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Nonlocal Perturbations of the Radial Wave Equation. Lax–Phillips Approach.

We establish sufficient conditions under which the scattering of the evolution system described by the radial wave equation can be investigated in the framework of the Lax–Phillips scattering scheme and study structures of the corresponding scattering matrices.

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Nonlocal Perturbations of the Radial Wave Equation. Lax–Phillips Approach.[1]

1 Introduction

The Lax–Phillips approach in the scattering theory lies on the boundary between the stationary and time-dependent methods and it is characterized by well-developed investigation tools that provide a simple relationship between the nature of a perturbation and properties of the corresponding scattering matrix.

However, the possibility of applying the Lax–Phillips approach for studying the scattering of evolution systems is limited by a lot of sufficiently rigid restrictions on free and perturbed evolutions. In view of this, as a rule, the Lax–Phillips approach is used for the investigation of various finite (local) perturbations of the wave equation or, dual to it, Schrödinger equation (see [1] and the bibliography therein). On the other hand, there exist only a few examples of application of the Lax–Phillips approach for studying nonlocal perturbations [2–4]. In this connection, the determination of new classes (types) of nonlocal perturbations of the wave equation, which can be studied in the Lax–Phillips framework, is of great interest.

In the present paper, we establish sufficient conditions under which the scattering of the evolution system described by the radial wave equation

$$\partial_t^2 f(s,t) = \partial_s^2 f(s,t) - \frac{k(k+1)}{s^2} f(s,t) - (Uf)(s,t), \quad k \in \mathbb{N} \cup 0 \tag{1}$$

with a nonlocal potential

$$(Uf)(s,t) = p(s) \int_0^\infty p(\tau) f(\tau,t) d\tau,$$

where a real function $p(s)$ belongs to $L_2(\mathbb{R}_+)$, can be investigated in the framework of the Lax–Phillips scattering scheme and study structures of the corresponding scattering matrices.

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2 Conditions of Applicability of the Lax–Phillips Approach

Taking into account the conditions on the function $p(s)$, we can write equation (1) as follows

$$\frac{d^2}{dt^2}f(t) = -\tilde{L}_k f, \tag{2}$$

where the operator

$$\tilde{L}_k f = l_k(f) = -\frac{d^2}{ds^2}f(s) + \frac{k(k+1)}{s^2}f(s) + p(s)(f,p)_{L_2(\mathbb{R}_+)}, \qquad k \in \mathbb{N} \cup 0 \tag{3}$$

with the domain of definition

$$D(\tilde{L}_k) = \{f(s) \in W_2^2(\mathbb{R}_+) \mid f(0) = 0\}, \quad \text{if} \quad k = 0$$

or

$$D(\tilde{L}_k) = \{f(s) \in L_2(\mathbb{R}_+) \mid l_k(f) \in L_2(\mathbb{R}_+)\}, \quad \text{if} \quad k \in \mathbb{N}$$

is a positive self-adjoint operator in $L_2(\mathbb{R}_+)$. In what follows, we always assume that the function $p(s)$ in (3) is a nonzero function.

It is naturally to assume that equation (2) defines a perturbed evolution if $p(s) \neq 0$ and a free evolution if $p(s) = 0$. Thus, a free evolution is described by the equation

$$\frac{d^2}{dt^2}f(t) = -L_k f, \tag{4}$$

where

$$L_k f = -\frac{d^2}{ds^2}f(s) + \frac{k(k+1)}{s^2}f(s), \quad D(L_k) = D(\tilde{L}_k). \tag{5}$$

Consider the following problem: *whether we can consider simultaneously equations (2) and (4) as, respectively, perturbed and free equations in the Lax–Phillips framework?*

We recall [5] that equation (4) can define a free evolution in the Lax–Phillips sense if and only if, for the group $W_{L_k}(t)$ of solutions of the Cauchy problem of (4), there exist orthogonal subspaces $D_\pm$ (so-called outgoing and incoming subspaces) of the energy space H_{L_k} such that

$$(a)\ \ W_{L_k}(\pm t)D_\pm \subset D_\pm \ \ (t \geq 0); \quad (b)\ \ \bigcap_{t\geq 0} W(\pm t)D_\pm = \{0\}; \quad (c)\ \ D_- \oplus D_+ = H_{L_k}.$$

Here, the energy space H_{L_k} is defined as the closure of the set of initial data

$$\left\{ \begin{pmatrix} u \\ v \end{pmatrix} \mid \forall u \in D(L_k),\ \forall v \in L_2(\mathbb{R}_+) \right\}$$

with respect to the energy norm

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2_{H_{L_k}} = (L_k u, u) + \|v\|^2.$$

Moreover, if equation (2) also defines a perturbed evolution in the Lax–Phillips sense, then the corresponding group $W_{\tilde{L}_k}(t)$ of solutions of the Cauchy problem for (2) possesses *the same subspaces* $D_\pm$ with properties (a) and (b).

The conditions of applicability of the Lax–Phillips approach were discussed in [6] for a general case of the abstract wave equation. It follows from [6] that conditions $(a)-(c)$, which guarantee that equation (4) is free in the Lax–Phillips sense, are equivalent to the following conditions on the operator L_k:

(i) *in the space* $L_2(\mathbb{R}_+)$, *there exists a simple maximal symmetric operator* B *such that the operator* L_k *is a positive self-adjoint extension of the symmetric operator* B^2;

(ii) $\quad (L_k f, f) = \|B^* f\|^2$ *for any* $f \in D(L_k)$.

Furthermore, it is easy to verify that if, simultaneously, equation (2) is perturbed in the Lax–Phillips sense, then, *for the same choice of* B, the operator $\tilde{L}_k$ satisfies only condition (i).

We note that incoming D_- and outgoing D_+ subspaces in $(a)-(c)$ are determined uniquely by the operator B and coincide with the closures of the sets

$$\left\{ \begin{pmatrix} f \\ -iBf \end{pmatrix} \mid \forall f \in D(B^2) \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} f \\ iBf \end{pmatrix} \mid \forall f \in D(B^2) \right\} \tag{6}$$

in the energy space H_{L_k}, respectively.[2]

Thus, we can formulate the problem given above as follows: *whether there exists a simple maximal symmetric operator* B *in* $L_2(\mathbb{R}_+)$ *for which the operators* L_k *and* $\tilde{L}_k$ *satisfy conditions* $(i), (ii)$ *and* (i), *respectively?*

Lemma 1. *The operators* L_k *and* $\tilde{L}_k$ *satisfy simultaneously condition* (i) *for a certain choice of* B *if and only if* $\tilde{L}_k = L_k$.

Proof. If L_k and $\tilde{L}_k$ satisfy condition (i), then $\tilde{L}_k f = L_k f$ for any function $f \in D(B^2)$. On the other hand, by virtue of (3) and (5), the last equality is true only on the subspace $\mathfrak{H}' := L_2(\mathbb{R}_+) \ominus < p(s) >$ of the space $L_2(\mathbb{R}_+)$. Therefore, $\mathfrak{H}' \supset D(B^2)$, and, in view of the density of $D(B^2)$ in $L_2(\mathbb{R}_+)$, we get that $\mathfrak{H}' = L_2(\mathbb{R}_+)$. Hence, $p(s) = 0$ and $\tilde{L}_k = L_k$.

[2] Without less of generality, we assume that $-i \in \rho(B)$.

Conversely, in the case where $\tilde{L}_k = L_k$, a simple maximal symmetric operator B in $L_2(\mathbb{R}_+)$ for which the operator L_k satisfies condition (i) (and, moreover, condition (ii) too) was constructed in [7] as follows:

$$B = T^{-1}\mathcal{B}T, \qquad D(B) = T^{-1} \overset{0}{W_2^1} (\mathbb{R}_+). \tag{7}$$

Here, $T = F_{sin}\Gamma_{k+1/2}$ is an unitary operator in $L_2(\mathbb{R}_+)$, $\mathcal{B} = i\frac{d}{ds}$ $(D(\mathcal{B}) = \overset{0}{W_2^1} (\mathbb{R}_+))$ is a simple maximal symmetric operator in $L_2(\mathbb{R}_+)$, and

$$(F_{sin}u)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(s)\sin sx ds, \quad (\Gamma_{k+1/2}u)(x) = \int_0^\infty \sqrt{sy}u(s)J_{k+1/2}(sy)ds$$

are the sine-Fourier and Hankel transformations, respectively. Lemma 1 is proved.

Lemma 1 states that we cannot consider the operators $\tilde{L}_k$ and L_k in the Lax–Phillips framework simultaneously. At the same time, the assumption that equation (4) defines a free evolution is natural and, as was shown in [7], equation (4) is free in the Lax–Phillips sense. Therefore, to avoid the trivial situation $\tilde{L}_k = L_k$, we must slightly modify condition (i) under which equation (2) defines a perturbed evolution in the Lax–Phillips sense.

We note that, in an implicit simplest form, the approach to the definition of a Lax–Phillips perturbed evolution proposed below was used by Lax and Phillips [5] and Pavlov [4] for the investigation of finite and nonfinite perturbations of the wave equation, respectively.

In what follows, we fix the simple maximal symmetric operator B in $L_2(\mathbb{R}_+)$ which is defined by (7) and consider the set $\mathcal{L}_B$ of operators B', each of which satisfies the following condition:

I. *There exists an invariant with respect to B subspace $\mathfrak{H}' \subset L_2(\mathbb{R}_+)$ such that the operator*

$$B' = B|_{D(B')}, \qquad D(B') = D(B) \cap \mathfrak{H}'$$

acting in $\mathfrak{H}'$ possesses the same defect numbers as B.

Definition 1. *The operator $\tilde{L}_k$ is called a Lax–Phillips perturbed operator if there exists an operator $B' \in \mathcal{L}_B$ such that $\tilde{L}_k$ is an extension (with the exit in the more wide space $L_2(\mathbb{R}_+)$) of the symmetric operator B'^2.*

Replacing in (6) B by B', it is easy to see that the condition of Definition 1 ensures the existence of subspaces $D'_\pm$ with properties (a) and (b) for the group $W_{\tilde{L}_k}(t)$ which are *parts* of the initial subspaces $D_\pm$ constructed by B.

Now, we establish necessary and sufficient conditions on the function $p(s)$ in (3) under which the operator $\tilde{L}_k$ is Lax–Phillips perturbed.

In $L_2(\mathbb{R}_+)$, we consider the isometric operator $G = (B - iI)(B + iI)^{-1}$, where B is defined by (7). A function $p(s) \in L_2(\mathbb{R}_+)$ is called *noncyclic* for G^* if the set

$$E_{p(s)} := \bigvee_{n=0}^{\infty} G^{*n} p(s)$$

is contained strictly in $L_2(\mathbb{R}_+)$, i.e., $E_{p(s)} \neq L_2(\mathbb{R}_+)$.

Theorem 1. *The operator $\tilde{L}_k$ is a Lax–Phillips perturbed operator if and only if the function $p(s)$ in (3) is noncyclic for G^*.*

Proof. Assume that $\tilde{L}_k$ is a Lax–Phillips perturbed operator. By virtue of condition **I**, there exists a nonzero subspace $\mathfrak{H}'$ such that

$$\tilde{L}_k u = B'^2 u = B^2 u = L_k u, \quad \forall u \in D(B^2) \cap \mathfrak{H}'.$$

Therefore, equalities (3) and (5) imply that $p(s) \in \mathfrak{H}_1 = L_2(\mathbb{R}_+) \ominus \mathfrak{H}'$.

Obviously, the subspace $\mathfrak{H}'$ is invariant with respect to G. Hence, the subspace $\mathfrak{H}_1$ is invariant with respect to G^*. Therefore, the relation $p(s) \in \mathfrak{H}_1$ yields

$$E_{p(s)} \subseteq \mathfrak{H}_1 \subset L_2(\mathbb{R}_+).$$

Thus, the function $p(s)$ is noncyclic for G^*.

Conversely, if the function $p(s)$ is noncyclic for G^*, then the nonzero subspace $\mathfrak{H}' = L_2(\mathbb{R}_+) \ominus E_{\mathsf{p}(s)}$ is invariant for the isometric operator G. Taking this fact into account, it is easy to verify that the operator

$$B' = -i(G + I)(G - I)^{-1}|_{\mathfrak{H}'}$$

satisfy condition **I**. Thus, $\tilde{L}_k$ is a Lax–Phillips perturbed operator. Theorem 1 is proved.

The condition that the function $p(s)$ must be noncyclic from Theorem 1 is not always convenient for the verification. Equivalent conditions can be formulated with the use of well-developed theory of cyclic elements in Hardy spaces (see [8] and the bibliography therein). We recall some results in the form convenient for our considerations. Similarly to [8], we denote by $\mathcal{R}$ the set of rational functions from the Hardy space H^2_{Π} in the upper half-plane. It is clear that any function $r(z) \in \mathcal{R}$ has a finite number of poles z_j in the lower half-plane $\Im m\ z < 0$.

Definition 2. [8] *The capacity of a function $r \in \mathcal{R}$ is called the quantity*

$$cap_z(r) = \sum k(z_j) \left(1 - \frac{|z_j + i|}{|z_j - i|}\right),$$

where the summation is taken over all poles z_j of the function $r(z)$ and $k(z_j)$ is order of the pole at the point z_j.

We denote by G the operator of multiplication by $\frac{i\delta+1}{i\delta-1}$ in the space H^2_Π. A function $\psi(\delta) \in H^2_\Pi$ is called *noncyclic* for G^* if

$$E_{\psi(\delta)} := \bigvee_{n=0}^{\infty} \mathsf{G}^{*n}\psi(\delta) \neq H^2_\Pi.$$

It is well known (see., for example, [8, p. 327]) that the operator

$$\Phi : f(e^{i\theta}) \to \psi(\delta) = \frac{\sqrt{2}}{1-i\delta} f\left(\frac{i\delta+1}{i\delta-1}\right) \quad (\delta \in \mathbb{R}), \qquad \forall f(e^{i\theta}) \in L_2(0, 2\pi) \tag{8}$$

maps isometrically the space $L_2(0, 2\pi)$ onto $L_2(\mathbb{R})$. Furthermore, the Hardy class H^2 in the unit disk is mapped onto the Hardy class H^2_Π in upper half-plane. Under this mapping, the operator $\mathcal{S}$ of multiplication by $e^{i\theta}$ in H^2 transforms into the operator G. Hence, the set of noncyclic elements for G^* is the image of the set of noncyclic elements for $\mathcal{S}^*$ under the mapping Φ. We note that the set $\mathcal{R}$ is the image of the set of rational functions f from the Hardy class H^2 under the mapping Φ. Furthermore, the capacity cap_w of the function f defined in [8, p. 57] either is equal to the capacity $cap_z(\Phi f)$ of the function Φf or differs from this capacity by one.

Using properties of the operator Φ mentioned above, we can formulate assertion (3) of the theorem on finite-dimensional approximation ([8, p. 57]) as follows:

Lemma 2. *The function $\psi(\delta) \in H^2_\Pi$ is noncyclic for G^* if and only if there exists a sequence of rational functions $r_n(\delta) \in \mathcal{R}$ such that $\lim_{n\to\infty} \|\psi - r_n\|_{L_2(\mathbb{R})} = 0$ and $\sup_{n\geq 1} cap(r_n) < \infty$.*

Taking relations (7) into account, it is easy to see that the operator

$$[\mathsf{Y}p](\delta) := \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{i\delta s}(Tp)(s)ds, \qquad \forall p(s) \in L_2(\mathbb{R}_+) \tag{9}$$

maps isometrically $L_2(\mathbb{R}_+)$ onto H^2_Π in such a way that the operator B transforms into the operator of multiplication by δ in H^2_Π and the operator G transforms into G.

Theorem 2. *The operator $\tilde{L}_k$ is a Lax–Phillips perturbed operator if and only if for the function $\psi(\delta) = [\mathsf{Y}p](\delta)$, where Y is defined by (9), there exists a sequence of rational functions $r_n(\delta) \in \mathcal{R}$ such that*

$$\lim_{n\to\infty} \|\psi - r_n\|_{L_2(\mathbb{R})} = 0 \quad \text{and} \quad \sup_{n\geq 1} cap(r_n) < \infty.$$

Proof. Since the mapping Y transfers G into G, we get that Y maps the set of noncyclic elements for G^* in the space $L_2(\mathbb{R}_+)$ onto the set of noncyclic elements for G^* in H^2_Π. Hence, taking into account Lemma 2, we obtain that Theorem 2 is a direct corollary of Theorem 1.

Corollary 1. *The set of functions $p(s) \in L_2(\mathbb{R}_+)$ that determine (by means of relation (3)) Lax–Phillips perturbed operators $\tilde{L}_k$ is a dense linear set of the first category in $L_2(\mathbb{R}_+)$.*

Proof. Corollary 1 follows from Theorem 2 and the Corollary in [8, p. 58].

3 The Construction of the Scattering Matrix

It follows from (3) and (5) that the operator $\tilde{L}_k$ can be considered as a one-dimensional perturbation of L_k. Hence, in view of well-known results of the Kato–Birman theory [1], the wave operators

$$Q_{\pm}(\tilde{L}_k, L_k) = \lim_{t\to\pm\infty} W_{\tilde{L}_k}(-t)W_{L_k}(t)$$

exist and are defined on the whole space H_{L_k}. Thus, the scattering operator

$$S_{(\tilde{L}_k,L_k)} = \Omega^*_+(\tilde{L}_k, L_k)\Omega_-(\tilde{L}_k, L_k). \tag{10}$$

is well posed as an operator in H_{L_k}. The aim of the present section is to investigate the structure of $S_{(\tilde{L}_k,L_k)}$ in the case where $\tilde{L}_k$ is a Lax–Phillips perturbed operator.

We start from the more detail investigation of the structure of the set $\mathcal{L}_B$ from Definition 1.

Let $\mathfrak{A}$ be the set of all isometric operators A in $L_2(\mathbb{R}_+)$ that admit the following representation:

$$A = g(B) := \int_{-\infty}^{\infty} g(\xi)dE_\xi, \tag{11}$$

where E_ξ is the spectral function of the operator B and $g(\lambda)$ is an inner function, i.e., a bounded analytic function in the upper half-plane and such that $|g(\delta)| = 1 \quad (\delta \in \mathbb{R})$. We note that, by virtue of the Plesner's results [9, 10], the right-hand side of (11) is well posed and determines an isometric operator in $L_2(\mathbb{R}_+)$.

Now, we show that the set $\mathcal{L}_B$ admits the following simple description where elements from $\mathfrak{A}$ are considered as parameters:

Lemma 3. *An operator B' belongs to $\mathcal{L}_B$ if and only if there exists an operator $A \in \mathfrak{A}$ such that*

$$B' = ABA^*, \qquad D(B') = AD(B). \tag{12}$$

Proof. By virtue of Proposition 1 in [11], an operator $A \in \mathfrak{A}$ commutes with B. Therefore, choosing $\mathfrak{H}'$ as $AL_2(\mathbb{R}_+)$, it is easy to verify that relations (12) define an operator B', which satisfies condition **I** and, hence, $B' \in \mathcal{L}_B$.

Conversely, let $B' \in \mathcal{L}_B$. By virtue of condition **I**, B' acts in a subspace $\mathfrak{H}'$ of $L_2(\mathbb{R}_+)$, which is invariant with respect to B. Therefore, taking into account the definition of Y (relation (9)), we get that the subspace $\mathfrak{L} = \mathsf{Y}\mathfrak{H}'$ of H^2_Π is invariant with respect to G. But then, by virtue of the Beurling theorem [12], there exists an inner function $g(\lambda)$ such that $\mathfrak{L} = g(\delta)H^2_\Pi$. Substituting this function into (11), we define an isometric operator A.

For any element $A\gamma \ \ (\gamma \in L_2(\mathbb{R}_+))$, using (7) and (11), we obtain

$$(TA\gamma)(s) = T(g(B)\gamma)(s) = g(\mathcal{B})(T\gamma)(s).$$

Using this equality, similarly to the proof of Theorem 1 in [11], we arrive at a conclusion that

$$[\mathsf{Y}(A\gamma)](\delta) = g(\delta)[\mathsf{Y}\gamma](\delta) \qquad (\forall \gamma \in L_2(\mathbb{R}_+)). \tag{13}$$

Therefore,

$$\mathsf{Y}\mathfrak{H}' = \mathfrak{L} = g(\delta)H^2_\Pi = g(\delta)\mathsf{Y}L_2(\mathbb{R}_+) = \mathsf{Y}AL_2(\mathbb{R}_+)$$

and, hence, $AL_2(\mathbb{R}_+) = \mathfrak{H}'$. Using the last equality and the fact that A commutes with B, we can write condition **I** in the form of (12). Lemma 3 is proved.

In what follows, we denote by B_A the operator $B' \in \mathcal{L}_B$ that is defined by (12). This operator acts in the space $AL_2(\mathbb{R}_+)$ which, in what follows, we shall denote by $\mathfrak{H}_A$.

Let $\tilde{L}_k$ be a Lax–Phillips perturbed operator. This means, in view of Definition 1 and Lemma 3, that there exists an isometric operator $A \in \mathfrak{A}$ such that $\tilde{L}_k$ is an extension of B^2_A, where B_A is a simple maximal symmetric operator acting in the subspace $\mathfrak{H}_A$ of $L_2(\mathbb{R}_+)$. We fix this operator A and consider the operator

$$L_A = AL_kA^*, \qquad D(L_A) = AD(L_k), \tag{14}$$

where L_k is defined by (3).

It follows from (12) and (14) that, similarly to L_k, the operator L_A satisfies conditions (i) and (ii) (with the substitution B_A for B) and, hence, the equation

$$u_{tt} = L_A u \tag{15}$$

defines a free evolution in the Lax–Phillips sense. Furthermore, as follows from the choice of B_A, the operator $\tilde{L}_k$ satisfies conditions (i) for the same operator B_A and, hence, equation (2) defines a perturbed evolution in the Lax–Phillips sense.

Thus, the groups $W_{L_A}(t)$ and $W_{\tilde{L}_k}(t)$ of solutions of the Cauchy problems for equations (14) and (2) *simultaneously* can be considered in the Lax–Phillips framework, moreover, the first of them defines a free evolution and the second group defines a perturbed evolution in the Lax–Phillips sense. In this case, in addition to a similar statement (10) that the corresponding scattering operator

$$S_{(\tilde{L}_k, L_A)} = \Omega_+^*(\tilde{L}_k, L_A)\Omega_-(\tilde{L}_k, L_A). \tag{16}$$

is well posed on the whole energy space H_{L_A}, which is defined by analogy with H_{L_k} (with the replacement of L_k by L_A and $L_2(\mathbb{R}_+)$ by $\mathfrak{H}_A$), we can state, using the general statements of the Lax–Phillips theory [5], that the image of $S_{(\tilde{L}_k, L_A)}$ in the spectral representation of free group $W_{L_A}(t)$ can be realized as an operator of multiplication by a scalar function $\mathfrak{S}_{(\tilde{L}_k, L_A)}(\delta)$ $(\delta \in \mathbb{R})$, which is the restriction to the real axis of a contracting function $\mathfrak{S}_{(\tilde{L}_k, L_A)}(z)$ $(\Im m\ z < 0)$ holomorphic in the lower half-plane (the Heisenberg scattering matrix). Moreover, using results of [7, 13], we obtain a simple algorithm for the construction of $\mathfrak{S}_{(\tilde{L}_k, L_A)}(z)$, which does not require the previous construction of the spectral representation of $W_{L_A}(t)$.

It is clear that condition (i) remains true for the operator L_k if we replace in it the operator B by B_A. Therefore, in view of Proposition 5.2 in [14, Chap. 4], there exist wave operators

$$Q_{\pm}(L_k, L_A) = \lim_{t \to \pm\infty} W_{L_k}(-t) W_{L_A}(t), \tag{17}$$

which are defined on the whole space H_{L_A}. In what follows, we assume that these operators satisfy the following condition

$$Q_{\pm}(L_k, L_A) H_{L_A} = H_{L_k}. \tag{18}$$

We note that condition (18) is natural in the sense that it ensures an equivalence of Lax–Phillips free evolutions, which are described by the groups $W_{L_k}(t)$ and $W_{L_A}(t)$.

It follows from (18) and the chain law [1, p. 29] that $\Omega_{\pm}(\tilde{L}_k, L_A) = \Omega_{\pm}(\tilde{L}_k, L_k)\Omega_{\pm}(L_k, L_A)$, Multiplying this equality by $\Omega_{\pm}^*(L_k, L_A)$ and passing to adjoint operators, we obtain

$$\Omega_{\pm}^*(\tilde{L}_k, L_k) = \Omega_{\pm}(L_k, L_A)\Omega_{\pm}^*(\tilde{L}_k, L_A).$$

Substituting the obtained equalities into (10) and taking into account (16), we arrive at a conclusion that

$$S_{(\tilde{L}_k, L_k)} = \Omega_+(L_k, L_A) S_{(\tilde{L}_k, L_A)} \Omega_-^*(L_k, L_A). \tag{19}$$

As was shown in [7], $L_k = B^* B$, where B is defined by (7). Therefore, the translation representation for the group $W_{L_k}(t)$ which is associated with subspaces $D_{\pm}$ of form (6) is defined by formula 4.18 in [14, p. 221]. Applying to this formula the Fourier transform in $L_2(\mathbb{R})$ and taking into account the definition (9) of Y, we conclude that an isometric operator $\mathcal{F} : H_{L_k} \to L_2(\mathbb{R})$ that, for any $u \in D(L_k)$ and $v \in L_2(\mathbb{R}_+)$, is defined by the formula

$$\mathcal{F}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\left(\mathsf{Y}[iBu+v](\delta) + \mathsf{Y}[iBu-v](-\delta)\right) \tag{20}$$

determines the spectral representation for $W_{L_k}(t)$ associated with $D_\pm$.

Similarly, the spectral representation for the group $W_{L_A}(t)$ associated with subspaces $D^A_\pm$, which are defined by (6) with the replacement of B by B_A, is determined by an isometric operator $\mathcal{F}_A : H_{L_A} \to L_2(\mathbb{R})$, which, for any $u \in D(L_A)$ and $v \in \mathfrak{H}_A$, is defined by the formula

$$\mathcal{F}_A\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\left(\mathsf{Y}_A[iB_Au+v](\delta) + \mathsf{Y}_A[iB_Au-v](-\delta)\right). \tag{21}$$

Here,

$$[\mathsf{Y}_Ap](\delta) := \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{i\delta s}(TA^*p)(s)ds, \quad \forall p(s) \in \mathfrak{H}_A$$

and T is the same as in (7).

Equalities (18) and (19) imply that

$$\mathfrak{S}_{(\tilde{L}_k,L_k)} := \mathcal{F}S_{(\tilde{L}_k,L_k)}\mathcal{F}^{-1} = \mathfrak{K}_+\mathfrak{S}_{(\tilde{L}_k,L_A)}(\delta)\mathfrak{K}_-^{-1}, \tag{22}$$

where $\mathfrak{K}_\pm = \mathcal{F}\Omega_\pm(L_k, L_A)\mathcal{F}_A^{-1}$ and $\mathfrak{S}_{(\tilde{L}_k,L_A)}(\delta) = \mathcal{F}_AS_{(\tilde{L}_k,L_A)}\mathcal{F}_A^{-1}$.

Lemma 4. *The operators $\mathfrak{K}_+$ and $\mathfrak{K}_-$ act in $L_2(\mathbb{R})$ as the operators of multiplication by the functions $g(\delta)$ and $g(-\delta)$, respectively.*

Proof. Taking into account (14), it is easy to see that the closure in H_{L_k} of the operator $\mathbb{A} : H_{L_k} \to H_{L_A}$ defined by the formula

$$\mathbb{A}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} Au \\ Av \end{pmatrix}, \quad \forall u \in D(L_k), \ \ \forall v \in L_2\mathbb{R}_+$$

is an isometric operator from H_{L_k} onto H_{L_A}.

In view of (20) and (21), taking into account the definitions of B_A and $\mathfrak{H}_A$, it is easy to establish that

$$\mathcal{F}_A\mathbb{A} = \mathcal{F}. \tag{23}$$

Hence, the operators $\mathfrak{K}_\pm$ in (22) can be represented in the form $\mathfrak{K}_\pm = \mathcal{F}\Omega_\pm(L_k, L_A)\mathbb{A}\mathcal{F}^{-1}$. Twice using (23), we obtain that

$$\mathcal{F}_A\mathbb{A}W_{L_k}(t)d = e^{i\delta t}\mathcal{F}d = e^{i\delta t}\mathcal{F}_A\mathbb{A}d = \mathcal{F}_AW_{L_A}(t)\mathbb{A}d \quad (\forall d \in H_{L_k}).$$

Therefore, $W_{L_A}(t)\mathbb{A} = \mathbb{A}W_{L_k}(t)$. The last equality and relations (17), (18) imply that the operators $\Omega_\pm(L_k, L_A)\mathbb{A}$ are unitary operators in H_{L_k} that commute with $W_L(t)$. Hence (see Corollary 4.1 in [5, Chap. 2]), the images $\mathfrak{K}_\pm$ of these operators in the spectral representation of $W_{L_k}(t)$ can be written as operators of multiplication by scalar functions $\mathfrak{K}_\pm(\delta)$.

Let $d_+ = \begin{pmatrix} u \\ iBu \end{pmatrix}$, where $u \in B^2$. Clearly, $d_+ \in D_+$ and

$$\mathbb{A}d_+ = \begin{pmatrix} Au \\ iABu \end{pmatrix} = \begin{pmatrix} v \\ iB_A v \end{pmatrix} \in D_A \qquad (v = Au \in D(B_A^2) = AD(B^2)).$$

Here, D_+^A is defined by (6) with the replacement of B by B_A. It follows from the first equality (for $\tilde{L} = L_k$ and $L = L_A$) of the proof of Proposition 5.2 in [14, Chap. 4] that $\Omega_+(L_k, L_A)\mathbb{A}d_+ = \mathbb{A}d_+$. Hence, taking (20) and (13) into account, we obtain

$$(\mathfrak{K}_+(\delta)(\mathcal{F}d_+)(\delta) = [\mathcal{F}\mathbb{A}d_+)](\delta) = i\sqrt{2}[\Upsilon ABu](\delta) = i\sqrt{2}g(\delta)[\Upsilon Bu](\delta) = g(\delta)(\mathcal{F}d_+)(\delta).$$

Thus $\mathfrak{K}_+(\delta) = g(\delta)$. Reasoning similarly, we obtain $\mathfrak{K}_-(\delta) = g(-\delta)$. Lemma 4 is proved.

Theorem 3. *Assume that $\tilde{L}_k$ is a Lax–Phillips perturbed operator and condition (18) is true. Then the image $\mathfrak{S}_{(\tilde{L}_k,L_k)}$ of the scattering operator $S_{(\tilde{L}_k,L_k)}$ in the spectral representation for the free group $W_{L_k}(t)$ acts as the operator of multiplication by the scalar function (scattering matrix)*

$$\mathfrak{S}_{(\tilde{L}_k,L_k)}(\delta) = \frac{g(\delta)}{g(-\delta)}\mathfrak{S}_{(\tilde{L}_k,L_A)}(\delta), \qquad \delta \in \mathbb{R},$$

where $g(\delta)$ is the restriction on the real axis of the inner function $g(\lambda)$ in the upper half-plane and $\mathfrak{S}_{(\tilde{L}_k,L_A)}(\delta)$ is the restriction on the real axis of the Heisenberg scattering matrix $\mathfrak{S}_{(\tilde{L}_k,L_A)}(z)$ holomorphic in the lower half-plane.

Proof. Theorem 3 follows directly from relation (22) and Lemma 4.

4 Examples of Lax–Phillips perturbed operators in the case of s-wave.

In the case of s-wave, the parameter k in (3) and (5) is equal to zero. Hence, the operator T in (7) is the identity operator and $B = \mathcal{B}$.

In the space $L_2(\mathbb{R}_+)$, we consider the orthonormal basis of Laguerre functions

$$q_n(s) = \frac{e^{s/2}}{n!}\frac{d^n}{ds^n}(s^n e^{-s}) \qquad (n = 0, 1 \ldots).$$

Using (9) (as $T = I$) and relation (25) in [15, p. 158], for any Laguerre function $q_n(s)$, we obtain

$$\psi(\delta) = [\mathsf{Y} q_n](\delta) = (-1)^n \sqrt{\frac{2}{\pi}} \frac{(1+2i\delta)^n}{(1-2i\delta)^{n+1}}.$$

It is clear that the function $\psi(\delta)$ belongs to the class $\mathcal{R}$. Hence, Theorem 2 yields that, for any partial Fourier sum

$$Q_m(s) = \sum_{n=0}^{m} \alpha_n q_n(s) \qquad (\alpha_n = (p, q_n)_{L_2(\mathbb{R}_+)}),$$

of an arbitrary function $p(s) \in L_2(\mathbb{R}_+)$, the operator

$$\tilde{L} = -\frac{d^2}{ds^2} f(s) + Q_m(s) \int_0^\infty Q_m(\tau) f(\tau) d\tau,$$

$D(\tilde{L}) = \{f(s) \in W_2^2(\mathbb{R}_+) \mid f(0) = 0\}$ is a Lax–Phillips perturbed operator.

Now, we present an example of $p(s)$, for which the corresponding operator

$$\tilde{L} = -\frac{d^2}{ds^2} f(s) + p(s)(f, p)_{L_2(\mathbb{R}_+)} \tag{24}$$

is not a Lax–Phillips perturbed operator (under the choice of the initial operator B as $\mathcal{B}$). In this case, as follows from the proof of Theorem 2, the function $\psi(\delta) = [\mathsf{Y} p](\delta)$ must be a cyclic function for G^* in H^2_Π or, which is equivalent, the function $\Phi^{-1}[\psi_k(\delta)]$ must be a cyclic function for $\mathcal{S}^*$ in H^2, where the transformation Φ is defined by (8).

For example, the function $f(e^{i\theta}) = (1 - e^{i\theta})^{1/2}$ is cyclic for $\mathcal{S}^*$ ([8, p. 50]). Hence, it follows from the definition of Φ, equality (9), and relation (3) in [15, p. 133] (for $\alpha = 1, p = -i\delta, \nu = 3/2$) that the function $p(s) = \mathsf{Y}^{-1}[\Phi f(e^{i\theta})] = \sqrt{s}e^{-s}$ does not define a Lax–Phillips perturbed operator in (24).

Let us consider the inner function $g(\lambda) = [(\lambda - i)(\lambda + i)^{-1}]^m$ $(m \in \mathbb{N})$ and, by means of (11), define the corresponding isometric operator

$$A = [(\mathcal{B} - iI)(\mathcal{B} + iI)^{-1}]^m. \tag{25}$$

Using the well-known ([16, p. 365]) relation $Aq_n(2s) = q_{n+m}(2s)$ for the Laguerre functions $q_n(2s)$ $(n \in \mathbb{N} \cup \{0\})$, we get that

$$< q_0(2s), \ldots, q_{m-1}(2s) >= L_2(\mathbb{R}_+) \ominus AL_2(\mathbb{R}_+).$$

It follows from the last relation and the definition of the Laguerre functions that an arbitrary function $p(s) = e^{-s} P_{m-1}(s)$, where $P_{m-1}(s)$ is a real polynomial of order $m - 1$, belongs to the subspace $L_2(\mathbb{R}_+) \ominus AL_2(\mathbb{R}_+)$. Therefore, if we choose A in the form (25), then the operator

$$\tilde{L} = -\frac{d^2}{ds^2}f(s) + e^{-s}P_{m-1}(s)\int_0^\infty e^{-\tau}P_{m-1}(\tau)f(\tau)d\tau \quad (m \in \mathbb{N}) \tag{26}$$

is an extension of ${B_A}^2$, where B_A is defined by (12) (for $B = \mathcal{B}$).

We note that condition (18) is equivalent to the condition of quasi-equivalence of the subspaces $D_\pm$ and $D_\pm^A$ proposed in [11]. Therefore, using Proposition 2 in [11], it is easy to verify that equality (18) is true if we choose A as (25). Thus, we can use Theorem 3 for the determination of the scattering matrix $\mathfrak{S}_{(\tilde{L},L_0)}(\delta)$, where $\tilde{L}$ is defined by (26).

Let for definiteness $m = 2$. Then the polynomial $P_1(s)$ in (26) has the form $P_1(s) = p + p_1 s \quad \{p, p_1\} \in \mathbb{R}$. Using the algorithm for the construction of Heisenberg scattering matrices proposed in [7, 13], we obtain

$$\mathfrak{S}_{(\tilde{L},L_A)}(\delta) = \frac{4(\delta+i)^4 - (p+p_1)^2(\delta+2i)^2 - p^2(\delta^2+1) + 2p_1^2 - (p+p_1)^2}{4(\delta-i)^4 - (p+p_1)^2(\delta-2i)^2 - p^2(\delta^2+1) + 2p_1^2 - (p+p_1)^2} I.$$

Therefore, by virtue of Theorem 3 and relation (25), we get

$$\mathfrak{S}_{(\tilde{L},L_0)}(\delta) = \frac{(\delta-i)^4}{(\delta+i)^4}\mathfrak{S}_{(\tilde{L},L_A)}(\delta).$$

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