y},\widetilde{Z}),\quad Z + \frac{h}{2i}Q_u(X,X',\widetilde{Y},\widetilde{Z})\right)$$

sends $\overline{\Omega_4(\mu_1-\lambda,\mu_2-\lambda)}$ *in itself and has a unique fixed point* $(\widetilde{Y},\widetilde{Z})$ *in this set. Let us denote this fixed point by* $(\widetilde{Y},\widetilde{Z}) = \gamma_h(X,X',Y,Z)$. *Then, under the conditions (5.12), the jacobian* J_h *of the map*

$$\Omega_4(\frac{\mu_1-\lambda}{2},\frac{\mu_2-\lambda}{2}) \ni (Y,Z) \to \gamma_h(X,X',Y,Z) \in \overline{\Omega_4(\mu_1-\lambda,\mu_2-\lambda)}$$

satisfies

$$(5.13)\qquad |J_h(X,X',Y,Z)| \le 2 \qquad \forall (Y,Z)\in\Omega_4(\frac{\mu_1-\lambda}{2},\frac{\mu_2-\lambda}{2}) \qquad \forall (X,X')\in\Omega_4(\lambda).$$

Moreover, we have, under the same conditions

$$(5.14)\qquad |e^{-i\frac{h}{2}\sigma(Q_u(X,X',\gamma_h(X,X',Y,Z)),Q'_u(X,X',\gamma_h(X,X',Y,Z)))}| \le 2 \;\text{ if }\; hM^2 \le \frac{1}{2}\ .$$

The proof of this lemma is elementary.

End of the proof of Lemma 5.5. Then the map $\mathbf{R}^4 \ni (Y,Z) \to \gamma_h(X,X',Y,Z) : \mathbf{R}^4 \to \mathbf{C}^4$ defines a contour of integration γ_h in $\mathbf{C}^4$, depending on (X,X'), and we can write, using this change of contour, the expression (5.11) under the form

$$e^{u(X,X')}I_h(e^{-u}f)(X,X',h) = (\pi h)^{-2}\int_{\gamma_h} e^{-\frac{2i}{h}\varphi(X,X',Y,Z,h)}f(X+Y,X'+Z)dYdZ$$

$$\dots = (\pi h)^{-2}\int_{\mathbf{R}^4} e^{-\frac{2i}{h}\varphi(X,X',\gamma_h(X,X',Y,Z,h)}f((X,X')+\gamma_h(X,X',Y,Z))J_h(X,X',Y,Z)dYdZ\ .$$

By the definition of γ_h, we have

$$\varphi(X,X',\gamma_h(X,X',Y,Z) = \sigma(Y,Z) + \dots$$

$$\dots + \frac{h^2}{4}\sigma\left(Q_u\left(X,X',\gamma_h(X,X',Y,Z)\right)\ ,\ Q'_u\left(X,X',\gamma_h(X,X',Y,Z)\right)\right)\ .$$

Since the point $(X,X')+\gamma_h(X,X',Y,Z)$ is in $\Omega_4(\mu_1,\mu_2)$ if (Y,Z) is in $\Omega_4(\frac{\mu_1-\lambda}{2},\frac{\mu_2-\lambda}{2})$, we have, using (5.13), (5.14) and Theorem 3.1,

$$(5.15)\qquad \left\|e^{u(.)}I(e^{-u}f)(.,h)\right\|_\lambda \le 4(1+\frac{2\sqrt{h}}{\mu-\lambda}e^{-(\mu-\lambda)^2/4h})^4||f||_{\mu_1,\mu_2}\ ,$$

if the conditions (5.8) are satisfied. The estimation (5.7) follows from (5.10) and (5.15).

Lemma 5.5 is not satisfactory since the constant in the inequality is very far from 1. Proposition 5.8, (which uses Lemma 5.7), will give a better constant.

Lemma 5.7. *With the notations of Lemma 5.5, the function $\psi(.,h)$ of Proposition 5.5 satisfies, for each integer $m \geq 2$, for each λ and λ' such that $0 < \lambda < \lambda' < \mu = \inf(\mu_1, \mu_2)$*

$$(5.16) \qquad ||\psi(.,h)||_\lambda \leq ||f||_{\lambda'} + \frac{A_m h^m}{(\lambda' - \lambda)^{2m}} \sup_{s \in [0,h]} ||\psi(.,s)||_{\lambda'} \qquad \text{if} \quad h \leq \varepsilon_m(R - R', \|\nabla u\|_R),$$

where A_m and ε_m are positive functions of m, $R - R'$, where $R' = \max(\mu_1, \mu_2)$ and $||\nabla u||_R$.

Proof. With the notations of Lemma 5.5, we know, by (3.6), that the function $I_h(e^{-u})(X, X')$ defined in (3.5) satisfies

$$(5.17) \qquad 2i \frac{\partial I_h(e^{-u})}{\partial h} = \left(\frac{\partial^2}{\partial x' \partial \xi} - \frac{\partial^2}{\partial x \partial \xi'}\right) I_h(e^{-u}) \ .$$

Moreover, the function $\psi_h(X, X') I_h(e^{-u})(X, X') = I_h(fe^{-u})(X, X')$ satisfies a similar equation. By proposition 5.3, we know that there exists an holomorphic function $v(X, X', h)$ satisfying (5.4)for h small enough, but perhaps in a smaller domain. It follows from (5.17), from the similar equation satisfied by $\psi_h I_h(e^{-u})$ and from (5.4) that

$$2i \frac{\partial v}{\partial h} = \frac{\partial^2 v}{\partial x' \partial \xi} - \frac{\partial^2 v}{\partial x \partial \xi'} - \frac{\partial v}{\partial x'} \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial \xi'} \ ,$$

$$2i \frac{\partial \psi}{\partial h} = \frac{\partial^2 \psi}{\partial x' \partial \xi} - \frac{\partial^2 \psi}{\partial x \partial \xi'} + \frac{\partial v}{\partial x'} \frac{\partial \psi}{\partial \xi} - \frac{\partial v}{\partial x} \frac{\partial \psi}{\partial \xi'} + \frac{\partial \psi}{\partial x'} \frac{\partial v}{\partial \xi} - \frac{\partial \psi}{\partial x} \frac{\partial v}{\partial \xi'} \ .$$

Therefore, setting $z = (z_1, z_2, z_3, z_4) = (x, \xi, x', \xi')$, we can write, for all $j \in \{1, \dots, m+1\}$:

$$(5.18) \qquad \partial_h^j \psi(z, h) = \sum_{1 \leq |\alpha| \leq 2j} b_{jk}(z, h) \partial_z^\alpha \psi(z, h) \ ,$$

where the $b_{j\alpha}$ are linear combinations of the products of the form $\partial_z^{\beta_1} v ... \partial_z^{\beta_k} v$, where $\beta_1, ..., \beta_k$ are multi-indices such that $|\beta_1| + ... + |\beta_k| = j - |\alpha|$. Let us explain now the estimates of the functions $b_{j\alpha}$ of (5.18), and when (5.18) is valid. Let $R'' = \frac{R+R'}{2}$. By Proposition 5.3, there are two constants ε_m and M_1, both depending only on $R - R'$ and $||\nabla u||_R$, such that the equality (5.4)is valid if $z \in \Omega_4(R'')$ and $h < \varepsilon_m$, and such that we have $||\nabla v(.,h)||_{R''} \leq M_1$ if $h \leq \varepsilon_m$. By the Cauchy inequalities, and by the form of the functions $b_{j\alpha}$ of (5.18), there exists $M_2 > 0$ (depending only on m, $R - R'$ and $||\nabla u||_R$) such that

$$(5.19) \qquad ||b_{j\alpha}(.,h)||_{R'} \leq M_2 \qquad \text{if} \quad 1 \leq j \leq m+1, \qquad 1 \leq |\alpha| \leq 2j \ .$$

By the Taylor formula and (5.18), we can write:

$$(5.20) \qquad \psi(z, h) = \sum_{0 \leq |k| \leq 2m} a_k(z, h) \partial_z^k \psi(z, 0) + h^{m+1} R_m(z, h) \ , \quad \forall z \in \Omega_4(\lambda) \ ,$$

where:

$$(5.21) \qquad a_k(z, h) = \sum_{\frac{|k|}{2} \leq j \leq m} b_{jk}(z, 0) \frac{h^j}{j!} \ , \quad \textit{if} \quad 1 \leq |k| \leq 2m \ ,$$

$$a_0(z,h) = 1 \ ,$$

and:

$$(5.22) \qquad |R_m(z,h)| \leq \sup_{0\leq s\leq h} |\partial_s^{m+1}\psi(z,s)| \leq \sup_{0\leq s\leq h} \sum_{1\leq |k|\leq 2m+2} |b_{m+1,k}(z,s)||\partial_z^k \psi(z,s)| \ .$$

By (5.20), (5.21), (5.19), and by the Cauchy inequalities which, combined with (5.22), give a bound of R_m, there exists a constant $B>0$, depending on m, $R-R'$ and M_2, such that, if $z\in\Omega_4(\lambda)$ and $h\in[0,\varepsilon_m]$:

$$|\psi(z,h)| \leq \sum_{0\leq |k|\leq 2m} \frac{B^{|k|}h^{\frac{|k|}{2}}}{k!}|\partial_z^k\psi(z,0)| + \frac{Bh^{m+1}}{(\lambda'-\lambda)^{2m+2}} \sup_{0\leq s\leq h} ||\psi(.,s)||_{\lambda'} \ .$$

By lemma 5.4 applied to $\rho = Bh^{\frac{1}{2}}$ and $\mu=\lambda'$, $p=2m$, we see easily that if $z\in\Omega_4(\lambda)$ and $2C_{2m}Bh^{\frac{1}{2}} < \lambda'-\lambda$, C_{2m} being the constant of lemma 5.4, we have, since $\psi(z,0)=f(z)$:

$$\sum_{0\leq |k|\leq 2m} \frac{B^{|k|}h^{\frac{|k|}{2}}}{k!}|\partial_z^k\psi(z,0)| \leq \left[1+\left(\frac{C_{2m}Bh^{\frac{1}{2}}}{\lambda'-C_{2m}Bh^{\frac{1}{2}}}\right)^{2m+1}\right] ||f||_{\lambda'} \ .$$

Hence, if $2C_{2m}Bh^{\frac{1}{2}} < \lambda'-\lambda$, there is A_m, as in the statement of the lemma, such that (5.16) is satisfied. If h verifies $2C_{2m}Bh^{\frac{1}{2}} \geq \lambda'-\lambda$, the inequality (5.16) is trivial, with another A_m.

Proposition 5.8. *Let u a function in $K^1(\Omega_4(R))$, ($R>0$), f a bounded holomorphic function on $\Omega_4(\mu_1,\mu_2)$, ($0<\mu_j<R$, $j=1,2$), and $\psi(.,h)$ the function of (5.6). Then, for each integer $m\geq 2$, and for each λ such that $0<\lambda<\mu=\inf(\mu_1,\mu_2)$, we have*

$$(5.23) \qquad ||\psi(.,h)||_\lambda \leq \left(1+\frac{K_m h^m}{(\mu-\lambda)^{2m}}\right) ||f||_{\mu_1,\mu_2} \ ,$$

if h satisfies the following conditions:

$$(5.24) \qquad \begin{cases} h||\nabla_{x',\xi'}u||_R \leq \frac{\mu_1-\lambda}{2} \\ h||\nabla_{x,\xi}u||_R \leq \frac{\mu_2-\lambda}{2} \end{cases} \quad \textit{and} \quad h\leq\varepsilon_m \ ,$$

where K_m and ε_m are positive functions of m, $R-R'$ ($R'=\max(\mu_1,\mu_2)$), and $||\nabla u||_R$, independent on all the other variables.

In the application to the proof of Theorem 5.1, the constants K_m and ε_m will sometimes be denoted by $K(m,R-R',||\nabla u||_R)$ and $\varepsilon(m,R-R',||\nabla u||_R)>0$, and we may assume that K (resp. ε) is an increasing (resp. decreasing) function of its last variable.

Proof. By lemma 5.5, there exists $\varepsilon_1(R-R',||\nabla u||_R)>0$ such that, setting $\lambda'=(\lambda+\mu)/2$:

$$(5.25) \qquad ||\psi(.,h)||_{\lambda'} \leq 64(1+\frac{2\sqrt{h}}{(\mu-\lambda')}e^{-(\mu-\lambda')^2/4h})^4||f||_{\mu_1,\mu_2} \ ,$$

if the relations (5.8) are satisfied, replacing λ by λ' and ε by ε_1. Now we shall distinguish two cases.

First case: $h(\mu-\lambda)^{-2} \leq 1$. By lemma 5.7, there exists ε_2 and A_m, depending only on m, $R-R'$ and $||\nabla u||_R) > 0$ such that, if $h \leq \varepsilon_2$, we have (5.16). Hence, by (5.25) and (5.16), we see that there exist ε_m and K_m, both depending only on $||\nabla u||_R$ and $R-R'$, such that (5.23) is satisfied if the conditions (5.24) are.

Second case: $h(\mu-\lambda)^{-2} \geq 1$. In this case,(5.23) is a direct consequence of (5.7).

The previous proposition allows us now to demonstrate the theorem 5.1.

Proof of theorem 5.1.

Notations. For all $k \leq n$, we consider the following open sets of $\mathbf{C}^{4n}$:

$$\Omega_{k,n}(\lambda,\mu_1,\mu_2) = \{(X,X') \in \mathbf{C}^{2n} \times \mathbf{C}^{2n}, \quad |\mathrm{Im}(X_j,X'_j)|_\infty < \lambda \text{ if } \ j<k, \ \ldots$$

$$\ldots |\mathrm{Im}X_j|_\infty < \mu_1 \text{ and } |\mathrm{Im}X'_j|_\infty < \mu_2 \text{ if } j \geq k \ .$$

If $\mu_1 = \mu_2 = \mu$, this set will be simply denoted by $\Omega_{k,n}(\lambda,\mu)$. If f and g are bounded holomorphic functions defined respectively on $\Omega_{k,n}(R',R)$ and $\Omega_{k,n}(\lambda,\mu_1,\mu_2)$, we set:

$$||f||_{R',R,k} = \sup_{X\in\Omega_{k,n}(R',R)} |f(X)| \quad \textit{and} \quad ||g||_{\lambda,\mu_1,\mu_2,k} = \sup_{X\in\Omega_{k,n}(\lambda,\mu_1,\mu_2)} |g(X)| \ .$$

If f has all his derivatives bounded on $\Omega_{k,n}(R',R)$, we set:

$$||\nabla f||_{R',R,k} = \max_{1\leq j\leq n}(\max(||\partial_{x_j} f||_{R',R,k}, ||\partial_{\xi_j} f||_{R',R,k}, ||\partial_{x'_j} f||_{R',R,k}, ||\partial_{\xi'_j} f||_{R',R,k})) \ .$$

If f is a bounded holomorphic function on $\Omega_{k,n}(R',R)$ (resp. on $\Omega_{k,n}(\lambda,\mu_1,\mu_2)$), we introduce the function $I_k(f)(.,h)$ defined by the following oscillatory integral:

$$I_k(f)(X,X',h) = (\pi h)^{-2}\int_{\mathbf{R}^4} e^{-\frac{2i}{h}\sigma(Y,Z)} f(X_1,\ldots,X_k+Y,\ldots,X_n,\ X'_1,\ldots,X'_k+Z,\ldots,X'_n)\, dYdZ.$$

By Theorem 3.1, we know that this function is holomorphic and bounded on $\Omega_{k+1,n}(R',R)$ (resp. on $\Omega_{k+1,n}(\lambda,\mu_1,\mu_2)$) for $h>0$.

First step: We consider the sequence of function $u_k(.,h)$ on $\Omega_{k,n}(R',R)$ with $k \in \{1,\ldots,n\}$, defined by:

$$u_1(X,X') = p_1(X) + p_2(X') \qquad (X,X') \in \Omega_{2n}(R) \times \Omega_{2n}(R) = \Omega_{1,n}(R',R) \ ,$$

and by the following induction relation if $1 \leq k \leq n-1$:

$$e^{-u_{k+1}(.,h)} = I_k(e^{-u_k})(.,h) \ .$$

With the constant C_m of proposition 5.3, we define the following sequence $(P_k(h))_{1\leq k\leq n}$ by:

$$P_k(h) = (1 + h^m C_m(R-R', 2||\nabla u_1||_R))^{k-1} \ ,$$

and we set $c_m = \left[C_m(R-R', 2||\nabla u_1||_R)^{-1} \ln 2\right]^{\frac{1}{m}}$.

With this choice for c_m, we remark that:

$$h \leq c_m n^{-\frac{1}{m}} \Rightarrow P_k(h) \leq 2 \ , \quad \forall k \in \{1,\ldots,n\} \ .$$

Owing to proposition 5.3, we can choose $\varepsilon_0 > 0$ such that the following property (H_k) is valid for $k \in \{1, \ldots, n\}$:

(H_k): If $nh^m \leq \varepsilon_0$, then:

$$||\partial_{x_j} u_k(.,h)||_{R',R,k} \leq P_k(h)||\partial_{x_j} u_1||_R \leq 2||\partial_{x_j} u_1||_R \ , \tag{5.26}$$

and likewise for the derivatives in ξ_j, x'_j and ξ'_j, particularly:

$$||\nabla u_k(.,h)||_{R',R,k} \leq P_k(h)||\nabla u_1||_R \leq 2||\nabla u_1||_R \ . \tag{5.27}$$

Moreover, the real part of $u_k(.,h)$ is lower bounded on $\Omega_{k,n}(R', R)$.

Second step: We consider another sequence of functions $\psi_k(.,h)$ on $\Omega_{k,n}(\lambda, \mu_1, \mu_2)$ with $k \in \{1, \ldots, n+1\}$, defined by:

$$\psi_1(X, X') = f_1(X)f_2(X') \qquad (X, X') \in \Omega_{2n}(\mu_1) \times \Omega_{2n}(\mu_2) = \Omega_{1,n}(\lambda, \mu_1, \mu_2) \ ,$$

and by the following induction relation if $1 \leq k \leq n$:

$$\psi_{k+1}(.,h) = \frac{I_k(e^{-u_k}\psi_k)(.,h)}{I_k(e^{-u_k})(.,h)} \ .$$

Using inequalities (5.26) and (5.27) and proposition 5.8, we verify by induction the following property (I_k), for $k \in \{1, \ldots, n+1\}$, where ε_1 is the constant of proposition 5.8:

(I_k) : If:

$$\begin{cases} h||\nabla_{x'_k,\xi'_k} u_1||_R \leq \frac{\mu_1-\lambda}{4} \\ h||\nabla_{x_k,\xi_k} u_1||_R \leq \frac{\mu_2-\lambda}{4} \end{cases} \quad \textit{and} \quad nh^m \leq \min(\varepsilon_0(m, R-R', ||\nabla u_1||_R), \varepsilon_1^m(m, R-R', 2||\nabla u_1||_R)) \ ,$$

then:

$$||\psi_k(.,h)||_{\lambda,\mu_1,\mu_2,k} \leq \left(1 + \frac{h^m K_m(R-R', 2||\nabla u_1||_R)}{(\mu-\lambda)^{2m}}\right)^{k-1} ||\psi_1||_{\lambda,\mu_1,\mu_2,1} \ .$$

The definition of u_1 implies that:

$$||\nabla_{x'_k,\xi'_k} u_1||_R \leq ||\nabla p_2||_R \ , \quad ||\nabla_{x_k,\xi_k} u_1||_R \leq ||\nabla p_1||_R \ .$$

Hence, for $k = n+1$, setting:

$$\varepsilon_m \ = \ \varepsilon(m, R-R', ||\nabla u_1||_R) \ = \ \min(\varepsilon_0(m, R-R', ||\nabla u_1||_R), \varepsilon_1^m(m, R-R', 2||\nabla u_1||_R)) \ ,$$

we obtain:

$$||\psi_{n+1}(.,h)||_{\lambda,\mu_1,\mu_2,n+1} \leq \left(1 + \frac{h^m K_m(R-R', 2||\nabla u_1||_R)}{(\mu-\lambda)^{2m}}\right)^{n} ||\psi_1||_{\lambda,\mu_1,\mu_2,1} \ ,$$

if h satisfies (5.3). By the definition of the sequence (ψ_k), we verify easily that:

$$\psi_h(f_1, f_2)(X) = \psi_{n+1}(X, X, h) \ ,$$

and with the above-mentioned conditions on h, we obtain finally (5.2), with K_m depending on $R - R'$, m and $\|\nabla p_j\|_R$, $(j = 1, 2)$.

6. Formal construction of the symbol in theorem 2.2.

The formal aspect of theorem 2.2 is very similar to the Théorème 2.2 of [1]. The only difference is that the standard calculus is now replaced by the Weyl one. For each functions f and g, C^∞ in $\mathbf{R}^{2n}$, let us set $(f \otimes g)(y, \eta, z, \zeta) = f(y, \eta) g(z, \zeta)$ and

$$a_k(f, g)(x, \xi) \;=\; \frac{i^k}{2^k k!} \left[\sigma\big((D_y, D_\eta),\ (D_z, D_\zeta)\big)^k (f \otimes g) \right](x, \xi, x, \xi)\ . \tag{6.1}$$

We know that, if f and g are bounded with all their derivatives

$$(f \sharp_h g)(X) \;\sim \sum_{k \geq 0} a_k(f, g)(X) h^k\ . \tag{6.2}$$

Definition 6.1. *For each integers j and $k \geq 1$, and for each functions p and $\Phi_0, \ldots, \Phi_{k-1}$ in $C^\infty(\mathbf{R}^{2n})$, we denote by $c_j^{(k)}(p, \Phi_0, \ldots; \Phi_{k-1})$ the coefficient of h^j in the polynomial $e^{\Phi} a_k(p, e^{-\Phi})$, where $\Phi = \Phi_0 + \ldots + h^{k-1}\Phi_{k-1}$. If $p = (p_n)$ is in $S(b)$ $(b > 0)$, we can define a sequence of functions $E_n^{(j)}(x, \xi, t)$ by $E_n^{(0)}(x, \xi, t) = t p_n(x, \xi)$ and, if $k \geq 1$, by*

$$\frac{\partial E_n^{(k)}}{\partial t} \;=\; \sum_{j=0}^{k-1} c_j^{(k-j)}(p, E_n^{(0)}, \ldots, E_n^{(k-1)}) \qquad E_n^{(k)}(x, \xi, 0) \;=\; 0\ . \tag{6.3}$$

For each $m \geq 1$, let us set

$$\Phi_n^{(m)}(x, \xi, t, h) = \sum_{j=0}^{m-1} E_n^{(j)}(x, \xi, t) h^j\ . \tag{6.4}$$

Proposition 6.2. *With the previous notations, for each $a \in]0, b[$, the sequence of functions $E_n^{(j)}(x, \xi, t)$ is in $S(a)$ and there exists a function $R_n^{(m)}(x, \xi, t, h)$ in $S(a)$ such that, we have*

$$\frac{\partial}{\partial t} e^{-\Phi_n^{(m)}(., h, t)} + \sum_{k=0}^{m-1} a_k\left(p_n, e^{-\Phi_n^{(m)}(., h, t)}\right) h^k \;=\; h^m e^{-\Phi_n^{(m)}(., h, t)} R_n^{(m)}(., h, t)\ . \tag{6.5}$$

Moreover, $E_n^{(0)}(x, \xi, t) = t p_n(x, \xi)$ and, for $j \geq 1$ and $m \geq 1$, and we have

$$\|E_n^{(j)}(., t)\|_a \leq n C_j \quad \text{and} \quad \|R_n^{(m)}(., t, h)\|_a \leq n C_m, \tag{6.6}$$

where C_j is independent on n, and the function defined by (6.4) satisfies

$$||\nabla \Phi_n^{(m)}(., t, h)||_R \leq M t\ , \quad \forall t \in [0, 1]\ , \quad \forall h \in]0, 1]\ . \tag{6.7}$$

Proof. Here, and many times in the sequel, we omit the subscript n. By the Lemma 9.1 of [1], if $0 < a < b$ and if p, $\Phi_0, \dots \Phi_{k-1}$ are in $S(b)$, if $\Phi = \Phi_0 + \dots + h^{k-1}h^{k-1}$, then $e^{\Phi} a_k(p, e^{-\Phi})$ is a polynomial in h, of degree depending only on k, with coefficients in $S(a)$. Moreover, there exists $C_k > 0$, depending only on k, and on the norms of $\|\nabla p\|_b$ and $\|\nabla \Phi_j\|_b$ $(0 \leq j \leq k-1)$ such that the coefficients $c_j^{(k)}$ of this polynomial satisfy

$$(6.8)\quad \|\nabla c_j^{(k)}(p, \Phi_0, \dots ; \Phi_{k-1})\|_a \leq C_m \quad \textit{and, if} \quad k \geq 1, \quad \|c_j^{(k)}(p, \Phi_0, \dots ; \Phi_{k-1})\|_a \leq nC_m \ .$$

Therefore, the left hand side of (6.5), multiplied by $e^{\Phi^{(m)}(.,h,t)}$, is a polynomial in h, of degree depending only on m, with coefficients satisfying inequalities like (6.8). The induction relation defining the $E^{(j)}$ shows that the coefficient of h^j in this polynomial vanishes if $j \leq m-1$. Therefore, we can write (6.5), (6.6) and (6.7).

We can write the first coefficients of the formal series : $E^{(0)}(x, \xi, t) = tp(x, \xi)$, $E^{(1)} = 0$ and

$$(6.9)\qquad E^{(2)} = \frac{t^2}{8} \sum_{jk} \left[\frac{\partial^2 p}{\partial x_j \partial x_k} \frac{\partial^2 p}{\partial \xi_j \partial \xi_k} - \frac{\partial^2 p}{\partial x_j \partial \xi_k} \frac{\partial^2 p}{\partial \xi_j \partial x_k} \right] + \dots$$

$$\dots + \frac{t^3}{24} \sum_{jk} \left[\frac{\partial^2 p}{\partial x_j \partial x_k} \frac{\partial p}{\partial \xi_j} \frac{\partial p}{\partial \xi_k} + \frac{\partial^2 p}{\partial \xi_j \partial \xi_k} \frac{\partial p}{\partial x_j} \frac{\partial p}{\partial x_k} - 2 \frac{\partial^2 p}{\partial x_j \partial \xi_k} \frac{\partial p}{\partial \xi_j} \frac{\partial p}{\partial x_k} \right] .$$

Now we have a formal approximation. Since we hope that $e^{-\Phi_n^{(m)}(.,h,t)}$ will be an approximation of the symbol of $e^{-tOp_h(p_n)}$, it will be useful to give a majorization of the function $r_n^{(m)}(., t, h)$ defined by

$$(6.10)\qquad \frac{\partial}{\partial t} e^{-\Phi_n^{(m)}(.,t,h)} = -p_n \sharp_h e^{-\Phi_n^{(m)}(.,t,h)} + e^{-\Phi_n^{(m)}(.,t,h)} r_n^{(m)}(., t, h) \ .$$

The following proposition of Sjöstrand will be used in the proof of the bounds of $r_n^{(m)}$ in (6.10), and several times later. Let us recall it now.

Proposition 6.3. (Proposition 1.1 of [15]). *Let a and b such that $0 < a < b$. Then, there exists $C > 0$ (independent on n) with the following property. Let $f(y, \zeta)$ be an holomorphic function in $\Omega_{2n}(b)$ such that $|\nabla f|_\infty$ is bounded in $\Omega_{2n}(b)$. Then we have*

$$(6.11)\qquad \|\sum_{j=1}^n \frac{\partial^2 f}{\partial y_j \partial \zeta_j}\|_a \leq Cn\|\nabla f\|_b \qquad \|\nabla \left(\sum_{j=1}^n \frac{\partial^2 f}{\partial y_j \partial \zeta_j} \right) \|_a \leq C\|\nabla f\|_b \ .$$

Moreover, if $u = (u_1, \dots u_n) \in \mathbf{C}^n$, and $\rho > 0$ such that $a + \rho |u|_\infty \leq b$, we have

$$(6.12)\qquad \|\sum_{j=1}^n u_j \frac{\partial f}{\partial y_j}\|_a \leq \frac{1}{\rho} \|f\|_{a+\rho|u|_\infty} \ .$$

For the majorization of r_n defined in (6.10), we shall need the following Proposition, which gives a bound of the error term in the composition of symbols, in our unusual situation.

Proposition 6.4. *If $0 < a < b$, if p and Φ are in $S(b)$, if a_k is defined in (6.1), if $m \geq 1$, there exists a family of functions $T^{(m)}(., p, \Phi, h)$ bounded in $\Omega_{2n}(a)$, and a constant ε_0, independent of n, such that*

$$p_n \sharp e^{-\Phi_n} = \sum_{k=0}^{m-1} a_k(p_n, e^{-\Phi_n}) h^k + e^{-\Phi_n} h^m T_n^{(m)}(., p_n, \Phi_n, h) \tag{6.13}$$

if $0 < h \leq \varepsilon_0/(1 + Log\ n)$. Moreover, there exists $C_m > 0$, depending only on a, b, m, and on $\|\nabla\Phi\|_b$ and $\|\nabla p\|_b$, but not on n and h such that, with the previous relations

$$\|T_n^{(m)}(., p_n, \Phi_n, h)\|_a \leq nC_m . \tag{6.14}$$

Proof. For each function $\Phi \in C^\infty(\mathbf{R}^{2n})$ let us set, for each $(x, \xi) \in \mathbf{R}^{2n}$

$$H_\Phi(x, \xi) = \left(-\frac{\partial\Phi}{\partial\xi}, \frac{\partial\Phi}{\partial x} \right) ,$$

and, for each X and Y in $\mathbf{R}^{2n}$

$$Q_\Phi(X) = \int_0^1 H_\Phi(X + tY)\ dt .$$

Thus we have

$$\Phi(X + Y) - \Phi(X) = \sigma(Q_\Phi(X, Y), Y) .$$

With these notations, we can write, for each $\Phi \in S(b)$

$$e^\Phi \Big[p \sharp_h e^{-\Phi} \Big](X) = (\pi h)^{-2n} \int_{\mathbf{R}^{4n}} e^{-\frac{2i}{h}\sigma(Y - \frac{ih}{2}Q_\Phi(X,Z), Z)} p(X + Y)\ dYdZ \tag{6.15}$$

$$\ldots = (\pi h)^{-2n} \int_{\mathbf{R}^{4n}} e^{-\frac{2i}{h}\sigma(Y,Z)} F_h(X, Y, Z)\ dYdZ ,$$

where

$$F_h(X, Y, Z) = p(X + Y + \frac{ih}{2} Q_\Phi(X, Z)) . \tag{6.16}$$

Of course, this is not a change of variables, but a change of contour. Let a , a_1 and a_2 such that $a < a_1 < a_2 < b$. We remark that there exists a constant $C > 0$, independent on n, such that, if X and $X + Z$ are in $\Omega_{2n}(a_2)$, then $|Q_\Phi(X, Z)|_\infty \leq C$. Therefore, the change of contour is allowed and, by Proposition 6.3, there is some other constant $C > 0$, independent on h, such that, for h small enough,

$$|\nabla F_h(X, Y, Z)|_\infty \leq C \tag{6.17}$$

if X, $X+Y$ and $X+Z$ are in $\Omega_{2n}(a_2)$. Let L be the following operator in $\mathbf{R}^{4n}$

$$L \;=\; \frac{1}{2i}\,\sigma(\partial_Y,\ \partial_Z).$$

For the integral defined in (6.15), we have, by Taylor formula

$$e^{\Phi_n}\Big[p_n\ \sharp_h e^{-\Phi_n}\Big](X) \;=\; \sum_{k=0}^{m-1}\frac{1}{k!}h^k(L^kF_h)(X,0,0)\ + h^m A_n^{(m)}(X,h)\ , \tag{6.18}$$

where

$$A_n^{(m)}(X,h) = \int_0^1 \frac{(1-\theta)^{m-1}}{(m-1)!}(\pi h)^{-2n}\int_{\mathbf{R}^{4n}} e^{-\frac{2i}{h}\sigma(Y,Z)}(L^mF_h)(X,\sqrt{\theta}Y,\sqrt{\theta}Z)dY\,dZ\,d\theta.$$

The coefficients in the asymptotic expansion (6.18) depend on h, but we can write, for each $k\le m-1$

$$(L^kF_h)(X,0,0) \;=\; \sum_{j=0}^{m-k-1} h^j g_{(j,k)}(X) \;+\; h^{m-k}R^{(m,k)}(X,h)\ , \tag{6.19}$$

where, by Proposition 6.3, $g_{(j,k)}$ are in $S(a)$ and $R^{(m,k)}$ satisfy

$$\|R_n^{(m,k)}\|_a \le C_m\ . \tag{6.20}$$

Thus, we obtain an asymptotic expansion of the left hand side of (6.18). We have also the expansion (6.2), which is very classical, but, in our situation, does not give a good estimation of the error term. By an identification of these two expansions, we obtain

$$\sum_{k=0}^{m-1}\frac{1}{k!}h^k(L^kF_h)(X,0,0)\ -\ e^{\Phi}\sum_{k=0}^{m-1}a_k\left(p,e^{-\Phi(.)}\right)h^k \;=\; h^m\sum_{k=0}^{m-1}\frac{1}{k!}R_n^{(m,k)}(X,h)\ . \tag{6.21}$$

It remains to give a bound of $A_n^{(m)}(X,h)$. By (6.17) and Proposition 6.3, there exists $C>0$, independent on n, such that the function F_h defined in (6.16) satisfies

$$|\sigma(D_Y,\ D_Z)^mF_h(X,Y,Z)|\le Cn \qquad |\nabla\Big(\sigma(D_Y,\ D_Z)^mF_h(X,Y,Z)\Big)|_\infty\ \le\ C$$

if X, $X+Y$ and $X+Z$ are in $\Omega_{2n}(a_1)$. Therefore, it follows from Theorem 3.1 that

$$\|A_n^{(m)}\|_a\ \le\ Cn\left(1+\frac{\sqrt{h}}{a_1-a}e^{-\frac{(a_1-a)^2}{h}}\right)^{4n}\ \le\ Cne^4 \tag{6.22}$$

if $0<h\le (a_1-a)^2/(1+Log\,n)$. The equality (6.13) is proved, with $T_n^{(m)}=A_n^{(m)}+\sum_{k=0}^{m-1}\frac{1}{k!}R_n^{(m,k)}$, and the estimation (6.14) follows from (6.22) and (6.20). The Proposition is proved.

If we apply this to the function $\Phi^{(m)}$ defined in (6.4), we obtain from (6.14) the majorization of the error term $r_n^{(m)}$ in (6.10), which was our aim.

Corollary 6.5. *With the notations of definition 6.1, if* $0 < R < b$, *the family of functions* $r_n^{(m)}(.,h,t)$ *defined in (6.10) is in* $S(R)$ *if* $0 < h \le \varepsilon_0(1 + Log\ n)^{-1}$ *(where* ε_0 *is independent of* n*), and* $t \in [0,1]$ *and satisfies*

$$\|r_n^{(m)}(.,h,t)\|_R \ \le\ A_m n h^m\ , \tag{6.23}$$

where A_m *is bounded independently on* n *and* h *satisfying the previous relation.*

7. Semi-group properties of the formal construction.

Our first proposition will show that the symbol $e^{-\Phi^{(m)}(t,h)}$, which is expected to be an approximation of the symbol of the exponential of an operator, has indeed approximately the properties of a semigroup.

Proposition 7.1. *Let* $m \ge 1$ *be an integer,* $R > 0$ *such that* $\Phi^{(m)}(t,h)$ *defined in (6.4) is in* $S(R)$*, and* $R' \in]0,R[$*. Then there exist* ε_m *and* C_m*, depending only on* R*,* R'*,* m *and* $\sup_n \|\nabla\Phi_n^{(m)}\|_R$*, but not of* n*, such that, if* $t_0 > 0$ *and* $t_1 > 0$ *and* $t_0 + t_1 \le 1$,

$$\left\| \frac{e^{-\Phi^{(m)}(t_1,h)} \sharp_h e^{-\Phi^{(m)}(t_0,h)}}{e^{-\Phi^{(m)}(t_1+t_0,h)}} \right\|_{R'} \le 1 + C_m n h^m\ , \tag{7.1}$$

if $nh^m \le \varepsilon_m$.

Proof. We know by Theorem 4.1 that, if $t_0 > 0$ and $t > 0$, there exists a function $\Psi^{(m)}(t,t_0,h) \in S(R')$ and constants $\varepsilon_m > 0$ and $C_m > 0$ such that:

$$e^{-\Psi^{(m)}(t,t_0,h)} = e^{-\Phi^{(m)}(t,h)} \sharp_h e^{-\Phi^{(m)}(t_0,h)}\ , \tag{7.2}$$

if $nh^m \le \varepsilon_m$, and that $\Psi^{(m)}(t,t_0,h)$ has an asymptotic expansion in powers of h:

$$\Psi^{(m)}(.,t,t_0,h) = \sum_{j=0}^{m-1} \psi^{(j)}(.,t,t_0) h^j \ +\ g^{(m)}(.,t,t_0,h)\ , \tag{7.3}$$

where $g^{(m)}(.,t,t_0,h)$ satisfies, if $nh^m \le \varepsilon_m$, and $0 < t_0 < t_0 + t \le 1$

$$||g^{(m)}(.,t_1,t_2,h)||_{R'} \le C_m n h^m\ . \tag{7.4}$$

Let us prove that the coefficients $\psi^{(j)}(.,t,t_0)$ are the same as those of $\Phi^{(m)}(t_0+t,h)$, and, for that, we shall show that they are solutions of the same Cauchy problem. If we differentiate (7.2) with respect to t, using (6.10) and (6.23), we obtain punctually (without taking care of the global estimation of the error term):

$$\frac{\partial}{\partial t} e^{-\Psi^{(m)}(t,t_0,h)} = -p \sharp_h e^{-\Psi^{(m)}(t,t_0,h)} \ +\ \mathcal{O}(h^m)\ .$$

By Proposition 6.4, this is equivalent to

$$\frac{\partial}{\partial t} e^{-\Psi^{(m)}(t,t_0,h)} \ =\ -\ \sum_{k=0}^{m-1} a_k(p, e^{-\Psi^{(m)}(t,t_0,h)}) h^k \ +\ \mathcal{O}(h^m)\ .$$

By Definition 6.1, this is equivalent, for each k, to

$$\frac{\partial \psi_k(.,t,t_0)}{\partial t} \;=\; \sum_{j=0}^{k-1} c_j^{(k-j)}(p,\psi_0(t,t_0),\ldots,\psi_{k-1}(t,t_0)) \;,$$

which is the system (6.3). Since $\psi^{(j)}(0,t_0) = E^{(j)}(t_0)$, the uniqueness of the solution of this Cauchy problem shows that

$$\psi^{(j)}(.,t,t_0) = E^{(j)}(.,t+t_0) \;, \quad \forall j \in \{1,\ldots,m-1\} \;, \quad \forall t \in \mathbf{R}^+ \;. \tag{7.5}$$

Therefore, using (7.3), (7.5) and the definition (6.4) of $\Phi^{(m)}(t_0,h)$, we see that

$$\Psi^{(m)}(.,t,t_0,h) \;=\; \Phi^{(m)}(.,t+t_0,h) \;+\; g^{(m)}(t,t_0,h) \;,$$

where $g(m)(t,t_0,h)$satisfies (7.4). The Proposition follows easily.

Now, we shall compose symbols of the form $e^{-\Phi^{(m)}(t_j,h)}f_j$, where f_j are bounded holomorphic functions. We begin by the composition of only two such factors. This study will be useful to solve an integral equation in Section 8.

Proposition 7.2. *Let $m \geq 2$ and $R > 0$ such that the function $\Phi^{(m)}(t,h)$ of Proposition 6.1 is in $S(R)$, and $M > 0$ a constant such that (6.7) is satisfied. Let α such that $0 < \alpha < R$.Then there exist $\varepsilon_m > 0$ et $K_m > 0$, both depending only on m, R, α and M, but not on n, with the following properties. If b_1, b_2 satisfy $0 < \alpha < \min(b_1,b_2) \leq \max(b_1,b_2) < R$, if f_1 and f_2 are holomorphic functions, bounded respectively on $\Omega_{2n}(b_1)$ and $\Omega_{2n}(b_2)$ and if $t_1 > 0$ and $t_2 > 0$ satisfy $t_1+t_2 \leq 1$, if we set $b = \min(b_1,b_2)$, then we have*

$$\left\| \frac{(e^{-\Phi^{(m)}(t_1,h)}f_1)\sharp_h(e^{-\Phi^{(m)}(t_2,h)}f_2)}{e^{-\Phi^{(m)}(t_1+t_2,h)}} \right\|_\alpha \leq 2||f_1||_{b_1}||f_2||_{b_2}\left(1+\frac{K_m h^{m+1}}{(b-\alpha)^{2m+2}}\right)^n \;, \tag{7.6}$$

if $nh^m \leq \varepsilon_m$ and h satisfies the following conditions:

$$\begin{cases} h||\nabla\Phi^{(m)}(t_2,h)||_R \leq \frac{b_1-a}{4} \\ h||\nabla\Phi^{(m)}(t_1,h)||_R \leq \frac{b_2-a}{4} \end{cases} . \tag{7.7}$$

We may assume that ε_m (resp. K_m) is a decreasing (resp. increasing) function of α.

Proof. By Theorem 5.1, with m replaced by $m+1$, there exist $\varepsilon'_m > 0$ and K'_m depending on m, R, α and M such that:

$$\left\| \frac{(e^{-\Phi^{(m)}(t_1,h)}f_1)\sharp_h(e^{-\Phi^{(m)}(t_2,h)}f_2)}{e^{-\Phi^{(m)}(t_1,h)}\sharp_h e^{-\Phi^{(m)}(t_2,h)}} \right\|_\alpha \leq ||f_1||_{b_1}||f_2||_{b_2}\left(1+\frac{K'_m h^{m+1}}{(b-\alpha)^{2m+2}}\right)^n \;, \tag{7.8}$$

if $nh^{m+1} \leq \varepsilon'_m$ and if the conditions of (7.7) are satisfied. By Proposition 7.1, there exist $\varepsilon''_m > 0$ and $K''_m > 0$, such that

$$\left\| \frac{e^{-\Phi^{(m)}(t_1,h)}\sharp_h e^{-\Phi^{(m)}(t_2,h)}}{e^{-\Phi^{(m)}(t_1+t_2,h)}} \right\|_\alpha \leq 1 + K''_m nh^m \tag{7.9}$$

if $nh^m \le \varepsilon''_m$. If nh^m is small enough, (7.6) follows from (7.8) and (7.9).

Now, we shall compose several factors instead of two like in the previous proposition, but the number of factors will be of the same order than the dimension.

Proposition 7.3. *Let $m \ge 2$ and $R > 0$ such that the function $\Phi^{(m)}(.,t,h)$ defined in (6.4) is in $S(R)$, and $M > 0$ a constant such that (6.7) is valid. Let $A \ge 1$ be an integer, and a, b such that $0 < a < b < R$. Then there exists $\varepsilon_m > 0$, depending only on m, M, A, a, b and R, with the following properties. If $f_1, \dots, f_k$ ($1 \le k \le An$), are bounded holomorphic functions in $\Omega_{2n}(b)$, if $\theta_i \ge 0$ ($1 \le i \le k$), if $h \in]0,1]$, and if the following condition is satisfied :*

$$\sup_{\alpha \le k} ||f_\alpha||_b \le \frac{1}{4e^2} \qquad nh^m \le \varepsilon_m \qquad \theta_1 + \ldots + \theta_k \le 1, \tag{7.10}$$

then we have

$$\left\| \frac{(e^{-\Phi^{(m)}(\theta_1,h)} f_1)\sharp_h \ldots \sharp_h (e^{-\Phi^{(m)}(\theta_k,h)} f_k)}{e^{-\Phi^{(m)}(\theta_1+\ldots+\theta_k,h)}} \right\|_a \le \left[\sup_{\alpha \le k} ||f_\alpha||_b \right]^{\frac{k+1}{2}}. \tag{7.11}$$

Proof. For $k \in \{1, \dots, An\}$, we set:

$$a_k = b - (b-a)\frac{\ln k}{\ln(An)} .$$

Therefore $a = a_{An} < \ldots < a_1 = b$. We shall prove (7.11) by induction on the number k of factors. More precisely, we shall prove, by this induction, the stronger inequality, if the constant ε_m in (7.10) is well chosen, for every functions $f_1, \dots, f_k$ holomorphic in $\Omega_{2n}(b)$ and satisfying (7.10), we have

$$\left\| \frac{(e^{-\Phi^{(m)}(\theta_1,h)} f_1)\sharp_h \ldots \sharp_h (e^{-\Phi^{(m)}(\theta_k,h)} f_k)}{e^{-\Phi^{(m)}(\theta_1+\ldots+\theta_k,h)}} \right\|_{a_k} \le \left[\sup_{\alpha \le k} ||f_\alpha||_b \right]^{\frac{k+1}{2}}. \tag{X_k}$$

For $k = 1$, (X_1) is trivial. Let us assume that $2 \le k \le An$ and that (X_i) is proved for all $i < k$. Let $f_1, \dots, f_k$ and $\theta_1, \dots, \theta_k$ satisfying (7.10) (where ε_m will be determined later). Then we can find an integer i such that $\frac{k}{2} \le i \le \frac{2}{3}k$. Let F and G be the following functions:

$$F = \frac{(e^{-\Phi^{(m)}(\theta_1,h)} f_1)\sharp_h \ldots \sharp_h (e^{-\Phi^{(m)}(\theta_i,h)} f_i)}{e^{-\Phi^{(m)}(\theta_1+\ldots+\theta_i,h)}} ,$$

$$G = \frac{(e^{-\Phi^{(m)}(\theta_{i+1},h)} f_{i+1})\sharp_h \ldots \sharp_h (e^{-\Phi^{(m)}(\theta_k,h)} f_k)}{e^{-\Phi^{(m)}(\theta_{i+1}+\ldots+\theta_k,h)}} .$$

Thus the left-hand side of (X_k) can be written $\|H\|_{a_k}$, where

$$H = \frac{(e^{-\Phi^{(m)}(\alpha_i,h)} F)\sharp_h (e^{-\Phi^{(m)}(\alpha_{k-i},h)} G)}{e^{-\Phi^{(m)}(\alpha_i+\alpha_{k-i},h)}} ,$$

with $\alpha_i = \theta_1 + \ldots + \theta_i$ and $\alpha_{k-i} = \theta_{i+1} + \ldots + \theta_k$.

With our choice of i, we have $i \geq k-i$, and therefore $a_k < a_i \leq a_{k-i}$. By using Proposition 7.2 with $\alpha = a_k$, $b_1 = a_i$, $b_2 = a_{k-i}$, we can find ε'_m and K'_m depending only on M, a, b, R and m, such that:

$$\|H\|_{a_k} \leq 2||F||_{a_i}\ ||G||_{a_{k-i}} \left(1 + \frac{K'_m h^{m+1}}{(a_i - a_k)^{2m+2}}\right)^n \ , \tag{7.12}$$

if:

$$hM \leq \frac{a_{k-i} - a_k}{4} \qquad hM \leq \frac{a_i - a_k}{4} \qquad nh^m \leq \varepsilon'_m \ . \tag{7.13}$$

Since we have:

$$a_{k-i} - a_k \ \geq \ a_i - a_k = \frac{b-a}{\ln(An)} \ln(\frac{k}{i}) \geq \frac{b-a}{\ln(An)} \ln(\frac{3}{2}) \ , \tag{7.14}$$

the two left conditions in (7.13) are satisfied if $Ane^{-C/h} \leq 1$, where $C = \frac{b-a}{4M} ln\left(\frac{3}{2}\right)$ and M is the constant in (6.7). Therefore there exists $\varepsilon''_m > 0$, depending only on m, M, a, b, R and A such that, if $nh^m \leq \varepsilon''_m$, the conditions (7.13) are satisfied and therefore (7.12) is valid. By (7.14), there exists $\varepsilon'''_m > 0$, depending only on the same parameters such that, if $nh^m \leq \varepsilon'''_m$, we have

$$\left(1 + \frac{K_m h^{m+1}}{(a_i - a_k)^{2m+2}}\right)^n \leq \left(1 + \frac{1}{n}\right)^n \leq e \ . \tag{7.15}$$

With our induction hypothesis, we have:

$$||F||_{a_i} \leq \lambda^{\frac{i+1}{2}} \qquad ||G||_{a_{k-i}} \leq \lambda^{\frac{k-i+1}{2}} \qquad if \qquad \lambda = \sup_{\alpha \leq k} ||f_\alpha||_b \leq \frac{1}{4e^2} \qquad nh^m \leq \varepsilon_m. \tag{7.16}$$

If we choose now $\varepsilon_m = \min(\varepsilon''_m, \varepsilon'''_m)$, the inequalities (7.12), (7.15) and (7.16) imply:

$$||H||_{a_k} \leq 2\lambda^{\frac{i+1}{2}} \lambda^{\frac{k-i+1}{2}} e \leq \lambda^{\frac{k+1}{2}} \qquad \text{if} \quad \lambda \leq \frac{1}{4e^2} \qquad \text{and} \qquad nh^m \leq \varepsilon_m \ .$$

Therefore, with this choice of ε_m, the property (X_k) is proved for all $k \leq An$, and the proposition itself follows since $a_k \geq a$.

8. End of the proof of theorem 2.2.

Theorem 2.2 will follow from the next Proposition (where $f\sharp_h g$ is the composition law defined in (3.1)).

Proposition 8.1. *Let $p = (p_n)_{n\geq 1}$ an element of $S(b)$ ($b > 0$), $a \in]0, b[$, and m an integer ≥ 2. Then there exists $\varepsilon_m > 0$ and $C_m > 0$, depending only on the family p and on a, b and m, and a sequence of holomorphic bounded functions $u = (u_n)(x, \xi, t, h)$ on $\Omega_{2n}(a)$ depending in a C^∞ way on $t \in [0, 1]$ such that, if $nh^m \leq \varepsilon_m$:*

$$\begin{cases} \dfrac{\partial u}{\partial t}(., t, h) = -p\,\sharp_h u(., t, h) \ , \\ \qquad u(., 0, h) = 1 \ . \end{cases} \tag{8.1}$$

Moreover u can be written:

$$u(.,t,h) = e^{-\Phi^{(m)}(.,t,h)+R^{(m)}(.,t,h)} , \quad if \quad nh^m \le \varepsilon_m , \tag{8.2}$$

where $\Phi^{(m)}(t,h) \in S(a)$ is the family of polynomials in h defined in (6.4), and $R^{(m)}(.,t,h)$ a family of holomorphic bounded functions on $\Omega_{2n}(a)$ such that:

$$||R^{(m)}(.,t,h)||_a \le C_m nh^m , \quad \forall t \in [0,1] \quad \text{if} \quad nh^m \le \varepsilon_m . \tag{8.3}$$

Proof. If $\Phi^{(m)}(.,t,h)$ is the function (polynomial in h) defined in (6.4), we shall find a solution of problem (8.1) of the following form:

$$u(.,t,h) = e^{-\Phi^{(m)}(t,h)} + \int_0^t e^{-\Phi^{(m)}(t-s,h)} \sharp_h [e^{-\Phi^{(m)}(s,h)} v_h(.,s)] ds , \tag{8.4}$$

where $v_h(.,t)$ will be a function in a suitable space. If we replace u by the previous expression in (8.1), using (6.10), we see that u is a solution if and only if v_h satisfies:

$$v_h - T_h(r_h)v_h = r_h , \tag{8.5}$$

where, for each functions $u(.,t)$ and $v(.,t)$ in suitable spaces, we set :

$$[T_h(u)v)] (.,t) = e^{\Phi^{(m)}(t,h)} \int_0^t [e^{-\Phi^{(m)}(t-s,h)} u(t-s)] \sharp_h [e^{-\Phi^{(m)}(s,h)} v(s)] ds , \tag{8.6}$$

and where $r_h(.,t)$ is defined in (6.10). It remains to find a space E in which some power of $T_h(r_h)$ will be a contractive map. For each a_1 and a_2 such that $0 < a_1 < a_2$, let $E(a_1, a_2)$ be the space of functions $v(.,t)$ ($t \in [0,1]$) such that, for each $t \in [0,1]$, $v(.,t)$ is a bounded holomorphic function in $\Omega_{2n}(A(t,a_1,a_2))$, where

$$A(t,a_1,a_2) = a_2 - t(a_2 - a_1), \tag{8.7}$$

and such that $v(.,t)$ depends continuously on t. The space $E(a_1,a_2)$ is endowed with the norm:

$$||v||_{E(a_1,a_2)} = \sup_{0 \le t \le 1} ||v(.,t)||_{A(t,a_1,a_2)} .$$

Let $R \in]a,b[$ that the family $\Phi^{(m)}(t,h)$ defined in (6.4) is in in $S(R)$. Even if $0 < a_1 < a_2 < R$, the map $T_h(r_h)$ is not bounded in $E(a_1,a_2)$, but we shall prove that $[T_h(r_h)]^{(2m+2)n+1}$ is contractive in $E(a_1,a_2)$ in this case. For that, we need the following lemma.

Lemma 8.2. *Let $\Phi^{(m)}$ be the function defined in (6.4) and satisfying (6.7) with $M > 0$. Let $u(.,t)$ be a function in $E(R',R')$ (R'), $[a_1,a_2]$ be a compact interval in $]0,R'[$ and $m \ge 2$ be an integer. Then :*

i)There exists $\varepsilon_m > 0$, depending only on m, R, R', a_1, a_2 and M, such that, if

$$nh^m \le \varepsilon_m \qquad \|u\|_{E(R',R')} \le \frac{1}{4e^2}, \tag{8.8}$$

the map $[T_h(u)]^{(2m+2)n+1}$ *is contractive in* $E(a_1, a_2)$*, with a norm* $\leq \frac{1}{2}$*.*
ii) If $0 < a_1 \leq a_2 < a_3 \leq a_4 \leq R'$*, there exists another* $\varepsilon_m > 0$*, depending only on the same parameters and on* a_3 *and* a_4*, such that, if* $u \in E(R', R')$*,* $v \in E(a_3, a_4)$*, if (8.8) is satisfied and if* $1 \leq k \leq (2m+2)n+1$*,*

$$\| \left[T_h(u)\right]^k v\|_{E(a_1,a_2)} \leq 2e \; \|u\|_{E(R',R')}^{(k+1)/2} \; \|v\|_{E(a_3,a_4)} \; . \tag{8.9}$$

Proof of the Lemma. First step (points i and ii). If $v \in E(a_1, a_2)$, we have, for all integer k and $t > 0$

$$\left[T_h(u)^k v\right](.,t) = \int_{\Delta^k(t)} G_k(t_1, \ldots, t_k, t; u, v) dt_1 \ldots dt_k \; , \tag{8.10}$$

where $\Delta^k(t) = \{(t_1, \ldots, t_k),\ 0 \leq t_1 \leq \ldots \leq t_k \leq t\}$ and, for each $(t_1, \ldots, t_k) \in \Delta^k(t)$, and each function $u \in E(R', R')$ and $v \in E(a_1, a_2)$, we set

$$\begin{aligned} G_k(t_1, \ldots, t_k, t; u, v) \;&=\; e^{\Phi^{(m)}(t,h)}[e^{-\Phi^{(m)}(t-t_k,h)} u(t-t_k)]\sharp_h \cdots \\ &\cdots \sharp_h[e^{-\Phi^{(m)}(t_2-t_1,h)} u(t_2-t_1)]\sharp_h[e^{-\Phi^{(m)}(t_1,h)} v(t_1)] \; . \end{aligned} \tag{8.11}$$

Let μ in $]0, R[$. By using Proposition 7.3 with $f_1 = u(t-t_k)$, ..., $f_k = u(t_2-t_1)$, $A = 2m+3$, we know that there exists $\varepsilon'_m > 0$, depending only on R, R', M (the constant such that the function $\Phi_n^{(m)}(t,h)$ defined in (6.4) satisfies (6.7) in $\Omega_{2n}(R)$), m, and μ, such that, if

$$\|u\|_{E(R',R')} \leq (4e^2)^{-1} \qquad nh^m \leq \varepsilon'_m \qquad k \leq (2m+3)n \; , \tag{8.12}$$

there exists a bounded holomorphic function $F_h(t_1, \ldots, t_k, u)$ in $\Omega_{2n}(\mu)$ satisfying

$$[e^{-\Phi^{(m)}(t-t_k,h)} u(t-t_k)]\sharp_h \ldots \sharp_h[e^{-\Phi^{(m)}(t_2-t_1,h)} u(t_2-t_1)] = e^{-\Phi^{(m)}(t-t_1,h)} F_h(t_1, \ldots, t_k, t, u) \; ,$$

$$||F_h(t_1, \ldots, t_k, t, u)||_\mu \leq \|u\|_{E(R',R')}^{\frac{k+1}{2}} \; .$$

If $u \in E(R', R')$ and $v \in E(\lambda, \mu)$ $(0 < \lambda < \mu < R')$, if $0 \leq t_1 ... \leq t_k \leq t \leq 1$, we have

$$G_k(t_1, ..., t_k, t, u, v) \;=\; e^{\Phi^{(m)}(t,h)} \left[\left(e^{-\Phi^{(m)}(t-t_1,h)} F_h(t_1, \ldots, t_k, t, u)\right) \; \sharp_h \; \left(e^{-\Phi^{(m)}(t_1,h)} v(t_1)\right)\right] \; ,$$

where F_h is the function defined above. Let $\alpha \in]0, \lambda[$. By Proposition 7.2, there exist $\varepsilon''_m > 0$ and $K''_m > 0$, depending only on M, α, λ, R and m, such that, if $\alpha \leq \sigma < A(t_1, \lambda, \mu)$,

$$\|G_k(t_1, ..., t_k, t, u, v)\|_\sigma \;\leq\; 2\|F_h(t_1, \ldots, t_k, t, u)\|_\mu \; \|v(t_1)\|_{A(t_1,\lambda,\mu)} \left(1 + \frac{K''_m h^{m+1}}{(A(t_1, \lambda, \mu) - \sigma)^{2m+2}}\right)^n$$

if

$$hM(t-t_1) \leq \frac{A(t_1, \lambda, \mu) - \sigma}{4} \qquad hMt_1 \leq \frac{\mu - \sigma}{4} \qquad nh^m \leq \varepsilon''_m \; , \tag{8.13}$$

and therefore, by the definition of $E(\lambda,\mu)$, if (8.12) and (8.13) are satisfied

$$\|G_k(t_1,...,t_k,t,u,v)\|_\sigma \;\le\; 2\|u\|_{E(R',R')}^{\frac{k+1}{2}}\;\|v\|_{E(\lambda,\mu)}\left(1+\frac{K_m h^{m+1}}{(A(t_1,\lambda,\mu)-\sigma)^{2m+2}}\right)^n \;.$$

Proof of the Lemma. Second step. Point i) If $v\in E(a_1,a_2)$, we apply the first step with $\lambda=a_1$, $\mu=a_2$, $k=(2m+2)n+1$, $\sigma=A(t,a_1,a_2)$. By the definition (8.7) of $A(t,\lambda,\mu)$, the two first conditions of (8.13) are satisfied if $hM\le\frac{a_2-a_1}{4}$. Therefore, we can find ε_m, as in the statement of i) such that, if $nh^m\le\varepsilon_m$

$$\|G_{(2m+2)n+1}(t_1,...,t_{(2m+2)n+1},t,u,v)\|_{A(t,a_1,a_2)} \;\le\; 2\|u\|_{E(R',R')}^{(m+1)n+1}\;\|v\|_{E(a_1,a_2)}\left(\frac{2}{(t-t_1)^{2m+2}}\right)^n \;.$$

We remark that

$$\int_{\{(t_2,\dots,t_{(2m+2)n+1}),\; t_1<t_2<\dots<t_{(2m+2)n+1}<t\}} dt_2\dots dt_{(2m+2)n+1} \;=\; \frac{(t-t_1)^{(2m+2)n}}{((2m+2)n)!}\;.$$

Therefore, by the expression (8.10), (8.11) of $T_h(u)^k$, it follows, for $k=(2m+2)n+1$, that, for some other ε_m and K_m, for each $u\in E(R,R)$ and $v\in E(a_1,a_2)$

$$\|\left[T_h(u)^{(2m+2)n+1}v\right](t)\|_{A(t,a_1,a_2)} \;\le\; t\|u\|_{E(R',R')}^{(m+1)n+1}\;\|v\|_{E(a_1,a_2)}\;\frac{2^{n+1}}{((2m+2)n)!}$$

if $nh^m\le\varepsilon_m$, $0<t<1$ and if the first condition of (8.12) is satisfied. The point i) follows easily.

Proof of the Lemma. Third step. Point ii). Now, we apply the first step, with $\lambda=a_3$, $\mu=a_4$, $\alpha=a_1$, and $\sigma=A(t,a_1,a_2)$. Since $A(t,a_3,a_4)-\sigma\ge a_3-a_2$, the conditions of (8.13) are satisfied if $hM\le\frac{a_3-a_2}{4}$ and $nh^m\le\varepsilon''_m$. If, moreover, the conditions of (8.12) are satisfied, it follows from the first step, with another K_m that

$$\|G_k(t_1,...,t_k,t,u,v)\|_{A(t,a_1,a_2)} \;\le\; 2\|u\|_{E(R',R')}^{\frac{k+1}{2}}\;\|v\|_{E(a_3,a_4)}\left(1+\frac{K_m h^{m+1}}{(a_3-a_2)^{2m+2}}\right)^n \;,$$

and the point ii) follows easily.

End of the proof of Proposition 8.1. Let a and b like in the Proposition 8.1, and choose a compact interval $[a_1,a_2]$ contained in $]a,b[$, and $R\in]a_2,b[$. We apply Lemma 8.2 i) with $u=r_h$ (the function defined in (6.10), which satisfies (6.23) in $\Omega_{2n}(R')$, with $R'<R$, for some $A_m>0$). If $A_m nh^m\le\frac{1}{4e^2}$, the second condition of (8.8) is satisfied. Therefore, if $nh^m\le\varepsilon_m$ (for some $\varepsilon_m>0$ independent of the dimension), the map $[T_h(r_h)]^{(2m+2)n+1}$ is contractive in $E(a_1,a_2)$, with a norm $\le\frac{1}{2}$. Hence, there exists $v_h\in E(a_1,a_2)$ such that

$$\left(I \;-\; T_h(r_h)^{(2m+2)n+1}\right)v_h \;=\; \sum_{k=0}^{(2m+2)n} T_h(r_h)^k r_h\;.$$

By Lemma 8.2 ii), applied with $a_3=a_4=\frac{a_2+R'}{2}$, we have, under the same type of conditions, if $k\le(2m+2)n$

$$\|T_h(r_h)^k r_h\|_{E(a_1,a_2)} \;\le\; 2e\left[\|r_h\|_{E(R',R')}\right]^{\frac{k+1}{2}+1}\;.$$

Therefore the solution v_h satisfies

$$\|v_h\|_{E(a_1,a_2)} \;\le\; 2\sum_{k=0}^{\infty} 2e(A_m n h^m)^{\frac{k+1}{2}+1} \; . \tag{8.14}$$

In order to prove that v_h satisfies (8.5), we set $w_h = (I - T_h(r_h))\, v_h \,-\, r_h$. Let $[a_1', a_2']$ be a compact interval in $]a, a_1[$. By Lemma 8.2 ii), under the same type of conditions, w_h is in $E(a_1', a_2')$ and satisfies $\big(I \,-\, T_h(r_h)^{(2m+2)n+1}\big)\, w_h \,=\, 0$. Since $T_h(r_h)^{(2m+2)n+1}$ is also contractive in $E(a_1', a_2')$ under the same type of conditions, it follows that $w_h = 0$, and therefore that v_h satisfies (8.5), and that u_h defined by (8.4) is a solution of the Cauchy problem (8.1). Equation (8.4) can be written

$$u(.\ ,\ h) \;=\; e^{-\Phi^{(m)}(.,h)}\,[1 + T_h(1)v_h] \;.$$

By (8.14) and Lemma 8.2 ii), we have $\|T_h(1)v_h\|_{E(a_1',a_2')} \le A_m n h^m \le \frac{1}{2}$ if nh^m is small enough, and therefore there exists an holomorphic function $R^{(m)}(., t, h)$ in $E(a_1', a_2')$ such that $1 + T_h(1)v_h = e^{R^{(m)}(.,h)}$ and

$$\|R^{(m)}(., h)\|_{E(a_1',a_2')} \;\le 2\|T_h(1)v_h\|_{E(a_1',a_2')} \;\le\; 2A_m n h^m \;.$$

Proposition is proved, and Theorem 2.2 is another equivalent formulation.

9. Application to thermodynamic limits.

At each point j of the lattice $\mathbf{Z}$, we consider a particle A_j described, when there is no interaction, by an hamiltonian $A(x,\xi) \in C^\infty(\mathbf{R}^{2k})$, where $k \ge 1$ is a fixed integer. The interaction between A_j and A_{j+1} will be described by a function $B(x,\xi,y,\eta) \in C^\infty(\mathbf{R}^{4k})$, ($A$, B and k are independent on j).

We assume that A (resp. B) extends to an holomorphic function in $\Omega_{2k}(a)$, (resp. $\Omega_{4k}(a)$), defined as in (1.4), that B is bounded in $\Omega_{4k}(a)$, and that A has bounded derivatives in $\Omega_{2k}(a)$, and that its real part is lower semi-bounded in this domain. We assume also that A and B are real for real (x,ξ), and that $B(y,\eta,x,\xi) = B(x,\xi,y,\eta)$, and that there exist $c > 0$, $R > 0$ and $\delta > 0$ such that

$$(x,\xi) \in \mathbf{R}^{4k}, \quad |(x,\xi)| \ge R \;\Rightarrow A(x,\xi) \;\ge c|(x,\xi)|^\delta.$$

We shall write now the hamiltonian describing the system of particles A_j ($|j| \le n$) with interaction. If there is no interaction between A_n and A_{-n}, this hamiltonian can be written, setting $X = (x,\xi)$ and denoting by $(X^{(-n)}, \dots, X^{(n)})$ the variable of $\mathbf{R}^{2k(2n+1)}$ (with $X^{(j)} = (x^{(j)}, \xi^{(j)}) \in \mathbf{R}^{2k}$)

$$\widetilde{p}_n(X) \;=\; \sum_{j=-n}^{n} A(X^{(j)}) \;+\; \sum_{j=-n}^{n-1} B(X^{(j)}, X^{(j+1)}) \;. \tag{9.1}$$

If there is an interaction between A_n and A_{-n} (case some formulas will be simpler), the hamiltonian becomes, setting $X^{(n+1)} = X^{(-n)}$,

$$p_n(X) \;=\; \sum_{j=-n}^{n} \Big(A(X^{(j)}) \;+ B(X^{(j)}, X^{(j+1)}) \Big) \;. \tag{9.2}$$

We see easily that the sequences (p_n) and $(\widetilde{p}_n)$ are in $S(a)$.

We denote by $P_n(h)$ and $\widetilde{P}_n(h)$ the $h-$pseudodifferential operator in $L^2(\mathbf{R}^{k(2n+1)})$, associated to the symbols p_n and $\widetilde{p}_n$ by the Weyl calculus (2.5). Given the sequence of operators $P_n(h)$, we define the thermodynamic limit as the limit, if it exists, of the sequence

$$\Lambda_n(t,h) \ := \ \frac{1}{(2n+1)} \ ln \ \left[(2\pi h)^{(2n+1)k} Tr \ \left(e^{-tP_n(h)} \right) \right] \tag{9.3}$$

(the same as in (1.7), but with t/h replaced by t), has a limit $\Lambda_\infty(t,h)$ when $n \to +\infty$. We define also a similar sequence $\widetilde{\Lambda}_n(t,h)$ for the operator $\widetilde{P}_n(h)$.

By theorem 2.2, we know that $e^{-tP_n(h)}$ is an $h-$pseudodifferential operator associated, by the Weyl calculus, to a symbol of the form $e^{-q_n(x,\xi,t,h)}$.

$$e^{-tP_n(h)} \ = \ Op_h \left(e^{-q_n(.,t,h)} \right) \ , \tag{9.4}$$

and we have also a symbol $\widetilde{q}_n(x,\xi,t,h)$ with similar properties for $\widetilde{P}_n(h)$.

The main results of this section are the theorems 9.1 (existence of the thermodynamic limit) and 9.3 (asymptotic expansion in powers of h).

Theorem 9.1. *With the previous notations, for each $h > 0$ and $t > 0$, the sequence $\Lambda_n(t,h)$ defined in (9.3) and its analogue $\widetilde{\Lambda}_n(t,h)$ for the operator $\widetilde{P}_n(h)$ have a common limit $\Lambda(t,h)$ when $n \to \infty$. There is a constant $C > 0$ such that*

$$|\Lambda_n(t,h) \ - \ \Lambda(t,h)| \ + \ |\widetilde{\Lambda}_n(t,h) \ - \ \Lambda(t,h)| \ \leq \ \frac{C}{n} \qquad \forall h \in]0,1], \qquad \forall t \in [0,1] \ . \tag{9.5}$$

The proof relies on the following Lemma.

Lemma 9.2. *There exists $C > 0$ such that, for each m and $n \geq 1$, for each h and t in* $]0,1]$

$$\Delta_{mn}(t,h) \ := \ |(2m+2n+1)\widetilde{\Lambda}_{m+n}(t,h) \ - \ (2m+1)\widetilde{\Lambda}_m(t,h) \ - \ (2n+1)\widetilde{\Lambda}_n(t,h)| \ \leq \ C \ . \tag{9.6}$$

Proof of the Lemma. For $m \geq 1$ and $n \geq 1$, let us denote by $(X,Y) = (X^{(-m-n)}, \dots, X^{(m+n)}, Y)$ the variable of $\mathbf{R}^{2k(2m+2n+2)}$. We define two functions in $\mathbf{R}^{2k(2m+2n+2)}$ by

$$a_{mn}(X,Y) \ = \ \widetilde{p}_{m+n}(X) \ + \ A(Y),$$

$$b_{mn}(X,Y) \ = \ B(X^{(-m-1)}, X^{(-m)}) \ + \ B(X^{(m)}, X^{(m+1)}) - B(X^{(-m-1)}, Y) \ - \ B(Y, X^{(m+1)}).$$

We denote by $A_{mn}(h)$ and $B_{mn}(h)$ the $h-$operators in $L^2(\mathbf{R}^{k(2m+2n+2)})$ associated to the symbols $a_{mn}(x,\xi)$ and $b_{mn}(x,\xi)$, and by $P_0(h)$ the $h-$operator associated to $A(y,\eta)$. We see easily that

$$(2m+2n+1)\widetilde{\Lambda}_{m+n}(t,h) \ = \ Log \ \left[(2\pi h)^{2k(m+n+1)} Tr \big(e^{-tA_{mn}(h)} \big) \right] \ - \ Log \left[(2\pi h)^k Tr \big(e^{-tP_0(h)} \big) \right],$$

$$(2m+1)\widetilde{\Lambda}_m(t,h) + (2n+1)\widetilde{\Lambda}_n(t,h) = \ Log \left[(2\pi h)^{2k(m+n+1)} Tr \big(e^{-tA_{mn}(h)+tB_{mn}(h)} \big) \right].$$

If we set $F(\theta) = Log\Big(Tr\big(e^{-tA_{mn}(h)+\theta tB_{mn}(h)}\big)\Big)$, we can write, for the left hand side $\Delta_{mn}(t,h)$ of (9.6)

$$\Delta_{mn}(t,h) \;\leq\; |F(1)-F(0)| \;+\; \left|Log\;\left[(2\pi h)^k Tr\left(e^{-tP_0(h)}\right]\right)\right|.$$

We obtain

$$|F(1)-F(0)| \leq \sup_{\theta\in[0,1]} |F'(\theta)| \;=\; \sup_{\theta\in[0,1]} \frac{|Tr\big(tB_{mn}(h)e^{-tA_{mn}(h)+t\theta B_{mn}(h)}\big)|}{|Tr\big(e^{-tA_{mn}(h)+t\theta B_{mn}(h)}\big)|}.$$

Since $e^{-tA_{mn}(h)+t\theta B_{mn}(h)}$ is a positive self-adjoint operator, we have

$$|Tr\big(B_{mn}(h)e^{-tA_{mn}(h)+t\theta B_{mn}(h)}\big)| \;\leq\; \|B_{mn}(h)\|\;|Tr\big(e^{-tA_{mn}(h)+t\theta B_{mn}(h)}\big)|.$$

Since the norm of $B_{mn}(h)$ is bounded independently on m, n and $h\in]0,1]$, the Lemma is proved.

Proof of Theorem 9.1. Since the sequence $\widetilde{\Lambda}_n(t,h)$ satisfies (9.6), it follows classically (cf. Helffer-Sjöstrand [6], lemma 2.5) that $\widetilde{\Lambda}_n(t,h)$ has a limit $\Lambda(t,h)$ when $n\to\infty$, and that

$$|\widetilde{\Lambda}_n(t,h) \;-\; \Lambda(t,h)| \;\leq\; \frac{C}{n} \qquad \forall n\geq 1 \quad \forall h\in]0,1].$$

The proof of Lemma 9.2 shows also that, for some constant $C>0$

$$|\Lambda_n(t,h) \;-\; \widetilde{\Lambda}_n(t,h)| \;\leq\; \frac{C}{n} \qquad \forall n\geq 1\ .$$

Therefore (9.5) is proved.

Let us prove now that $\Lambda(t,h)$ has an asymptotic expansion in powers of h. In order to define the first term of this expansion, we need the following integral operator $S_0(t)$ in $L^2(\mathbf{R}^{2k})$ defined by

$$(S_0(t)u)(X) \;=\; \int_{\mathbf{R}^{2k}} e^{-\frac{t}{2}(A(X)+2B(X,Y)+A(Y))}\; u(Y)\;dY \qquad \forall u\in L^2(\mathbf{R}^{2k})\ . \tag{9.7}$$

This operator, which is somewhat similar to the Kac operator studied by Helffer [3] and Helffer-Ramond [4], is self-adjoint, of trace class. The norm $\|S_0(t)\|$ is a simple eigenvalue of $S_0(t)$ (by Krein-Rutman theorem), and all other eigenvalues have a modulus strictly smaller than $\|S_0(t)\|$.

Theorem 9.3. *The thermodynamic limit $\Lambda(t,h)$ of Theorem 9.1 has an asymptotic expansion when $h\to 0$*

$$\Lambda(t,h) \;\sim\; \sum_{j\geq 0}\gamma_j(t)h^j\ , \tag{9.8}$$

where $\gamma_j(t)$ are real numbers, and $\gamma_0(t) = Log(\|S_0(t)\|)$, where $S_0(t)$ defined in (9.7).

If $B=0$, (when there is no interaction), the 'classical' thermodynamic limit $\gamma_0(t)$ is given by

$$\gamma_0(t) \;=\; Log\;\left(\int_{\mathbf{R}^{2k}} e^{-tA(X)}dX\right).$$

We shall not define explicitely the others coefficients $\gamma_j(t)$ but, for each integer m, we shall define an analytic function of h (denoted below by $\frac{1}{d_m} ln\ \lambda_m(t,h)$, and we shall prove later (lemma 9.5) that the difference between this function and $\Lambda(t,h)$ is $\mathcal{O}(h^m)$. The function $\lambda_m(t,h)$ will be an eigenvalue of an operator $S_m(t,h)$, and the construction of this operator will be our first step.

Construction of the operator $S_m(t,h)$.

By theorem 2.2, we know that $q_n(x,\xi,t,h)$, (the symbol satisfying (9.4)), has the asymptotic expansion (2.8) with coefficients $E_n^{(j)}(x,\xi,t)$. By the construction of section 6, we know that $E_n^{(0)} = tp_n$, $E_n^{(1)} = 0$, and that $E_n^{(2)}$ is given like in (6.9)), where p is replaced by p_n defined in (9.2). By (6.9) and (9.2), we see that there is a function $f^{(2)} \in C^\infty(\mathbf{R}^{8k} \times \mathbf{R})$ such that

$$E_n^{(2)}(x,\xi,t) \ = \ \sum_{j=1}^{n} f^{(2)}(X^{(j)}, X^{(j+1)}, X^{(j+2)}, X^{(j+3)}, t).$$

We have set $X^{(n+j)} = X^{(-n+j-1)}$ for $j \geq 1$. As a function of t, $f^{(2)}$ is a polynomial. As a function of X, it is holomorphic with bounded derivatives in $\{X \in \mathbf{C}^{8k}, |Im\ X|_\infty < a\}$. More generally, if we follow the induction in the proof of Proposition 6.1, we see that, for each integer m, there is some integer $d_m \geq 1$, and a function $F^{(m)}(.,t,h) \in C^\infty(\mathbf{R}^{2k(1+d_m)})$ such that the function $\Phi_n^{(m)}$ defined in (6.4) can be written

$$\Phi_n^{(m)}(x,\xi,t,h) \ = \ \sum_{j=-n}^{n} F^{(m)}(X^{(j)}, \dots, X^{(j+d_m)}, t, h) \ . \tag{9.9}$$

For example, we have $d_0 = d_1 = 1$, $d_2 = 3$. The approximation $\Phi_n^{(m)}$ can be written in the form (9.9), but not in the unique way. For $m = 1$, $F^{(1)}$ (which is independent on h) can be written

$$F^{(1)}(X,t) \ = \ t\left(\frac{A(X^{(1)})}{2} \ + \ B(X^{(1)}, X^{(2)}) \ + \ \frac{A(X^{(2)})}{2} \right).$$

For m arbitrary and $h = 0$, we can take, for all X in $\mathbf{R}^{2k(1+d_m)}$,

$$F^{(m)}(X,t,0) \ = \ \frac{t}{d_m+1} \sum_{j=1}^{1+d_m} A(X^{(j)}) \ + \ \frac{t}{d_m} \sum_{j=1}^{d_m} B(X^{(j)}, X^{(j+1)}). \tag{9.10}$$

Now, we introduce the operator $S_m(t,h)$ in $L^2(\mathbf{R}^{2kd_m})$ defined by

$$(S_m(t,h)u)(Y) \ = \ \int_{\mathbf{R}^{2kd_m}} e^{-\Psi_m(X,Y,t,h)} u(X)\ dX \qquad \forall u \in L^2(\mathbf{R}^{2kd_m}), \tag{9.11}$$

where we set here $X = (X^{(1)}, \dots, X^{(d_m)})$, $Y = (Y^{(1)}, \dots, Y^{(d_m)})$, $X^{(j+d_m)} = Y^{(j)}$ and:

$$\Psi_m(X,Y,t,h) \ = \ \sum_{j=1}^{d_m} F^{(m)}(X^{(j)}, \dots, X^{(j+d_m)}, t, h). \tag{9.12}$$

Lemma 9.4. *For each $t > 0$, the operator $S_m(t,h)$ is of trace class, with a trace norm bounded independently on h small enough, and it has, for h small enough, a simple eigenvalue $\lambda_m(t,h)$ such* *that $\lambda_m(t,0) = \|S_0(t)\|^{d_m}$, and all other eigenvalues are strictly smaller in modulus.In other words, there exist ρ_1 and ρ_2 such that $0 < \rho_1 < \rho_2$, $|\lambda_m(t,h)| \geq \rho_2$ and $\sigma(S_m(t,h)) \setminus \{\lambda_m(t,h)\} \subset B(0,\rho_1)$ for h small enough.*

Proof. We shall prove that $S_m(t,0)$ is isospectral to the power $S_0(t)^{d_m}$ of $S_0(t)$ defined in (9.7). We remark that, for each integer n, $S_0(t)$ satisfies

$$Tr(S_0(t)^n) \;=\; \int_{\mathbf{R}^{2kn}} e^{-tG_n(X)} dX,$$

where

$$G_n(X) \;=\; B(X^{(n)}, X^{(1)}) \;+\; \sum_{j=1}^{n} A(X^{(j)}) \;+\; \sum_{j=1}^{n-1} B(X^{(j)}, X^{(j+1)}) \qquad \forall X \in \mathbf{R}^{2nk}.$$

Similarly, $S_m(t,h)$ defined in (9.11) is of trace class and, for each integer q, we have

$$Tr(S_m(t,h)^q) \;=\; \int_{\mathbf{R}^{2kqd_m}} e^{-\theta_q(X,t,h)} dX \;, \tag{9.13}$$

where, setting now $X^{(qd_m+j)} = X^{(j)}$ for $j \geq 1$

$$\theta_q(X,t,h) \;=\; \sum_{j=1}^{qd_m} F^{(m)}(X^{(j)}, \ldots, X^{(j+d_m)}, t, h) \;. \tag{9.14}$$

It follows from (9.10) that $\theta_q(X,t,0) = tG_{qd_m}(X)$ for all X in $\mathbf{R}^{2kqd_m}$, and therefore $S_m(t,0)$ is related to the operator $S_0(t)$ of (9.7), for each integer q, by

$$Tr(S_m(t,0)^q) \;=\; Tr(S_0(t)^{qd_m}).$$

Therefore, $S_m(t,0)$ is isospectral to $S_0(t)^{d_m}$, hence has $\|S_0(t)\|^{d_m}$ as a simple eigenvalue, and all others eigenvalues have a modulus strictly smaller. Since $S_m(t,h)$ depends analytically on h, by the classical theory of perturbation of a single eigenvalue, $S_m(t,h)$ has an eigenvalue $\lambda_m(t,h)$ which depends analytically on h, such that $\lambda_m(t,0) = \|S_0(t)\|^{d_m}$.

For h small enough, *Log* $\lambda_m(t,h)$ is well defined. Since m is arbitrary, the theorem 9.3 will follow from

Lemma 9.5. *With the previous notations, we have, for each integer m, and for h small enough*

$$\Lambda(t,h) \;=\; \frac{1}{d_m} Log\ (\lambda_m(t,h)) \;+\; \mathcal{O}(h^m) \;. \tag{9.15}$$

Proof. First step. It is natural to approximate $\Lambda_n(t,h)$ (defined in (9.3)), by

$$\Lambda_n^{(m)}(t,h) \;=\; \frac{1}{2n+1}\ Log\ \left[\int_{\mathbf{R}^{2k(2n+1)}} e^{-\Phi_n^{(m)}(x,\xi,t,h)}\ dx d\xi \right], \tag{9.16}$$

where $\Phi_n^{(m)}$ is the approximation of (6.4). We shall prove that

$$\left|\Lambda_n^{(m)}(t,h)\ -\frac{1}{d_m}Log\ (\lambda_m(t,h))\right|\ \leq\ \frac{C}{k}\qquad if\quad 2n+1=kd_m \tag{9.17}$$

for some constant C. When $2n+1=kd_m$, we have

$$\Phi_n^{(m)}(X^{(-n)},\dots X^{(n)},t,h)=\theta_k(X^{(-n)},\dots X^{(n)},t,h),$$

where θ_k is the function defined in (9.14), and therefore

$$\Lambda_n^{(m)}(t,h)\ =\ \frac{1}{kd_m}\ Log\ \left[\ Tr\ (S_m(h)^k)\right]\qquad if\qquad 2n+1=kd_m.$$

By Lemma 9.4, we have, for some $C>0$

$$\left|\frac{1}{k}Log\left(Tr\ (S_m(h)^k)\right)\ -\ Log\ (\lambda_m(t,h))\right|\ \leq\ \frac{C}{k}$$

and (9.17) follows.

Second step. Theorem 2.2 shows that there is $K_m>0$ and $\varepsilon_m>0$, independent on n, such that, for each h small enough, n and t satisfying $0\leq t\leq 1$ and $nh^m\leq\varepsilon_m$, and for real (x,ξ), we have

$$|q_n(x,\xi,t,h)\ -\ \Phi_n^{(m)}(x,\xi,t,h)|\ \leq\ K_m n h^m\ , \tag{9.18}$$

where q_n is the symbol satisfying (9.4) and $\Phi_n^{(m)}$ is its approximation, defined in (6.4). Since we used the Weyl calculus and since $e^{-tP_n(h)}$ is self-adjoint, the function $q_n(x,\xi,t,h)$ is real-valued. Therefore, it follows from (9.18) that, for some constants $\varepsilon_m>0$ and $K_m>0$

$$nh^m\ \leq\ \varepsilon_m\ \Longrightarrow |\Lambda_n(t,h)\ -\ \Lambda_n^{(m)}(t,h)|\ \leq\ K_m h^m\ . \tag{9.19}$$

For h small enough, let $n=n(h)$ be an integer such that $2n+1$ is a multiple of d_m, and $\frac{\varepsilon_m}{2}\leq nh^m\leq\varepsilon_m$, where ε_m is a constant such that (9.19) is valid. Then we have

$$|\Lambda(t,h)\ -\ \frac{1}{d_m}Log\ \lambda_m(t,h))|\ \leq\ |\Lambda(t,h)\ -\ \Lambda_n(t,h)|\ +\ |\Lambda_n(t,h)\ -\ \Lambda_n^{(m)}(t,h)|\ +\ \dots$$

$$\dots\ +\ |\Lambda_n^{(m)}(t,h)\ -\ \frac{1}{d_m}Log\ \lambda_m(t,h))|$$

Using (9.5), (9.19) and (9.17), it follows that

$$|\Lambda(t,h)\ -\ \frac{1}{d_m}Log\ \lambda_m(t,h))|\ \leq\ K_m h^m\ +\ \frac{2C}{n}\ \leq\ \left(K_m+\frac{4C}{\varepsilon_m}\right)h^m\ .$$

The lemma is proved, and Theorem 9.3 follows directly.

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